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Centers of Γ-isotypity in abelian groups

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Centers of Γ-Isotypity in Abelian Groups.

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All groups in this paper are assumed to be abelian groups. We follow the terminology and notation of [1]. Let $\mathbb{P}$ be the set of all primes in the natural order; denote by $\mathcal{K}$ the class of all sequences $(\alpha_p)_{p\in\mathbb{P}}$, where each α_p is either an ordinal or the symbol ∞ which is considered to be larger than any ordinal. Let G be a group and $\Gamma = (\alpha_p)_{p\in\mathbb{P}} \in \mathcal{K}$. A subgroup H of G is said to be Γ-isotype in G if $p^\beta H = H \cap p^\beta G$ for every prime p and for every ordinal $\beta \leqslant \alpha_p$. If $\alpha_p = 0$, $\alpha_p = 1$, $\alpha_p = \omega$, $\alpha_p = \infty$ for every prime p then Γ-isotype subgroups are precisely subgroups, neat subgroups, pure subgroups, isotype subgroups respectively.

Let G be a group. If H is a subgroup of G, then each H-high subgroup of G is neat in G though not necessarily pure in G. The subgroup H is said to be a center of purity in G (J. D. Reid [5]) if every H-high subgroup of G is pure in G. The question of determining all centers of purity (J. M. Irwin [2]) was settled by R. S. Pierce [4] (see also J. D. Reid [5]). The class of all groups in which every subgroup is a center of purity (i.e. in which every neat subgroup is pure) was described by C. Megibben [3]. The results of R. S. Pierce and C. Megibben were generalized by V. S. Rochlina [6].

The purpose of this paper is to determine all centers of Γ-isotypity in G and describe the class of all groups in which every subgroup is a center of Γ-isotypity.

DEFINITION. Let G be a group and $\Gamma \in \mathcal{K}$. A subgroup H of G will be called a center of Γ-isotypity in G (a center of isotypity in G) if every H-high subgroup of G is Γ-isotype (isotype) in G.

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The necessary and sufficient conditions for a subgroup H of G to be a center of Γ-isotypity in G are contained already in the following lemma (compare with Proposition 2.1 [5], Lemma [4] and Lemma 2 [6]).

LEMMA. Let G be a group, H a subgroup of G and $\Gamma = (\alpha_p)_{p\in \mathrm{P}} \in \mathcal{K}$. Then there is a H-high subgroup of G that is not Γ-isotype in G iff there are a prime p, an ordinal $\beta < \alpha_p$ and elements $m \in G$, $h \in H[p]$ with the following properties:

(i) $O(m) = \infty$ or $O(m) = p^j$, where $j > 1$;

(ii) $h_p^*(m) = h_p^*(h) < \beta \leqslant h_p^*(h + m)$;

(iii) $\langle m \rangle \cap H = 0$.

PROOF. Let M be an H-high subgroup of G that is not Γ-isotype in G. Then there is a prime p and an ordinal $\alpha \leqslant \alpha_p$ such that $p^\alpha M \neq$ $\neq M \cap p^\alpha G$; let α be the least ordinal with this property. Obviously, $\alpha = \beta + 1 > 1$. Let $x \in M \cap p^\alpha G \setminus p^\alpha M$; $x = px'$, where $x' \in p^\beta G$. Since $px' \in M \cap pG = pM$, there is $m_1 \in M$ such that $px' = pm_1$. Hence $x' - m_1 \in G[p] = M[p] \oplus H[p]$, i.e. $x' = m_1 + m_2 + h$, where $m_2 \in M[p]$ and $h \in H[p]$. If $h \in p^\beta G$ then $n = m_1 + m_2 \in M \cap p^\beta G = p^\beta M$ and $x = px' = pn \in p^\alpha M$, a contradiction. Hence

$$h_p^*(n) = h_p^*(h) < \beta \leqslant h_p^*(h + n)\,.$$

If n is of infinite order then write $m = n$ and we are through. Otherwise denote by m the p-component of n. Now $h_p^*(m) = h_p^*(n)$ and $h_p^*(h + m) = h_p^*(h + n)$. If $O(m) = p$ then $x = pn \in p^\alpha M$, a contradiction. Consequently $O(m) = p^j$, where $j > 1$.

Conversely, suppose that there are a prime p, an ordinal $\beta < \alpha_p$ and elements $m \in G$, $h \in H[p]$ with the properties (i)-(iii). If m is of infinite order then write $n = 0$. If $O(m) = p^j$, where $j > 1$, then write $n = p^{j-1}m$; hence

$$n = p^{j-1}(m + h) \in p^\beta G[p] \setminus H\,.$$

In the both cases, there is a subgroup S such that $n \in S$ and

$$p^\beta G[p] = S \oplus (p^\beta G[p] \cap H)\,.$$

Further, there is a subgroup M_0 containing S such that

$$G[p] = M_0 \oplus H[p] .$$

Now, $\langle M_0, m\rangle \cap H = 0$. For, if $m_0 + zm = h'$, where $m_0 \in M_0$, $h' \in H$ and z is an integer, then

$$pzm = ph' \in \langle m\rangle \cap H = 0 .$$

If $O(m) = \infty$ then $z = 0$ and $h' \in M_0 \cap H[p] = 0$. If $O(m) = p^j$, where $j > 1$, then $z = p^{j-1}z'$ and

$$m_0 + zm = m_0 + z'n = h' \in M_0 \cap H[p] = 0 .$$

Let M be an H-high subgroup of G containing $\langle M_0, m\rangle$. If $pm \in \in p^{\beta+1}M$ then there is $m' \in p^\beta M$ such that $pm = pm'$. Hence

$$m' - m \in M \cap G[p] = M \cap (M_0 \oplus H[p]) = M_0 ,$$

$$h - m' + m \in p^\beta G[p] ,$$

i.e.

$$h - m' + m = s + x ,$$

where $s \in S$ and $x \in p^\beta G[p] \cap H$. Consequently,

$$h - x = s + m' - m \in M_0 \cap H[p] = 0$$

and $h = x \in p^\beta G$, a contradiction. Hence $pm \notin p^{\beta+1}M$. On the other hand,

$$pm = p(m + h) \in p^{\beta+1}G \cap M ,$$

i.e. M is not Γ-isotype in G.

THEOREM. Let G be a group, H a subgroup of G and $\Gamma = (\alpha_p)_{p\in \mathrm{P}} \in \mathcal{K}$. Then H is a center of Γ-isotypity in G iff for every prime p one of the following two conditions is satisfied:

(i) $H[p] \subset \bigcap_{\gamma<\alpha_p} p^\gamma G$.

(ii) There is an ordinal $\gamma < \alpha_p$ such that $(p^\gamma G + H)/H$ is a torsion group and $p^{\gamma+2}G[p] \subset H[p] \subset p^\gamma G$.

Proof. Suppose that for a prime p neither (i) nor (ii) holds. Hence there is an ordinal $\beta < \alpha_p$ such that $H[p] \not\subset p^\beta G$; let β be the least ordinal with this property. Obviously, $\beta = \gamma + 1$ and $H[p] \subset p^\gamma G$. Let $h \in H[p]$ such that $h_p^*(h) = \gamma$. By assumption, either $(p^\gamma G + H)/H$ is not torsion or $p^{\gamma+2}G[p] \not\subset H[p]$. In the first case there is an element $g \in p^\gamma G$ of infinite order such that $\langle g\rangle \cap H = 0$. In the second case there is an element $u \in p^{\gamma+2}G[p] \setminus H[p]$; write $u = pv$, where $v \in p^\beta G$. If we denote $m = pg - h$, resp. $m = v - h$, then m is of infinite order, resp. $O(m) = p^2$; in the both cases $\langle m\rangle \cap H = 0$. Further,

$$h_p^*(m) = h_p^*(h) = \gamma < \beta \leqslant h_p^*(h + m)$$

and by lemma, H is not a center of Γ-isotypity in G.

Conversely, suppose that for every prime p one of the two conditions (i), (ii) holds. If H is not a center of Γ-isotypity in G then by lemma, there are a prime p, an ordinal $\beta < \alpha_p$ and elements $m \in G$, $h \in H[p]$ such that

$$O(m) = \infty \qquad \text{or} \qquad O(m) = p^j, \qquad \text{where } j > 1\,,$$
$$h_p^*(m) = h_p^*(h) < \beta \leqslant h_p^*(h + m)\,,$$
$$\langle m\rangle \cap H = 0\,.$$

Since $h \notin p^\beta G$, for the prime p the condition (i) is not satisfied. Hence there is an ordinal $\gamma < \alpha_p$ such that $(p^\gamma G + H)/H$ is a torsion group and $p^{\gamma+2}G[p] \subset H[p] \subset p^\gamma G$. Consequently, $h \in p^\gamma G$, $\beta > \gamma$ and $m \in p^\gamma G$. Since the group $(p^\gamma G + H)/H$ is torsion, $O(m) = p^j$, where $j > 1$. Now,

$$0 \neq p^{j-1}m = p^{j-1}(h + m) \in p^{\beta+1}G[p] \subset p^{\gamma+2}G[p] \subset H[p]\,,$$

which contradicts with $\langle m\rangle \cap H = 0$. Hence H is a center of Γ-isotypity in G.

Corollary 1. Let G be a group. A subgroup H of G is a center of isotypity in G iff for every prime p one of the following two conditions holds:

(i) $H[p] \subset p^\infty G$.

(ii) There is an ordinal γ such that $(p^\gamma G + H)/H$ is torsion and $p^{\gamma+2}G[p] \subset H[p] \subset p^\gamma G$.

COROLLARY 2 (R. S. Pierce [4]). Let G be a group. A subgroup H of G is a center of purity in G iff for each prime p one of the following two conditions holds:

(i) $H[p] \subset p^{\omega} G$.

(ii) G/H is torsion and there is an integer $n \geqslant 0$ such that $p^{n+2}G[p] \subset H[p] \subset p^n G$.

PROPOSITION. Let G be a group and $\Gamma = (\alpha_p)_{p \in \mathbb{P}} \in \mathcal{H}$. Then the following are equivalent:

(i) Each subgroup of G is a center of Γ-isotypity in G.

(ii) Each neat subgroup of G is Γ-isotype in G.

(iii) For every prime p either

(a) $G[p] \subset \bigcap\limits_{\gamma < \alpha_p} p^{\gamma} G$ or

(b) G is torsion and there is an integer n, $0 \leqslant n < \alpha_p$, such that $p^{n+2} G_p = 0$ and $G[p] \subset p^n G$.

PROOF. The equivalence of the assertions (i) and (ii) is trivial. By theorem, (iii) implies (i). Suppose that the assertion (iii) is not true, i.e. for a prime p neither (a) nor (b) holds. Thus G_p is not divisible, there is an integer $m \geqslant 0$ such that $G[p] \subset p^m G$ and $G[p] \not\subset p^{m+1} G$; hence $m + 1 < \alpha_p$. Let $g \in G[p]$ be an element of p-height m, write $H = \langle g \rangle$. Obviously $H[p] \not\subset \bigcap\limits_{\gamma < \alpha_p} p^{\gamma} G$, $H[p] \subset p^m G$, $H[p] \not\subset p^{m+1} G$. Since either G is not torsion or $p^{m+2} G_p \neq 0$, H is not a center of Γ-isotypity in G by theorem.

Note that $p^{n+2} G_p = 0$ and $G[p] \subset p^n G$ iff G_p is a direct sum of cyclic groups of orders p^{n+1} and p^{n+2}.

COROLLARY 3. Let G be a group and $\Gamma = (\alpha_p)_{p \in \mathbb{P}} \in \mathcal{H}$. If $\alpha_p \geqslant \omega$ for each prime p then the following are equivalent:

(i) Each subgroup of G is a center of Γ-isotypity in G.

(ii) For every prime p either G_p is divisible or G is torsion and there is an integer $n \geqslant 0$ such that G_p is a direct sum of cyclic groups of orders p^{n+1} and p^{n+2}.

(iii) Each neat subgroup of G is pure in G.

(iv) Each neat subgroup of G is isotype in G.

PROOF. By proposition, the assertions (i) and (ii), (ii) and (iii), (ii) and (iv) are equivalent.

The equivalence of the assertions (ii) and (iii) from corollary 3 was in the first time proved by C. Megibben [3].

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