Rendiconti *del* Seminario Matematico *della* Università di Padova

C. Menini

Linearly compact rings and strongly quasi-injective modules

Rendiconti del Seminario Matematico della Università di Padova, tome 65 (1981), p. 251-262

<http://www.numdam.org/item?id=RSMUP_1981__65__251_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 65 (1980)

Linearly Compact Rings and Strongly Quasi-Injective Modules.

C. MENINI (*)

Introduction.

Throughout this paper, all rings are associative with identity $1 \neq 0$ and all modules are unitary.

Let R be a ring. A left R-module ${}_RK$ is called *strongly quasi-injective* (for short s.q.i.) if given any submodule B of ${}_RK$, a morphism $f\colon B \to {}_RK$ and an element $x \in K \setminus B$, f extends to an endomorphism $\bar{f}$ of ${}_RK$ such that $(x)\bar{f} \neq 0$.

The notion of s.q.i. module comes from the study of dualities, induced by topological bimodules, between a category of abstract modules and a category of topological modules, where it plays a central role (cf. [2]).

Investigating on the concept of s.q.i. module, the following question naturally arises. Let ${}_RK$ be a s.q.i. module, $A = \mathrm{End}({}_RK)$. When is K_A s.q.i.? The study of this problem leads to the following characterization of linearly compact rings.

THE MAIN THEOREM. *Let R be a left linearly topologized ring with respect to a ring topology τ, let $\mathcal{F}$ be the filter of open left ideals of R and let $\mathcal{C}_{\mathcal{F}}$ be the hereditary pretorsion class of left R-modules associated with $\mathcal{F}$. The following statements are equivalent.*

(*) Indirizzo dell'A.: Istituto di Matematica dell'Università di Ferrara, Via Machiavelli 35, 44100 Ferrara (Italy).

Lavoro eseguito nell'ambito della attività dei gruppi di ricerca matematica del C.N.R.

(a) *R is linearly compact in the topology τ.*

(b) *If ${}_RK$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$ and $A = \mathrm{End}\,({}_RK)$, then ${}_RK_A$ is faithfully balanced and K_A is quasi-injective.*

(c) *There exists a faithfully balanced module ${}_RK_A$ such that ${}_RK$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$ and K_A is quasi-injective.*

(d) *Let ${}_RU$ be a minimal cogenerator of $\mathcal{C}_{\mathcal{F}}$, $T = \mathrm{End}\,({}_RU)$. Then ${}_RU_T$ is faithfully balanced and both the modules ${}_RU$ and U_T are s.q.i.*

Moreover, if condition (d) *is fulfilled,* T *is linearly compact in its U-adic topology*

(See below for explained definitions.)

Some results obtained in [4] for discrete linearly compact rings are here extended to the general case.

As an application of our results, we get a quick proof of Leptin's theorem which characterizes a linearly compact ring with zero Jacobson radical as a cartesian product of endomorphism rings of vector spaces.

A structure theorem on faithfully balanced modules ${}_RK_A$ which are s.q.i. both on R and A, obtained as intermediate result, has an intrinsic interest (cf. Theorem 10).

I would like to thank Prof. A. Orsatti for his helpful suggestions.

Some conventions and notations. Let R be a ring. R-Mod will denote the category of left R-modules and Mod-R that of right R-modules. The notation ${}_RM$ will be used to emphasize that M is a left R-module. Morphisms between modules will be written on the opposite side to that of the scalars and the composition of morphisms will follow this convention. For every $M \in R$-Mod, $E_R(M)$, or simply $E(M)$, will denote the injective envelope of M in R-Mod and Soc $({}_RM)$, or simply Soc (M), the socle of M. If L is a subset of ${}_RM \in R$-Mod, we denote by $\mathrm{Ann}_R\,(L)$ the annihilator of L in R:

$$\mathrm{Ann}_R\,(L) = \{r \in R\colon\ rx = 0 \ \text{for every}\ x \in L\}\,.$$

If $L = \{x\}$, we will simply write $\mathrm{Ann}_R\,(x)$.

If J is a left ideal of R, we define the annihilator of J in M, $\mathrm{Ann}_M\,(J)$, by setting:

$$\mathrm{Ann}_M\,(J) = \{x \in M\colon\ rx = 0 \ \text{for every}\ r \in J\}\,.$$

The annihilator in R of $\mathrm{Ann}_M(J)$ will be denoted by $\mathrm{Ann}_R\,\mathrm{Ann}_M(J)$.

Analogous notations will be used for right modules.

$\boldsymbol{N}$ will denote the set of positive integers.

1. To begin with, let us recall some definitions.

Let R be a ring and let $M \in R$-Mod. ${}_RM$ is *quasi-injective* (for short q.i.) if for every submodule $L \leqslant {}_RM$ and every morphism $f: L \to {}_RM$, f extends to an endomorphism $\bar{f}$ of ${}_RM$. ${}_RM$ is a *self-cogenerator* if, for every $n \in \boldsymbol{N}$, given a submodule L of ${}_RM^n$ and an element $x \in M^n \setminus L$, there exists a morphism $f\colon {}_RM^n \to {}_RM$ such that $(L)f = 0$ and $(x)f \neq 0$. Clearly if ${}_RM$ is both quasi-injective and selfcogenerator, then ${}_RM$ is strongly quasi-injective. The converse is true as well (cf. [2], Corollary 4.5).

Let ${}_RK_A$ be a bimodule. ${}_RK_A$ is *faithfully balanced* if $A \cong \mathrm{End}\,({}_RK)$ and $R \cong \mathrm{End}\,(K_A)$ canonically.

Let R be a ring and let $M \in R$-Mod. The *M-topology of R* is defined by taking as a basis of neighbourhoods of 0 in R the annihilators in R of the finite subsets of M. It is easy to check that this topology is a left linear ring topology on R.

Finally recall that a linearly topologized left module M over a discrete ring R is said to be *linearly compact* if M is Hausdorff and if any finitely solvable system of congruences $x \equiv x_i \bmod X_i$, where the X_i are closed submodules of ${}_RM$, is solvable.

2. PROPOSITION. *Let R be a ring, ${}_RK \in R$-Mod a selfcogenerator, $A = \mathrm{End}\,({}_RK)$. If R is linearly compact in the K-topology, then K_A is quasi-injective.*

PROOF. Cf. [4], Prop. 3.4 *a*).

Let R be a ring, τ a left linear ring topology on R, $\mathcal{F}$ the filter of open left ideals of R. The left exact preradical in R-Mod associated with $\mathcal{F}$, $t_{\mathcal{F}}$, is defined by setting, for every $M \in R$-Mod:

$$t_{\mathcal{F}}(M) = \{x \in M\colon\ \mathrm{Ann}_R(x) \in \mathcal{F}\}\,.$$

The hereditary pretorsion class of R-Mod associated with $\mathcal{F}$ is defined by setting

$$\mathcal{C}_{\mathcal{F}} = \{M \in R\text{-Mod}\colon\ M = t_{\mathcal{F}}(M)\}\,.$$

3. LEMMA. *Let R be a left linearly topologized ring with respect to a ring topology τ, let $\mathcal{F}$ be the filter of open left ideals of R and let ${}_RK$*

be a cogenerator of $\mathcal{C}_{\mathcal{F}}$. *For every closed left ideal* J *of* R *it is*

$$\mathrm{Ann}_R \mathrm{Ann}_K (J) = J .$$

PROOF. Let $r \in R \setminus J$. There is an open left ideal L of R such that $L \geq J$ and $r \notin L$. Since ${}_RK$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$, there is a morphism $f\colon R/L \to {}_RK$ such that $(r + L)f \neq 0$. Hence there is an $x \in {}_RK$ such that $Lx = 0$ and $rx \neq 0$. Thus $Jx = 0$ and therefore $r \notin \mathrm{Ann}_R \mathrm{Ann}_K (J)$.

4. PROPOSITION. *Let* ${}_RK_A$ *be a faithfully balanced bimodule, let* τ *be a left linear Hausdorff ring topology on* R *and let* $\mathcal{F}$ *be the filter of open left ideals of* R. *Assume that* ${}_RK$ *is a cogenerator of* $\mathcal{C}_{\mathcal{F}}$ *and that* K_A *is quasi-injective. Then* R *is linearly compact in the topology* τ.

PROOF. The following technique is due to Müller (cf. [3], Lemma 4). Let $(J_i)_{i\in I}$ be a family of closed left ideals of R and let

$$(1) \qquad x \equiv r_i \pmod{J_i} \qquad (r_i \in R)$$

be a finitely solvable system of congruences in R. Set $L = \sum_{i\in I} \mathrm{Ann}_K (J_i)$. L is a submodule of K_A. Define a morphism $g\colon L \to K_A$ by setting $g\left(\sum_{i\in F} x_i\right) = \sum_{i\in F} r_i x_i$, where F is a finite subset of I and, for every $i \in F$, $x_i \in \mathrm{Ann}_K (J_i)$. Since (1) is finitely solvable, g is well defined. Since K_A is quasi-injective, g extends to an endomorphism of K_A. Since ${}_RK_A$ is faithfully balanced, this endomorphism is the left multiplication by an element $r \in R$ so that we have, for every $i \in I$, $r - r_i \in \mathrm{Ann}_R \mathrm{Ann}_K (J_i)$. By Lemma 3, $\mathrm{Ann}_R \mathrm{Ann}_K (J_i) = J_i$ for every $i \in I$, thus (1) is solvable.

Let R be a ring and let τ be a left linear ring topology on R. The *Leptin topology* τ^* of τ is the ring topology on R defined by taking as a basis of neighbourhoods of 0 in R the cofinite open left ideals of R. Recall that a left ideal of R is *cofinite* if it is a finite intersection of completely irreducible left ideals of R. A left ideal I of R is *completely irreducible* if R/I is an essential submodule of the injective envelope $E(S)$ of a left simple R-module S.

Let $\mathcal{F}$ be the filter of open left ideals of R. In the following $\mathcal{S}$ will always denote a system of representatives of the isomorphism classes of the simple left R-modules and $\mathcal{S}_{\mathcal{F}}$ the intersection $\mathcal{S} \cap \mathcal{C}_{\mathcal{F}}$.

Let ${}_RU$ be the minimal cogenerator of $\mathcal{C}_{\mathcal{F}}$. It is well known that

$$ {}_RU = t_{\mathcal{F}}\Big(\bigoplus_{S\in\mathcal{S}} E(S)\Big) = \bigoplus_{S\in\mathcal{S}} t_{\mathcal{F}}(E(S)) $$

and hence, in our notations, it is:

$$ {}_RU = \bigoplus_{S\in\mathcal{S}_{\mathcal{F}}} t_{\mathcal{F}}(E(S)) \,. $$

5. LEMMA. *Let R be a left linearly topologized ring with respect to a ring topology τ, let $\mathcal{F}$ be the filter of open left ideals of R and let ${}_RU$ be the minimal cogenerator of $\mathcal{C}_{\mathcal{F}}$. Then the U-topology of R coincides with the Leptin topology τ^* of τ.*

PROOF. Let $x\in{}_RU$. Then $\operatorname{Ann}_R(x)$ is open and cofinite in R. Conversely, let $J\in\mathcal{F}$ such that $E(R/J)=E(S)$ where $S\in\mathcal{S}$. Since $J\in\mathcal{F}$, $R/J\in\mathcal{C}_{\mathcal{F}}$ so that $R/J\leqslant t_{\mathcal{F}}(E(R/J)) = t_{\mathcal{F}}(E(S))\leqslant{}_RU$.

6. LEMMA. *In the hypothesis of Lemma above, let ${}_RK$ be a cogenerator of $\mathcal{C}_{\mathcal{F}}$. Then the K-topology of R is equivalent to τ (i.e. they have the same closed ideals).*

PROOF. Let J be a left ideal of R which is closed in the K-topology of R. Since ${}_RK\in\mathcal{C}_{\mathcal{F}}$, J is closed in τ. Conversely assume J closed in τ. J is an intersection of open completely irreducible ideals of $\mathcal{F}$. Thus, by Lemma 5, J is closed in the U-topology of R. Since ${}_RK$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$, it contains the minimal cogenerator ${}_RU$. Hence the U-topology of R is contained in the K-topology and thus J is closed in the K-topology of R.

Let ${}_RK_A$ be a bimodule over the rings R and A. We say that R *separates points and (finitely generated) submodules of K_A* if for every (finitely generated) submodule L of K_A and for every $x\in K\setminus L$, there is an $r\in R$ such that $r(L)=0$ and $rx\neq 0$.

7. LEMMA. *Let R be a ring, ${}_RK\in R$-*Mod*, $A=$ *End* $({}_RK)$. If ${}_RK$ is quasi-injective, then R separates points and finitely generated submodules of K_A.*

PROOF. Let L be a finitely generated submodule of K_A and let $y\in K$. Assume that $\operatorname{Ann}_R(y)\geqslant\operatorname{Ann}_R(L)$ and let $\{x_1, \dots, x_n\}$ be a finite system of generators of L_A. Consider the element $x=(x_1, \dots, x_n)\in K^n$

and define a morphism $f: Rx \to Ry$ by setting $(rx)f = ry$ $(r \in R)$. f is well defined since $rx = 0$ means $r \in \bigcap_{i=1}^{n} \mathrm{Ann}_R(x_i) = \mathrm{Ann}_R(L) \leqslant \mathrm{Ann}_R(y)$ by assumption. Since ${}_RK$ is q.i. and by Proposition 6.6 [2], f extends to a morphism $\bar{f}: {}_RK^n \to {}_RK$. Hence there are $a_1, \ldots, a_n \in A$ such that $y = (x)f = (x)\bar{f} = \sum_{i=1}^{n} x_i a_i \in L$.

8. Lemma. *Let M be a left linearly topologized R-module over the discrete ring R. Assume that M is linearly compact and let Y be an open submodule of ${}_RM$, $(X_i)_{i\in I}$ a family of closed submodules of ${}_RM$. If M/Y is finitely embedded and $\bigcap_{i\in I} X_i \subseteq Y$, then there is a finite subset F of I such that $\bigcap_{i\in F} X_i \subseteq Y$.*

Proof. M/Y is finitely embedded means that there is a finite number $Y_1, \ldots, Y_n$ of modules of M such that $Y = Y_1 \cap \ldots \cap Y_n$ and, for each i, $E(M/Y_i)$ is the injective envelope of a simple left R-module S_i. The same proof of Lemma 2 [3] shows that for every $j = 1, \ldots, n$ there is a finite subset F_j of I such that $\bigcap_{i\in F_j} X_i \subseteq Y_j$. Setting $F = \bigcup_{i=1}^{n} F_j$ we get $\bigcap_{i\in F} X_i \subseteq Y$.

9. Let R and A be rings and let ${}_RK_A$ be a faithfully balanced bimodule such that both ${}_RK$ and K_A are strongly quasi-injective. Let $\mathcal{F}$ be the filter of left ideals of R which are open in the K-topology of R and let ${}_R\mathfrak{T}$ be the set of maximal left ideals of R belonging to $\mathcal{F}$. Let $P \in {}_R\mathfrak{T}$. R/P is a left simple R-module belonging to $\mathcal{C}_{\mathcal{F}}$, i.e. $R/P \in \mathcal{S}_{\mathcal{F}}$. Let $\mathcal{G}$ be the filter of right ideals of A which are open in the K-topology of A. By statements d) and e) in 6.9 [2] it follows that for every $P \in {}_R\mathfrak{T}$, set $S = R/P$ and $S^* = \mathrm{Hom}_R(S, K) = \mathrm{Ann}_K P$, S^* is a right simple submodule of K_A and moreover each simple submodule of K_A has this form. Since K_A is strongly quasi-injective, K_A is a cogenerator of $\mathcal{C}_{\mathcal{G}}$ (cf. [2], Theorem 6.7), thus K_A contains a copy of each right simple A-module in $\mathcal{C}_{\mathcal{G}}$. *Therefore the right simple A-modules of $\mathcal{C}_{\mathcal{G}}$ are precisely those of the form* $\mathrm{Ann}_K P$ *where* $P \in {}_R\mathfrak{T}$. Moreover, by Lemma 3, for each $P \in {}_R\mathfrak{T}$, $P = \mathrm{Ann}_R \mathrm{Ann}_K(P)$.

Let $(S_\lambda)_{\lambda\in\Lambda}$ be a system of representatives of the isomorphisms classes of the left simple R-modules of $\mathcal{C}_{\mathcal{F}}$. Note that if $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$ then $S_\lambda^* \neq S_\mu^*$. Let $\sigma(S_\lambda)$, $\lambda \in \Lambda$, be the isotypical component of $\mathrm{Soc}({}_RK)$ with respect to S_λ and write $\sigma(S_\lambda) = S_\lambda^{(\nu_\lambda)}$ where ν_λ is a suitable cardinal number and $S_\lambda^{(\nu_\lambda)}$ denotes the direct sum of ν_λ copies of S_λ. By Proposition 6.10 [2], Soc $({}_RK)$ = Soc (K_A), Soc $({}_RK)$ is essential in ${}_RK$ and Soc (K_A) is essential in K_A. Moreover it is Soc $(K_A) = \bigoplus_{\lambda\in\Lambda} \sigma(S_\lambda^*)$ and $\sigma(S_\lambda) = \sigma(S_\lambda^*)$. Finally, for every $\lambda \in \Lambda$, $\sigma(S_\lambda^*) = S_\lambda^{*(\mu_\lambda)}$ where μ_λ is a suitable cardinal number. The cardinal numbers ν_λ and μ_λ $(\lambda \in \Lambda)$ are uniquely determined by ${}_RK_A$.

10. THEOREM. *Let ${}_RK_A$ be a faithfully balanced bimodule over the rings R and A such that ${}_RK$ is strongly quasi-injective. The following statements are equivalent*:

(a) *K_A is strongly quasi-injective.*

(b) *R is linearly compact in the K-topology and R separates points and submodules of K_A.*

(c) *R is a linearly compact in the K-topology and* Soc $({}_RK)$ *is essential in ${}_RK$.*

If these conditions hold, then A is linearly compact in the K-topology and moreover, using the notations of 9., it is

$$ {}_RK \cong \bigoplus_{\lambda\in\Lambda} t_{\mathcal{F}}(E_R(\sigma(S_\lambda))) = \bigoplus_{\lambda\in\Lambda} [t_{\mathcal{F}}(E_R(S_\lambda))]^{(\nu_\lambda)} $$

and

$$ K_A \cong \bigoplus_{\lambda\in\Lambda} t_{\mathcal{G}}(E_A(\sigma(S_\lambda^*))) = \bigoplus_{\lambda\in\Lambda} [t_{\mathcal{G}}(E_A((S_\lambda^*)))]^{(\mu_\lambda)} . $$

PROOF. $(a) \Rightarrow (b)$ follows by Proposition 4, since, as we remarked in 9., ${}_RK$ is a cogenerator of $\mathcal{C}_{\mathcal{F}}$.

$(b) \Rightarrow (a)$ follows by Proposition 2.

$(a) \Rightarrow (c)$. By (b) R is linearly compact in the K-topology and since ${}_RK_A$ is faithfully balanced with both ${}_RK$ and K_A s.q.i., Soc $({}_RK)$ is essential in ${}_RK$, as we recalled in 9.

$(c) \Rightarrow (b)$. First of all, let us prove that ${}_RK \leqslant \bigoplus_{\lambda\in\Lambda} t_{\mathcal{F}}(E(S_\lambda))^{(\nu_\lambda)}$. Let $x \in K$. Rx is linearly compact discrete and hence Soc (Rx) is a direct sum of a finite number of left simple R-modules $S_1, \ldots, S_n$. By hypothesis, Soc $({}_RK)$ is essential in ${}_RK$. Hence Soc (Rx) is essential in Rx. It follows that

$$(1) \qquad Rx \leqslant \bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))$$

and hence the claimed inclusion is proved.

Let us prove that R separates points and submodules of K_A. Let $L \leqslant K_A$ and let $x \in K$. Assume that $\mathrm{Ann}_R(x) \geqslant \mathrm{Ann}_R(L)$. Note that, by (1), $R \diagdown \mathrm{Ann}_R(x) \cong Rx$ is finitely embedded. Hence, by Lemma 8, there is a finite subset $F \subseteq L$ such that $\mathrm{Ann}_R(x) \geqslant \bigcap_{l \in F} \mathrm{Ann}_R(l)$. Thus, by Lemma 7, x belongs to the submodule of K_A spanned by F and hence $x \in L$.

Let us assume that the equivalent conditions (a), (b) and (c) hold. We have already seen in the proof of (c) $\Rightarrow$ (b) that

$$_RK \leqslant \bigoplus_{\lambda \in \Lambda} [t_{\mathcal{F}}(E(S_\lambda))]^{(\nu_\lambda)} .$$

Obviously, it is clear that for every $\lambda \in \Lambda$,

$$[t_{\mathcal{F}}(E(S_\lambda))]^{(\nu_\lambda)} \leqslant t_{\mathcal{F}}(E(\sigma(S_\lambda))) .$$

Since $_RK$ is s.q.i., $_RK$ is an injective cogenerator of $\mathcal{C}_{\mathcal{F}}$ (cf. [2], Theorem 6.7). Thus it is straightforward to prove that $\bigoplus_{\lambda \in \Lambda} t_{\mathcal{F}}(E(\sigma(S_\lambda))) \leqslant {}_RK$. Hence we get the following chain of inclusions:

$$_RK \leqslant \bigoplus_{\lambda \in \Lambda} [t_{\mathcal{F}}(E(S_\lambda))]^{(\nu_\lambda)} \leqslant \bigoplus_{\lambda \in \Lambda} t_{\mathcal{F}}(E(\sigma(S_\lambda))) \leqslant {}_RK$$

and therefore the first chain of inclusions is proved.

In view of remarks in 9. and by symmetry, the analogous equalities hold for K_A.

11. Corollary. *Let $_RK_A$ be a faithfully balanced bimodule such that both $_RK$ and K_A are s.q.i. Let $_R\mathfrak{I}$ be the set of left maximal ideals of R which are open in the K-topology of R and let $J(R)$ be the Jacobson radical of R. Then*

$$J(R) = \bigcap \{P\colon P \in {}_R\mathfrak{I}\} .$$

In particular, $J(R)$ is closed in the K-topology of R.

PROOF. Let Z_A denote the socle of K_A. As we recalled in 9., it is $\mathrm{Ann}_R(Z_A) = \bigcap\{P: P \in {}_R\mathcal{T}\} \geqslant J(R)$.

Let $a \in \mathrm{Ann}_R(Z_A)$ and, by way of contradiction, assume that $a \notin J(R)$. Thus there is a left maximal ideal Q of R such that $a \notin Q$. Hence $Ra + Q = R$ and therefore $1 = ra + q$, where $r \in R$ and $q \in Q$. Since $a \in \mathrm{Ann}_R(Z_A)$, for every $x \in Z_A$ it is $qx = (1 - ra)x = x$. Thus, since Z_A is essential in K_A, q, as endomorphism of K_A, is injective and Im $(q) = K_A$. Let q': $\mathrm{Im}(q) \to K$ be the left inverse of the corestriction of q to Im (q). Since K_A is q.i., q' extends to an endomorphism of K_A. Thus $1 \in Q$. Contradiction.

12. REMARK. Let R be a linearly compact ring with respect to a left linear topology τ and $\mathcal{F}$ be the filter of open left ideals of R. Let ${}_RU = \bigoplus_{S \in \mathcal{S}_\mathcal{F}} t_\mathcal{F}(E(S))$ be the minimal cogenerator of $\mathcal{C}_\mathcal{F}$ and denote by $\mathcal{F}^*$ the filter of left ideals which are open in the U-topology of R, i.e. in the Leptin topology of τ, τ^* (cf. Lemma 5). Clearly, a left simple R-module belongs to $\mathcal{C}_\mathcal{F}$ if and only if it belongs to $\mathcal{C}_{\mathcal{F}^*}$. Moreover for each simple left R-module S, $t_\mathcal{F}(E(S)) = t_{\mathcal{F}^*}(E(S))$. In fact, since $\mathcal{F}^* \subseteq \mathcal{F}$, $t_{\mathcal{F}^*}(E(S)) \leqslant t_\mathcal{F}(E(S))$. On the other hand,

$$t_\mathcal{F}(E(S)) \leqslant {}_RU \in \mathcal{C}_{\mathcal{F}^*}.$$

In particular ${}_RU$ is also the minimal cogenerator of $\mathcal{C}_{\mathcal{F}^*}$.

PROOF OF THE MAIN THEOREM. $(a) \Rightarrow (b)$. Let ${}_RK$ be a cogenerator of $\mathcal{C}_\mathcal{F}$ and let $A = \mathrm{End}({}_RK)$. By Lemma 6, the K-topology of R is equivalent to τ and hence R is linearly compact in the K-topology too. Thus, since ${}_RK$ is a selfcogenerator, by Corollary 7.4 [2], $R = \mathrm{End}(K_A)$ and therefore ${}_RK_A$ is faithfully balanced. By Proposition 2, K_A is q.i.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (a)$. Since ${}_RK$ is faithful, the K-topology of R is Hausdorff. By Lemma 6, τ is equivalent to the K-topology of R and hence τ is Hausdorff too. Thus, by Proposition 4, R is linearly compact.

$(d) \Rightarrow (c)$ is trivial.

$(a) \Rightarrow (d)$. Let us remark, first of all, that in view of Lemma 5, R is linearly compact in the U-topology. Let us now proceed by steps.

1) ${}_RU$ *is q.i.* Set $E = E\left(\bigoplus_{S \in \mathcal{S}_{\mathcal{F}}} S\right)$. Let us prove that ${}_RU = t_{\mathcal{F}}(E)$. From this the claim will follow for ${}_RU$ will be a fully invariant submodule of ${}_RE$. Since ${}_RU = \bigoplus_{S \in \mathcal{S}_{\mathcal{F}}} t_{\mathcal{F}}(E(S))$, it is clear that ${}_RU \leqslant t_{\mathcal{F}}(E)$. Conversely, let $x \in t_{\mathcal{F}}(E)$. Then $\mathrm{Ann}_R(x) \in \mathcal{F}$ and hence $Rx \simeq R/\mathrm{Ann}_R(x)$ is linearly compact discrete. Thus Soc (Rx) is a direct sum of a finite number of (non-isomorphic) left simple R-modules. Since $x \in E$, Soc (Rx) is essential in Rx. Thus $x \in t_{\mathcal{F}}(E(\mathrm{Soc}\,(Rx))) \leqslant {}_RU$.

2) ${}_RU$ *is s.q.i.* By Remark 12 and by Theorem 6.7 [2].

3) ${}_RU_T$ *is faithfully balanced.* Since R is linearly compact in the U-topology, it is complete. Thus, since ${}_RU$ is a selfcogenerator, by Corollary 7.4 [2], $R = \mathrm{End}\,(U_T)$.

4) U_T *is s.q.i.* Note that Soc $({}_RU)$ is essential in ${}_RU$. Then the claim follows by Theorem 10.

13. Corollary. *Let R be a left linearly compact ring with respect to a ring topology τ, let ${}_R\mathcal{S}$ be the set of open left maximal ideals of R and let $J(R)$ be the Jacobson radical of R. Then*

$$J(R) = \bigcap \{P\colon\; P \in {}_R\mathcal{S}\}\,.$$

In particular, $J(R)$ is closed in R.

Proof. Follows by The Main Theorem, by Corollary 11 and by Remark 12.

The idea of the following application is due to Prof. A. Orsatti.

14. Theorem (Leptin [1]). *Let R be a left linearly topologized ring with respect to a ring topology τ. Assume that R is linearly compact and that the Jacobson radical of R, $J(R)$, is zero. Then R, endowed with the Leptin topology of τ, is topologically isomorphic to a topological product $\prod_{\lambda \in \Lambda} \mathrm{End}_{D_\lambda}(V_\lambda)$ where, for every $\lambda \in \Lambda$, V_λ is a vector space over the division ring D_λ and $\mathrm{End}_{D_\lambda}(V_\lambda)$ is endowed with the finite topology.*

(Zelinsky [5]). *Moreover if τ has two-sided ideals as a basis of neighbourhoods of zero, each V_λ has finite dimension over D_λ.*

Proof. Let $\mathcal{F}$ be the filter of open left ideals of τ and let ${}_RU$ be the minimal cogenerator of $\mathcal{C}_{\mathcal{F}}$. Set $A = \mathrm{End}\,({}_RU)$. By The Main

THEOREM, ${}_RU_A$ is faithfully balanced and both the modules ${}_RU$ and U_A are s.q.i. Suppose that Soc (U_A) is strictly contained in U_A. Then, since U_A is s.q.i. and $R = \mathrm{End}\,(U_A)$, there is a non zero element $r \in R$ such that $r(\mathrm{Soc}\,(U_A)) = 0$. Thus, by 9., by Remark 12 and by Corollary 13, r belongs to the Jacobson radical of R. Hence $\mathrm{Soc}\,(U_A) = U_A$. Since $\mathrm{Soc}\,(U_A) = \mathrm{Soc}\,({}_RU)$ (cf. 9)., we get ${}_RU = \bigoplus_{\lambda \in \Lambda} S_\lambda$, where $(S_\lambda)_{\lambda \in \Lambda}$ is a system of representatives of the isomorphism classes of the left simple R-modules of $\mathcal{C}_{\mathcal{F}}$. Thus each S_λ is fully invariant in ${}_RU$ and hence A is canonically isomorphic to the ring product $\prod_{\lambda \in \Lambda} D_\lambda$ where, for each $\lambda \in \Lambda$, $D_\lambda = \mathrm{End}_R\,(S_\lambda)$ is a division ring. Of course such a product acts componentwise over U so that the action of A over each S_λ naturally identifies with that of D_λ. Recall that, by 9., $S_\lambda = \sigma(S_\lambda^*)$. Moreover, since $R = \mathrm{End}\,(U_A)$, each S_λ is fully invariant submodule of U_A. Therefore we get the natural algebraic isomorphisms

$$\mathrm{End}\,(U_A) \simeq \prod_{\lambda \in \Lambda} \mathrm{End}_A\,(S_\lambda) = \prod_{\lambda \in \Lambda} \mathrm{End}_{D_\lambda}\,(S_\lambda)\,. \tag{1}$$

Now, since ${}_RU$ is a selfcogenerator, by Corollary 7.4 [2], End (U_A), endowed with the finite topology, is isomorphic to the completion of R in the U-topology. Since R is linearly compact in τ, the first statement follows easily by Lemma 5, as soon as we note that the finite topology of End (U_A) corresponds, through the isomorphisms (1), to the product topology of the finite topologies on the $\mathrm{End}_{D_\lambda}\,(S_\lambda)$, $\lambda \in \Lambda$.

Assume now that τ has two-sided ideals as a basis of neighbourhoods of 0. Fix $\lambda \in \Lambda$ and let $P \in {}_R\mathcal{T}$ such that $R/P \simeq S_\lambda$. Since $P \in \mathcal{F}$, P contains an open two-sided ideal. Since $\mathrm{Ann}_R\,(S_\lambda)$ is the largest two-sided ideal contained in P, it follows that $\mathrm{Ann}_R\,(S_\lambda) \in \mathcal{F}$. Let $\{e_i\}_{i \in I}$ be a basis of S_λ as a vector space over D_λ. Then $\mathrm{Ann}_R\,(S_\lambda) = \bigcap_{i \in I} \mathrm{Ann}_R\,(e_i)$. Since $(\mathrm{Ann}_R\,(e_i))_{i \in I}$ is a family of open coprimary left ideals of R and R is linearly compact, it is easy to check that the diagonal map $R/\mathrm{Ann}_R\,(S_\lambda) \to \prod_{i \in I} R/\mathrm{Ann}_R\,(e_i)$ of the canonical maps $R/\mathrm{Ann}_R\,(S_\lambda) \to R/\mathrm{Ann}_R\,(e_i)$ $(i \in I)$, is an isomorphism. Since $R/\mathrm{Ann}_R\,(S_\lambda)$ is linearly compact discrete, I must be finite.

15. REMARKS. 1) In the hypothesis of Theorem above, if τ is the discrete topology, then R is semisimple artinian ([5]). In fact, since ${}_RU$ is linearly compact discrete (cf. [3], Th. 1), Λ is finite.

2) In the hypothesis of Theorem above, if τ has two-sided ideals as a basis of neighbourhoods of zero, then $\tau = \tau^*$. In fact let L be an open two-sided ideal of R. Then R/L is a discrete linearly compact ring with zero Jacobson radical. By 1) above, R/L is artinian. Thus L is cofinite.

REFERENCES

[1] H. LEPTIN, *Linear Kompakte Moduln und Ringe - I*, Math. Z., **62** (1955), pp. 241-267; *II*, **66** (1957), pp. 289-327.

[2] C. MENINI - A. ORSATTI, *Good dualities and strongly quasi-injective modules*, to appear in Annali di Matematica Pura ed Applicata.

[3] B. J. MÜLLER, *Linear compactness and Morita Duality*, J. of Algebra, **16** (1970), pp. 60-66.

[4] A. ORSATTI - V. ROSELLI, *A characterization of discrete linearly compact rings by means of duality*, to appear in Rendiconti del Seminario Matematico dell'Università di Padova.

[5] D. ZELINSKY, *Linearly compact modules and rings*, Am. J. of Math., **75** (1953), pp. 79-90.

Manoscritto pervenuto in redazione il 12 dicembre 1980.