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Periodic Solutions for Second Order Differential Systems with Damping.

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Introduction.

In this paper we deal with the problem of the existence of periodic solutions for the differential systems of the Rayleigh type

$$x'' + F(x') + G(x) = h(t, x, x') \tag{1}$$

where h is bounded and periodic in t.

In our approach we consider some conditions, on F and G, which extend to the systems those used by G. E. H. Reuter in [15]. In 1952 he proved a boundedness theorem for the solutions of the second order scalar differential equation

$$x'' + F(x') + G(x) = h(t)$$

with F, G, h continuous and h bounded. He also obtained the existence of periodic solutions, for any periodic forcing term, under the

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assumption

$$\lim_{|x'|\to+\infty} F(x')\cdot \operatorname{sign} x' = +\infty = \lim_{|x|\to+\infty} G(x)\cdot \operatorname{sign} x$$

(see also [18, page 459]).

We remark that the assumption of Reuter about the damping term F is physically significant: in the applications we often have to consider a dissipation term $F(x')$ which behaves as a (positive) power of $|x'|$ for $|x'|$ large enough.

Hypotheses on the order of growth for the damping terms have already been considered by several authors. For instance, we recall that in the study of the wave equation

$$u'' - \Delta u + F(u') = h$$

the case $a|x'|^{\varrho-1}x' \leqslant F(x') \leqslant b|x'|^{\varrho-1}x'$ was dealt with by J. L. Lions - W. Strauss [8] and G. Andreassi - G. Torelli [1] for the Cauchy problem and by G. Prodi [11] and G. Prouse [12], [13], for the existence of periodic solutions.

Also in the case of ordinary differential equations there are some physical and engineering applications where a polynomial growth for the dissipative term is considered, as, for instance, in the scalar equation (of Rayleigh type)

$$x'' + |x'|x' + qx' + x - P^2x^3 = r\cdot\sin(\omega t)\,,$$

which arises in the problem of the vibrations of a suspended wire and which was studied by J. Cecconi [4] and F. Stoppelli [19] (for a recent result on this equation, see also L. Sanchez [17]).

Recently R. Reissig [14] (see also J. R. Ward [20]) considered the system (1) with F linear and symmetric and G. Caristi - S. Invernizzi [3] assumed that $F(x') = \beta|x'|^{\varrho-1}x'$ $(\beta > 0)$, with $\varrho > 1$. All of these authors reached their results using the « non resonance » hypothesis that $p\cdot I \leqslant G'(x) \leqslant q\cdot I$, where I is the identity matrix and p, q are real numbers, $\omega^2 n^2 < p \leqslant q < \omega^2(n+1)^2$, $(n = 0, 1, \ldots)$, as in the theorem of A. C. Lazer - D. A. Sánchez [7].

In a forthcoming paper [21], we prove that, under a « sign » condition on the scalar product $(F(x')|x')$ and a growth restriction on G (which must be weakly nonlinear), an existence result can be obtained in such a way as to extend to systems a theorem by A. Ascari [2] (see also [18, page 499]).

Here, in the sequel, we shall show that, when the rate of increase of $(F(x')|x')$ is of the same type as the ϱ-th power of $|x'|$, $(\varrho > 1)$, then we get the existence of periodic solutions of (1) if, for G, we assume a « sign » condition as in the Reuter theorem. More precisely, we shall prove a rather general result which permits us to extend the theorem of Reuter to the systems, in the part concerning the existence of periodic solutions. Moreover, we shall show that, when a polynomial expression for F and G is considered, also a result of J. Mawhin [9, Théoreme 5.2, page 373] can be extended to the case of systems.

We prove our existence result using the Leray-Schauder continuation method in its simplest version, namely by finding a-priori bounds for the periodic solutions of the « homotopic » equation

$$x'' + \lambda F(x') + (1-\lambda)\alpha\cdot x + \lambda G(x) = \lambda h(\cdot, x, x') \tag{2}$$

where $\lambda \in (0, 1]$ and $\alpha \neq 0$ is a sufficiently small real number.

The main result.

Here we are interested in the existence of T-periodic solutions $(T > 0)$ for the differential system

$$x''(t) + F(x'(t)) + G(x(t)) = h(t, x(t), x'(t)) \,. \tag{1}$$

Henceforth we shall assume that $F, G\colon R^m \to R^m$ are continuous and there exists a *continuously differentiable* scalar function $\mathscr{g}$,

$$\mathscr{g}\colon R^m \to R\,, \qquad \text{such that} \qquad G = \operatorname{grad} \mathscr{g}\,.$$

Moreover, we shall suppose that $h\colon R \times R^m \times R^m \to R^m$ is continuous, T-periodic with respect to its first variable (i.e. $h(t+T, x, y) = h(t, x, y)$, for every t, x, y) and bounded by a constant D $(D > 0)$: $|h(t, x, y)| \leqslant D$, for every t, x, y.

We use the symbol $(\cdot|\cdot)$ for the euclidean inner product in R^m and $|\cdot|$ for the corresponding norm. R_+ will denote the set of the non negative real numbers. If $x\colon R \to R^m$ is a continuous T-periodic function, we set

$$|x|_\infty = \max_{[0,T]} |x(t)|\,, \qquad |x|_q = \Bigl(\int_0^T |x(t)|^q\,dt\Bigr)^{1/q}, \qquad 1 \leqslant q < \infty\,.$$

THEOREM 1. *Let F, G, h be continuous and let $G = \operatorname{grad} g$. Let h be bounded by D. Let us suppose that there exists a continuous function H, $H\colon R^m \to R_+$, satisfying to*

$$\text{(i)} \qquad \liminf_{|x|\to+\infty} H(x)/|x| \geqslant A \qquad (A > 0)$$

such that

$$\text{(ii)} \qquad \limsup_{|y|\to+\infty} |F(y)|/H(y) < +\infty$$

and, either

$$(j) \qquad \liminf_{|y|\to+\infty} \bigl(F(y)|y\bigr)/H(y) \geqslant B \qquad (B > 0)$$

or

$$(j') \qquad \limsup_{|y|\to+\infty} \bigl(F(y)|y\bigr)/H(y) \leqslant -B \qquad (B > 0)$$

hold. Finally, let us suppose that, for every positive constant M it is

$$\text{(k)} \qquad \lim_{\substack{|x|\to+\infty \\ |y|\leqslant M}} \frac{\bigl(G(x+y)|x\bigr)}{|x|} = +\infty \qquad (\text{or } -\infty)\,,$$

uniformly with respect to y. Then the system (1) *has a T-periodic solution, provided that*

$$A \cdot B > D\,.$$

(Note that from (i) it follows that $H(y) > 0$ for y large enough and so (ii) and (j) or (j') are meaningful.)

Before proving Theorem 1, we give some applications of our result.

Using an «angle condition» between $F(x')$ and x', $G(x)$ and x, we get the following more expressive corollary. (For different results which have been obtained using an angle condition between $G(x)$ and x, see also [6] and, for the first order systems, see [5].)

COROLLARY 1. *Let F, G, h be continuous and let $G = \operatorname{grad} \mathcal{g}$. Assume*

$$(a) \qquad \liminf_{|y|\to+\infty} |F(y)| \geqslant A \qquad (A > 0)$$

and, either

$$(b) \qquad \liminf_{|y|\to+\infty} \big(F(y)|y\big)/|F(y)|\,|y| \geqslant B \qquad (0 < B \leqslant 1)$$

or

$$(b') \qquad \limsup_{|y|\to+\infty} \big(F(y)|y\big)/|F(y)|\,|y| \leqslant -B \qquad (0 < B \leqslant 1)\,.$$

Moreover, let

$$(c) \qquad \lim_{|x|\to+\infty} |G(x)| = +\infty$$

and, either

$$(d) \qquad \liminf_{|x|\to+\infty} \big(G(x)|x\big)/|G(x)|\,|x| > 0$$

or

$$(d') \qquad \limsup_{|x|\to+\infty} \big(G(x)|x\big)/|G(x)|\,|x| < 0\,.$$

Then (1) *has a T-periodic solution for every bounded forcing term h such that*

$$\sup |h(\cdot, \cdot, \cdot)| < A \cdot B\,.$$

PROOF OF COROLLARY 1. Define $H(y) = |F(y)|\,|y|$. Then, by an easy computation, (a) implies (i), (b) implies (j) and (b') implies (j'). Moreover, also (ii) is satisfied in virtue of the definition of $H(y)$. Now assume that (d) holds; let us fix $M > 0$ and let $|y| \leqslant M$. Then, from $|x| \to +\infty$, it follows that $|x + y| \to +\infty$ and $|G(x + y)| \to +\infty$, uniformly with respect to y. Let us consider the equality

$$\frac{\big(G(x+y)|x\big)}{|G(x+y)|\,|x+y|} = \frac{\big(G(x+y)|x+y\big)}{|G(x+y)|\,|x+y|} - \frac{\big(G(x+y)|y\big)}{|G(x+y)|\,|x+y|}$$

and take lim inf (respectively lim · sup, if we assume the second angle condition (d') on G) of both sides, for $|x| \to +\infty$ and $|y| \leqslant M$. Then, since

$$\lim_{|x|\to+\infty} \frac{(G(x+y)|y)}{|G(x+y)|\,|x+y|} = 0\,,$$

uniformly with respect to y, we get

$$\liminf_{\substack{|x|\to+\infty\\ |y|\leqslant M}} \frac{(G(x+y)|x)}{|G(x+y)|\,|x+y|} \geqslant \varepsilon > 0\,,$$

for some (positive) ε, (respectively $\limsup ... \leqslant -\varepsilon < 0$, if (d')), uniformly with respect to y, $|y| \leqslant M$. Let us observe now that

$$\lim_{\substack{|x|\to+\infty\\ |y|\leqslant M}} |x+y|/|x| = 1$$

(uniformly in y) and conclude that

$$\liminf_{\substack{|x|\to+\infty\\ |y|\leqslant M}} \frac{(G(x+y)|x)}{|G(x+y)|\,|x|} \geqslant \varepsilon > 0\,.$$

Finally, we recall that $|G(x+y)| \to +\infty$ and so we have (k). Then we apply Theorem 1 and the corollary is proved. (Note that a similar argument gives condition (k), with $-\infty$ as limit, when the second angle condition (d') is assumed for the vector field G.) Q.E.D.

Remark 1. Observe that if $F(x') = \beta|x'|^{\varrho-1}x'$, with $\beta > 0$ and $\varrho > 0$, then F verifies the assumptions of Corollary 1; theorefore our result applies to a class of dampings which are of physical interest. Let us observe also that if we change the condition (a) into

$$(a') \qquad \lim_{|y|\to+\infty} |F(y)| = +\infty\,,$$

then Corollary 1 gives the existence of T-periodic solutions of (1) for every bounded forcing term. Moreover if we assume that the direction of $F(y)$ is the same (or the opposite) as that of y for y large enough, then (b) (respectively (b')) is satisfied with $B = 1$. Obviously also (d) or (d') holds if we assume for G the same condition on the direction as assumed for F.

Hence, if the dimension of R^m is 1 $(m = 1)$, so that (1) becomes a scalar equation, then all the hypotheses about F and G of Corollary 1 (using (a') instead of (a)), can be summarized by writing

$$\lim_{|y|\to+\infty} F(y)\cdot \operatorname{sign} y = +\infty \qquad (\text{or } -\infty)\,.$$

$$\lim_{|x|\to+\infty} G(x)\cdot \operatorname{sign} x = +\infty \qquad (\text{or } -\infty)\,.$$

Thus we have as consequences a theorem of J. Mawhin [9, Théoreme 5.2, page 373], taking for F and G two polynomials of odd degree, and the above mentioned Reuter Theorem, in the part concerning the existence of periodic solutions.

At last we give another possible condition $\big($instead of (c) and (d) (or (d'))$\big)$, on the vector field G, which ensures the validity of (k) and which is easy to verify. That is the strong monotonicity, namely

$$(m) \qquad \big(G(x)-G(y)|x-y\big) \geqslant L|x-y|^2$$

for some $L > 0$.

In fact, if (m) holds, then

$$\big(G(x+y)|x\big)/|x| = \big(G(x+y)-G(y)|x\big)/|x| + \big(G(y)|x\big)|x| \geqslant L|x| - |G(y)|\,,$$

and taking the limit as $|x| \to +\infty$ we get (k) since $|G(y)|$ is bounded when $|y|$ is so.

Proof of Theorem 1. Let us denote by $\mathscr{C}^k_T$ $(k = 0, 1, 2)$ the (Banach) space of the T-periodic functions $R \to R^m$, of class $\mathscr{C}^k$, equipped with the norm $\max\{|x^{(i)}|_\infty,\ i = 0, \dots, k\}$. If $u, v \in \mathscr{C}^0_T$, we set $(u, v)_2 = \int_0^T \big(u(t)|v(t)\big)\,dt$, the $\mathscr{L}^2$-scalar product of u and v.

Assume that the hypothesis (k) holds with $+\infty$ as limit; fix a positive real number α sufficiently small so that the linear homogeneous scalar equation $x'' + \alpha\cdot x = 0$ has no nontrivial T-periodic solution. (If $-\infty$ appears in (k), then we choose $\alpha < 0$.) Then, in the framework of the Leray-Schauder topological degree theory [10], [16], it is well known that, in order to get the existence of the (T-periodic) solutions of (1), it is sufficient to prove that every (T-periodic) solution of (2), for any $\lambda \in (0, 1]$, is bounded in the $\mathscr{C}^1_T$-norm by some constant (which is independent by λ and x). (For the same approach, with more details in the functional setting, see [21].)

Let $x \in \mathscr{C}^2_T$ be a solution of (2) for some $\lambda \in (0, 1]$.

Let us take the $\mathscr{L}^2$-scalar product by x' of both members of (2). Observe that $(x'', x')_2 = (x, x')_2 = 0$ and, from $G = \operatorname{grad} g$, also $(G(x), x')_2 = 0$. Then, dividing by λ ($\lambda > 0$), we have:

$$(F(x'), x')_2 = (h(\cdot, x, x'), x')_2 . \tag{3}$$

Remark that, from (j), or (j'), it follows that for any ε, $0 < \varepsilon < B$, there exists a positive constant M_ε such that

$$\text{either} \qquad (F(y)|y) \geqslant (B - \varepsilon) H(y) - M_\varepsilon , \tag{4}$$

$$\text{or} \qquad (F(y)|y) \leqslant -(B - \varepsilon) H(y) + M_\varepsilon \tag{4'}$$

holds (respectively), for every y in R^m.

Then, from (3) and (4) (or (4')), using the Hölder inequality $|h(\cdot, x, x'), x')_2| \leqslant D|x'|_1$ (where D is a bound for h), we get

$$(B - \varepsilon)\int_0^T H(x'(t))\, dt \leqslant D|x'|_1 + TM_\varepsilon . \tag{5}$$

Remark that, because of (i), for any ε, $0 < \varepsilon < A$, there exists a positive constant N_ε such that

$$H(z) \geqslant (A - \varepsilon)|z| - N_\varepsilon \tag{6}$$

holds, for every z in R^m.

Then, from (5) and (6) we deduce (7):

$$(B - \varepsilon)(A - \varepsilon)|x'|_1 \leqslant D|x'|_1 + T\big(M_\varepsilon + N_\varepsilon(B - \varepsilon)\big) . \tag{7}$$

From (7) and the condition $A \cdot B > D$, we immediately have that $|x'|_1$ is bounded by some positive constant, say C_1, as soon as we choose ε sufficiently small:

$$|x'|_1 \leqslant C_1 . \tag{8}$$

From (8) and (5) we also get

$$\int_0^T H(x'(t))\,dt \leqslant C_2 \tag{9}$$

for some positive constant C_2.

Let $\bar{x} = \frac{1}{T}\int_0^T x(t)\,dt$, the mean value of x and set $u(t) = x(t) - \bar{x}$. For the r-th coordinate in R^m it is $|u_r(t)| \leqslant \int_0^T |u_r'(s)|\,ds = \int_0^T |x_r'(s)|\,ds$ (as every u_r has mean value 0); since $\int_0^T |x_r'(s)|\,ds \leqslant |x'|_1$, we easily deduce, using also (8), that there exists a positive constant M, such that

$$|u|_\infty \leqslant M\,. \tag{10}$$

Let us take the $\mathscr{L}^2$-scalar product of both sides of (2) by the constant periodic (vector) function $\bar{x}$. Observe that $(x'', \bar{x})_2 = 0$, $(x, \bar{x})_2 = T|\bar{x}|^2 \geqslant 0$ and therefore

$$\lambda(G(x), \bar{x})_2 = \lambda(h(\cdot, x, x'), \bar{x})_2 - \lambda(F(x'), \bar{x})_2 - (1-\lambda)\alpha T|\bar{x}|^2\,.$$

Hence, since $\alpha > 0$, dividing by $\lambda > 0$ we get

$$\begin{aligned}(G(x), \bar{x})_2 &\leqslant (h(\cdot, x, x'), \bar{x})_2 - (F(x'), \bar{x})_2 \leqslant \\ &\leqslant |(h(\cdot, x, x), \bar{x})_2| + |(F(x'), \bar{x})_2| \leqslant \\ &\leqslant DT|\bar{x}| + |\bar{x}|\Big(\int_0^T |F(x'(t))|\,dt\Big)\,.\end{aligned} \tag{11}$$

From (ii) it follows that there exist two positive constant, W and V, such that

$$|F(y)| \leqslant W\cdot H(y) + V \tag{12}$$

holds for every y in R^m.

Then, from (11) and (12), using the a-priori bound (9), we get

$$(G(x), \bar{x})_2 \leqslant (TD + C_2 W + TV)|\bar{x}|\,. \tag{13}$$

Now either $\bar{x} = 0$ (and so it is bounded) or we can divide (13) by $|\bar{x}|$ and have that the ratio $(G(x), \bar{x})_2/|\bar{x}|$ is bounded by some constant; call it C_3.

The last result can be written also in the following way:

$$\int_0^T \Big(\big(G(\bar{x} + u(t))\,|\bar{x}\big)/|\bar{x}|\Big)\,dt \leqslant C_3\,.$$

If we now use the mean value theorem for the continuous periodic scalar function $t \leadsto \big(G(\bar{x} + u(t))\,|\bar{x}\big)/|\bar{x}|$, we know that there is some point t_0 in the interval $[0, T]$ such that

$$\big(G(\bar{x} + u(t_0))\,|\bar{x}\big)/|\bar{x}| \leqslant C_3/T\,, \tag{14}$$

with $|u(t_0)| \leqslant M$, as in (10).

Comparing (14) with hypothesis (k) we obtain that $|\bar{x}|$ must be bounded. Finally, from the bounds for $|\bar{x}|$ and $|u|_\infty$ we get a bound for x.

$$|x|_\infty \leqslant C_4\,. \tag{15}$$

Then, from (15) and the continuity of G, we have that $G(x)$ is bounded in the supremum norm (and also in the $\mathscr{L}^1$-norm).

Moreover, from (9) and (12) we have already seen that $F(x')$ is bounded in the $\mathscr{L}^1$-norm, and so we conclude that, in the equation (2), all the terms with x or x' are bounded in the $\mathscr{L}^1$-norm. Then (from (2)) also x'' is bounded in the $\mathscr{L}^1$-norm:

$$|x''|_1 \leqslant C_5\,. \tag{16}$$

Since $|x'|_\infty \leqslant |x''|_1 + |x'|_1/T$ from (8) and (16), we finally obtain the required bound for x' in the supremum norm:

$$|x'|_\infty \leqslant C_6\,. \tag{17}$$

Estimates (15) and (17) prove that every periodic solution of (2), for any λ, is bounded in $\mathscr{C}^1_T$ and so the thesis is proved.

With obvious modifications it is possible to deal with the case in which the limit in (k) is $-\infty$. Q.E.D.

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