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On a Decomposition of a von Neumann Algebra.

A. B. THAHEEM (*)

SUMMARY - Let M be a von Neumann algebra and S, T be *-automorphisms of M satisfying an operator equation: $Sx + S^{-1}x = Tx + T^{-1}x$ for all x in M. Assume further that S and T commute. Then it is shown that M admits a decomposition by a central projection e in M such that $S(Me) = T(Me)$ and $S(M(1-e)) = T^{-1}(M(1-e))$.

1. Introduction.

In [4] we associate a strongly continuous one-parameter group of *-automorphisms $\{S_t: t \in R\}$ of a von Neumann algebra M with a positive map ψ_S of M by the formula:

$$\psi_S(x) = \int_{-\infty}^{+\infty} \frac{1}{e^{\pi t} + e^{-\pi t}} S_t(x)\, dt \qquad (x \in M) .$$

We show there that ψ determines the group almost completely in the sense that if $\{S_t: t \in R\}$ and $\{T_t: t \in R\}$ are such groups then $\psi_S = \psi_T$

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implies that

$$(1) \qquad S_t x + S_t^{-1} x = T_t x + T_t^{-1} x \quad \text{for all } x \in M \text{ and } t \in R\,.$$

We use Arveson's theory of spectral subspaces of automorphism groups [1, 5] to solve operator equation (1). We essentially obtain a decomposition of M by a central projection $e \in M$ with

$$S_t(ex) = T_t(ex) \quad \text{and} \quad S_t\big((1-e)x\big) = T_t^{-1}\big((1-e)x\big)$$

for all $x \in M$ and $t \in R$.

This paper deals with a natural generalization of operator equation (1). Let S, T be any *-automorphisms of M satisfying the equation

$$(2) \qquad Sx + S^{-1}x = Tx + T^{-1}x \quad (x \in M)\,.$$

Now there is no spectral subspace theory of S and T similar to the one for automorphism groups in [1, 5]. However, assuming the commutativity of S and T, we also obtain here a central decomposition of M by a central projection $e \in M$ with

$$S(Me) = T(Me) \quad \text{and} \quad S\big(M(1-e)\big) = T^{-1}\big(M(1-e)\big)\,.$$

If M is a factor then $S = T$ or $S = T^{-1}$.

The approach here is simpler than that of [4] but the advantage there is that we do not assume the commutativity of automorphism groups to obtain the decomposition of M in solving equation (1).

The result is parallel to a well-known result of Kadison [2] which says that if θ is a Jordan isomorphism of a von Neumann algebra M, then there is a central projection $p \in M$ such that θ is an isomorphism of Mp and anti-isomorphism of $M(1-p)$.

This paper is organised in the following fashion: Section 2 contains the decomposition of M relative to operator equation (2). In section 3 we consider a special case where the additional condition of commutativity of S and T may be suppressed to obtain the decomposition of M. In other words, in this case, equation (2) automatically implies the commutativity of S and T. We conclude the paper with an example of two isometries satisfying equation (2) on a Hilbert space but they do not commute. Therefore, it is important to have the structure more than the isometries on a Hilbert space.

2. Decomposition of a von Neumann algebra.

Let M be a von Neumann algebra and S, T be *-automorphisms of M satisfying

$$(2) \qquad Sx + S^{-1}x = Tx + T^{-1}x \quad \text{for all } x \in M .$$

We shall write (2) as: $S + S^{-1} = T + T^{-1}$.

Denote by $N(T) = \{x \in M : Tx = 0\}$, the null space of T and $R(T) = \{x \in M : x = Ty \text{ for some } y \in M\}$, the range space of T.

Definition 2.1. Put

$$M_1 = \{x \in M : Sx = Tx\} = N(S - T) ,$$

$$M_2 = \{x \in M : Sx = T^{-1}x\} = N(S - T^{-1}) .$$

Set $M_1 \cap M_2 = M_0$. Then M_0 is non-empty because $1 \in M_0$ and $M_0 = \{x \in M : Sx = Tx = S^{-1}x = T^{-1}x\}$. Also M_1, M_2 and M_0 are von Neumann algebras. From equation (2) it is easy to see that $S(M_1) = M_1$, $T(M_1) = M_1$, $S(M_2) = M_2$ and $T(M_2) = M_2$.

Lemma 2.2. If $x \in M_1$ and $y \in M_2$, then

$$(Sx - S^{-1}x)(Sy - S^{-1}y) = 0 .$$

Proof. By assumption

$$(S + S^{-1})(xy) = (T + T^{-1})(xy) .$$

This implies that

$$(Sx)(Sy) + (S^{-1}x)(S^{-1}y) - (Sx)(S^{-1}y) - (S^{-1}x)(Sy) = 0 ,$$

therefore, $(Sx - S^{-1}x)(Sy - S^{-1}y) = 0$.

Since S leaves M_1 invariant so does S^2. Furthermore, $N(S^2 - 1) = N(S - S^{-1})$ and $R(S^2 - 1) = R(S - S^{-1})$. Then, by [3], the weakly closed subalgebra K_1 generated by $R(S^2 - 1)$ is a twosided ideal in M_1. If e is the central projection in M_1 such that $K_1 = M_1 e$ then $f = 1 - e$ is the largest projection such that $S(fx) = S^{-1}(fx)$ for all $x \in M_1$.

Let e_1, f_1 be the projections corresponding to S^2 on M_2 then by lemma 2.2, $ee_1 = e_1 e = 0$. Since S leaves $R(S - S^{-1})$ invariant and hence K_1, therefore, $Se = e$. Further, $Se = e = Te = T^{-1} e$. This implies that $e \in M_2$. We shall prove that e is in the center of M_2 and eventually in the center of M.

LEMMA 2.3. $fM_1 \subseteq M_0$, $f_1 M_2 \subseteq M_0$.

PROOF. For any $x \in M_1$, we have $S(fx) = S^{-1}(fx) = T^{-1}(fx)$. That is $fx \in N(S - T^{-1}) = M_2$ and hence $fx \in M_1 \cap M_2 = M_0$. Similarly $f_1 M_2 \subseteq M_0$.

LEMMA 2.4. $M_1 + M_2$ is weakly dense in M.

PROOF. $M_1 = N(S - T) = N(S^{-1} - 1)$, then it is easy to see that $R(S^{-1}T - 1) \subseteq N(S - T^{-1}) = M_2 \cdot S^{-1}T$ is again a *-automorphism, therefore, by [3, proposition 1], $M_1 + M_2$ is weakly dense in M.

We are now able to show the following

LEMMA 2.5. The projections e, e_1 are in the center of M.

PROOF. Since $f_1 M_2 \subseteq M_0$, therefore, e commutes with all the elements of $f_1 M_2$. Also $eM_1 \cdot e_1 M_2 = e_1 M_2 \cdot eM_1 = 0$, therefore, e commutes with all the elements of M_2. Since $M_1 + M_2$ is dense in M and therefore, by continuity e is in the center of M. Similarly e_1 is also in the center of M.

We notice that $eM_2 \subseteq M_0 \subseteq M_1$, therefore, $M_1 + M_2$ is dense in M implies that $eM \subseteq M_1$. Similarly $e_1 M \subseteq M_2$ and $(1 - e - e_1) M \subseteq M_0$.
We combine the above lemmas to prove the following

THEOREM 2.6. Let S, T be *-automorphisms of a von Neumann algebra such that $Sx + S^{-1}x = Tx + T^{-1}x$ for all $x \in M$. Assume that S and T commute, then there exists a central projection e in M such that $S(ex) = T(ex)$ and $S((1 - e)x) = T^{-1}((1 - e)x)$.

PROOF. Obviously $M = eM + e_1 M + (1 - e - e_1) M$. Since $eM \subseteq M_1$, therefore, $S(ex) = T(ex)$. Also, $(1 - e) M = e_1 M + (1 - e - e_1) M$, $e_1 M \subseteq M_2$ and $(1 - e - e_1) M \subseteq M_0$, therefore $S((1 - e)x) = T^{-1}((1 - e)x)$.

COROLLARY 2.7. If M is a factor then $Sx = Tx$ or $Sx = T^{-1}x$ for all x in M.

3. Commutativity of S and T.

In this section we consider a special case where the additional condition of commutativity of S and T may be dropped from the hypothesis of theorem 2.6 to get the decomposition.

Let A be a commutative Banach algebra and S an automorphism of A. Let A^* denote the Banach space of all continuous linear functionals on A and S^* the adjoint of S. Let Δ be the set of all complex homomorphisms of A and $\hat{A}$ the set of all Gelfand transforms $\hat{x}$, for $x \in A$. It is easy to see that $S^*(\Delta) = \Delta$, $T^*(\Delta) = \Delta$, and $Sx + S^{-1}x = Tx + T^{-1}x$ if and only if $S^*x^* + S^{*-1}x^* = T^*x^* + T^{*-1}x^*$ for all $x \in A$ and $x^* \in A^*$.

THEOREM 3.1. Let A be a commutative Banach algebra and S, T be automorphisms with $Sx + S^{-1}x = Tx + T^{-1}x$ for all $x \in A$. Then S and T commute.

PROOF. Put

$$P_1 = \{x^* \in A^* \colon S^*x^* = T^*x^*\},$$

$$P_2 = \{x^* \in A^* \colon S^*x^* = T^{*-1}x^*\}.$$

Obviously P_1 and P_2 are invariant under both S^* and T^*. Put

$$L_1 = P_1 \cap \Delta, \quad L_2 = P_2 \cap \Delta.$$

L_1 and L_2 are also invariant under S^* and T^*. Further, $L_1 \cup L_2 \subseteq \Delta$. We shall prove that, in fact, $L_1 \cup L_2 = \Delta$.

Let $h \in \Delta$ and $x \in A$, then

$$\hat{x}(S^*h) + \hat{x}(S^{*-1}h) = \hat{x}(T^*h) + \hat{x}(T^{*-1}h) \qquad \text{... (i)}$$

Suppose that $h \notin L_1 \cup L_2$, then

$$\left.\begin{array}{l} S^*h \neq T^*h \\ \text{and} \\ S^*h \neq T^{*-1}h \ (\text{or } S^{*-1}h \neq T^*h) \end{array}\right] \qquad \text{... (ii)}$$

Now there are two possibilities for S^*h and $S^{*-1}h$ in the sense that

either $\qquad (a) \quad S^*h = S^{*-1}h\,,$

or $\qquad (b) \quad S^*h \neq S^{*-1}h\,.$

Now S^*h, $S^{*-1}h$, T^*h, $T^{*-1}h \in \Delta$. Therefore, we can find a suitable $\hat{x}_0 \in \hat{A}$ $(x_0 \in A)$ such that in case (a)

$$\hat{x}_0(S^*h) = \hat{x}_0(S^{*-1}h) = \lambda \neq 0 \qquad (\lambda \text{ aconstant})\,,$$

and
$$\hat{x}_0(T^*h) = \hat{x}_0(T^{*-1}h) = 0\,.$$

This combined with (ii) contradicts (i). Similarly, we can choose $\hat{y}_0$ in $\hat{A}$ such that in case (b)

$$\hat{y}_0(S^*h) = \mu \neq 0 \qquad \text{and} \qquad \hat{y}_0(S^{*-1}h) = \hat{y}_0(T^*h) = \hat{y}_0(T^{*-1}h) = 0\,.$$

Again this together with (ii) contradicts (i).

Therefore, in any case we must have

$$S^*h = T^*h \qquad \text{or} \qquad S^*h = T^{*-1}h\,.$$

This shows that $L_1 \cup L_2 = \Delta$.

It is immediate that S^* and T^* commute on Δ and, therefore, for any $h \in \Delta$ and $x \in A$, we have

$$h(STx - TSx) = 0\,.$$

This implies that $STx - TSx \in \operatorname{Ker}(h)$ and hence $(STx - TSx) \in \in \operatorname{radical}(A) = \{0\}$, therefore, $STx = TSx$ for all $x \in A$. This completes the proof of the theorem.

The following example shows that it is essential to have algebraic structure to get the decomposition of theorem 2.6 (and also the commutativity of theorem 3.1).

EXAMPLE 3.2. Let H be a Hilbert space, λ a complex number with $|\lambda| = 1$ and $0 < E, F < 1$ be projections on H. Define two linear mappings S and T on H as:

$$Sx = \lambda Ex + \bar{\lambda}(1 - E)x ,$$
$$Tx = \lambda Fx + \bar{\lambda}(1 - F)x \qquad (x \in H) .$$

S and T are invertible isometries on H with $S^{-1}x = \bar{\lambda}Ex + \lambda(1 - E)x$ and $Sx + S^{-1}x = Tx + T^{-1}x$.

If E and F are chosen such that E and F do not commute, then, evidently $ST \neq TS$.

We conclude the paper with the following

PROBLEM. For what class of algebras the condition of commutativity of automorphisms in the hypothesis of theorem 2.6 turns out to be redundant?

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