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Comparison Results of Reaction-Diffusion Equations with Delay in Abstract Cones.

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1. Introduction.

Differential equations and differential inequalities containing functionals is of great importance in problems of biomathematical medicine, chemistry, heat flow and population growth. Many of these applications lead to an equation, which is of parabolic structure, in the sense that the equation would be parabolic if the function in it were replaced by a known function. A special case of this, which is known as reaction diffusion equation occurs in studies of population genetics [2, 9], conduction of nerve impulses [2, 4], chemical reactions [1, 3] and several other biological questions [1, 8].

In this paper we give comparison theorems related to parabolic differential inequalities with delay and flow in variance results for the parabolic differential equation with delay in a Banach space. We have also included a generalization of the classical Müller's theorem. Our results are generalizations of the results in [5]. See also [7] for different type of comparison theorems concerning parabolic differential inequalities with delay in R^n.

2. Preliminary results.

Let Ω be a bounded domain in R^n and let $G = [t_0 - \tau, t_0] \times \bar{\Omega}$ and $H = (t_0, \infty) \times \Omega$, $t_0 \geqslant 0$. Suppose the boundary ∂H of H is split

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into two parts ∂H_0, ∂H_1 such that $\partial H = \partial H_0 \cup \partial H_1$, $\{t_0\} \times \partial \Omega \subset \partial H_0$, and $\partial H_0 \cap \partial H_1 = \emptyset$.

Let E be a real Banach space with $\|\cdot\|$. A one k is a proper subset of E such that if $v, w \in k$, $\lambda \in R^+$, then $v + w$, $\lambda v \in k$. Throughout this paper we will consider a closed cone k and its interior k^0 and we assume that k^0 is nonempty. The cone k induces a partial ordering on E defined by $u \leqslant v$ iff $v - u \in k$ and $u < v$ iff $v - u \in k^0$.

Let k^* be the set of all continuous linear functionals c on E such that $c(u) \geqslant 0$ for all $u \in k$, and let k_0^* be the set of all continuous linear functionals c on E such that $c(u) > 0$ for all $u \in k^0$.

Let $\tau > 0$ be a given real number and $\mathcal{C} = C[[-\tau, 0], E]$ and $\mathcal{C}_\Omega = C[[-\tau, 0] \times \bar{\Omega}, E]$ denote the Banach spaces of continuous functions with the norm of $\varphi \in \mathcal{C}$, $\varphi(\cdot, x) \in \mathcal{C}_\Omega$ given by

$$\|\varphi\|_0 = \max \|\varphi(s)\| \qquad \text{and} \qquad \|\varphi(\cdot, x)\| = \max \|\varphi(s, x)\|$$

respectively.

A vector ν is said to be an outernormal at $(t, x) \in \partial H_1$, if $(t, x - h\nu) \in \in t \times \Omega$ for small $h > 0$. The outernormal derivative is then given by

$$\frac{\partial u}{\partial \nu}(t, x) = \lim_{h \to 0} \frac{u(t, x) - u(t, x - h\nu)}{h} .$$

We shall always assume that an outernormal exists on ∂H_1 and the functions in question have outernormal derivatives on ∂H_1.

If $u \in C[G \cup \bar{H}, E]$ and $\underset{s \in [-\tau, 0]}{u(t + s, x)} \in \mathcal{C}_\Omega$ for $t \in [t_0, \infty)$ and also if the partial derivatives $\partial u/\partial t$, $u_x (= \partial u/\partial x)$, $u_{xx} (= \partial^2 u/\partial x^2)$ exist and are continuous in H, then we shall say that $u(t, x)$ belongs to class Z.

From now on we denote $\underset{s \in [-\tau, 0]}{u(t + s, x)}$ as $u_t(\cdot, x)$.

We state below Mazur's theorem which is needed in our comparison theorems.

THEOREM 2.1 (Mazur). Let k be a cone with nonempty interior k^0. Then

(i) $u \in k$ is equivalent to $c(u) \geqslant 0$ for all $c \in k^*$.

(ii) $u \in \partial k$ implies that there exists a $c \in k_0^*$ such that $c(u) = 0$.

A function $f \in C[[t_0, \infty) \times \Omega \times E \times E^n \times E^{n^2} \times \mathcal{C}_\Omega, E]$ is said to be quasimonotone nondecreasing in u for fixed t, x, φ belonging to

$[t_0, \infty)$, Ω, $\mathcal{C}_\Omega$ respectively, if for any $u, v \in E$, $u_x, v_x \in E^n$, $u_{xx}, v_{xx} \in E^{n^2}$ such that $u \leqslant v$, $c(u) = c(v)$, $c(u_{x_i}) = c(v_{x_i})$ for $i = 1, \ldots, n$, and

$$\sum_{i,j=1}^{n} \gamma_i \gamma_j c(u_{x_i x_j} - v_{x_i x_j}) \leqslant 0 \quad \text{for } c \in k_0^*$$

implies

$$c\big(f(t, x, u, u_x, u_{xx}, \varphi(\cdot, x))\big) \leqslant c\big(f(t, x, v, v_x, v_{xx}, \varphi(\cdot, x))\big) .$$

For the case $E = R^n$ and $k = R^n_+$, the quasimonotone condition on f implies that $f_i\big(t, x, u, u_x, u_{xx}, \varphi(\cdot, x)\big) = f_i\big(t, x, u, u^i_x, u^i_{xx}, \varphi(\cdot, x)\big)$ for each i, $1 \leqslant i \leqslant N$.

A function $f \in C[[t_0, \infty) \times \Omega \times E \times E^n \times E^{n^2} \times \mathcal{C}_\Omega, E]$ is said to be monotone nondecreasing in $\varphi(\cdot, x)$ if for $u \in E$, $u_x \in E^n$, $u_{xx} \in E^{n^2}$, $\varphi(\cdot, x) \leqslant \psi(\cdot, x)$ implies that for $c \in k_0^*$

$$c\big(f(t, x, u, u_x, u_{xx}, \varphi(\cdot, x))\big) \leqslant c\big(f(t, x, u, u_x, u_{xx}, \psi(\cdot, x))\big) .$$

Remark 2.1. We give similar definitions of quasimonotonicity and monotonicity of a function $g \in C[[t_0, \infty) \times E \times \mathcal{C}, E]$.

3. In this section we develop the theory of differential inequalities related to parabolic differential equations with delay.

Theorem 3.1. Suppose that

(i) $v, w \in z$, $f \in C[[t_0, \infty) \times \Omega \times E \times E^n \times E^{n^2} \times \mathcal{C}_\Omega, E]$, f is quasimonotone nondecreasing in u relative to k and monotone nondecreasing in $\varphi(\cdot, x)$ relative to k, and

$$\frac{\partial v}{\partial t} \leqslant f\big(t, x, v, v_x, v_{xx}, v_t(\cdot, x)\big) ,$$

$$\frac{\partial w}{\partial t} \geqslant f\big(t, x, w, w_x, w_{xx}, w_t(\cdot, x)\big) , \quad \text{on } [t_0, \infty) \times \Omega$$

where $v = v(t, x)$, $w = w(t, x)$;

(ii) (a) $\quad v_{t_0}(\cdot, x) < w_{t_0}(\cdot, x) \quad$ for $x \in \bar{\Omega}$,

(b) $\quad v(t, x) < w(t, x) \quad$ on ∂H_0,

(c) $\quad \dfrac{\partial u(t, x)}{\partial \nu} < \dfrac{\partial w(t, x)}{\partial \nu} \quad$ on ∂H_1.

Then $v(t, x) < w(t, x)$ on $[t_0, \infty)\times\bar{\Omega}$, if one of the inequalities in (i) is strict.

PROOF. Assume that one of the inequalities in (i) is strict. Consider $m(t, x) = v(t, x) - w(t, x)$. It is enough to show that $m(t, x) < 0$. If it were not true, there would exist a (t_1, x_1) and $c \in k_0^*$ such that $m(t, x) < 0$ on $[t_0, t_1)\times\bar{\Omega}$, $m(t_1, x_1) \leqslant 0$ and $c(m(t_1, x)) \leqslant 0$. It is easy to see that the function $c(m(t_1, x))$ has its maximum at x_1 which is equal to zero. Clearly $(t_1, x_1) \notin G \cup \partial H_0$ because of (ii) (a) and (b). Also $(t_1, x_1) \notin \partial H_1$, for we would then have

$$\lim_{h\to 0} c\left(\frac{m(t_1, x_1) - m(t_1, x_1 - hv)}{h}\right) \geqslant 0 .$$

This contradicts (ii) (c). Hence, $(t_1, x_1) \in H$. Let $c \in k_0^*$ be such that $m(t_1, x_1) \leqslant 0$, $c(m(t_1, x_1)) = 0$, $c(m_{x_i}(t_1, x_1)) = 0$ and for $i = 1, 2, \ldots, n$, and $\sum_{i,j=1}^{n} \lambda_i \lambda_j \quad c(m_{x_i x_j}(t_1, x_1) \leqslant 0$ for $\lambda \in R^n$ and $m_{t_1}(\cdot, x) \leqslant 0$. This implies that

$$v(t_1, x_1) \leqslant w(t_1, x_1)\,, \quad c(v(t_1, x_1)) = c(w(t_1, x_1))\,, \quad c(v_{x_i}(t_1, x_1)) = c(w_{x_i}(t_1, x_1))$$

and

$$\sum_{i,j=1}^{n} \lambda_i \lambda_i c(v_{x_i x_j}(t_j, x_1) - w_{x_i x_j}(t_1, x_1)) \leqslant 0 \qquad \text{for } \lambda \in R^n$$

and $v_{t_1}(\cdot, x) \leqslant w_{t_1}(\cdot, x)$. Thus, in view of (i), it follows that

$$\begin{aligned} c\left(\frac{\partial m}{\partial t}(t_1, x_1)\right) &= c\left(\frac{\partial v}{\partial t}(t_1, x_1) - \frac{\partial w}{\partial t}(t_1, x_1)\right) \\ &< c\Big(\big(f(t_1, x_1, v(t_1, x_1), v_x(t_1, x_1), v_{xx}(t_1, x_1), v_{t_1}(\cdot, x)\big) - \\ &\qquad - f\big(t_1, x_1, w(t_1, x_1), w_x(t_1, x_1), w_{xx}(t_1, x_1), w_{t_1}(\cdot, x)\big)\Big) \\ &\leqslant 0 \end{aligned}$$

using quasimonotonicity of f in u relative to k and monotonicity of f in $\varphi(\cdot, x)$ relative to k. That is $c((\partial m/\partial t)(t_1, x_1)) < 0$. However $c(m(t_1 - h, x_1) - m(t_1, x_1)) < 0$ for $h > 0$ sufficiently small. It therefore follows that $c((\partial m/\partial t)(t_1, x_1)) \geqslant 0$, which leads to a contradiction. This completes the proof.

REMARK 3.1. The conclusion of the above theorem is not valid if one of the inequalities in (i) is not strict. However, we can dispense with the strict inequality needed in Theorem 3.1 of (i), (ii) if in addition f satisfies the following condition,

(C_0) $z \in Z$, $z > 0$ on $G \cup \bar{H} - \partial H_1$, $(\partial z/\partial \nu)(t, x) \geqslant \beta > 0$ on ∂H_1 and for sufficiently small $\varepsilon > 0$, either

$$(a) \quad \frac{\varepsilon \partial z}{\partial t} > f(t, x, v, v_x, v_{xx}, v_t(\cdot, x)) - f(t, x, v - \varepsilon z, v_x - \varepsilon z_x, v_{xx} - \varepsilon z_{xx}, v_t(\cdot, x) - \varepsilon z_t(\cdot, x))$$

or

$$(b) \quad \frac{\varepsilon \partial z}{\partial t} > f(t, x, w + \varepsilon z, w_x + \varepsilon z_x, w_{xx} + \varepsilon z_{xx}, w_t(\cdot, x) + + \varepsilon z_t(\cdot, x)) - f(t, x, w, w_x, w_{xx}, w_t(\cdot, x))$$

on $[t_0, \infty) \times \Omega$, where $v, w \in Z$.

THEOREM 3.2. Let the assumptions (i) of Theorem 3.1 hold. Suppose further that the condition C_0 is satisfied. Then the relations

$$\text{(ii)} \quad (a) \quad v_{t_0} \leqslant w_{t_0} \qquad \text{for } x \in \bar{\Omega}$$

$$(b) \quad v(t, x) \leqslant w(t, x) \qquad \text{on } \partial H_0$$

$$(c) \quad \frac{\partial v}{\partial \nu}(t, x) \leqslant \frac{\partial w}{\partial \nu}(t, x) \qquad \text{on } \partial H_1$$

imply $v(t, x) \leqslant w(t, x)$ on $[t_0, \infty) \times \bar{\Omega}$.

PROOF. Assume that the condition (a) of C_0 holds. Consider $\tilde{v} = v - \varepsilon z$ where $\varepsilon > 0$ is sufficiently small. We have

$$\frac{\partial \tilde{v}}{\partial t} = \frac{\partial v}{\partial t} - \frac{\varepsilon \partial z}{\partial t} < f(t, x, \tilde{v}, \tilde{v}_x, \tilde{v}_{xx}, \tilde{v}_t(\cdot., x)) \qquad \text{on } [t_0, \infty) \times \Omega$$

using $C_0(a)$. Also $\tilde{v}(t, x) < w(t, x)$ on $G \cup \partial H_0$, and

$$\frac{\partial \tilde{v}}{\partial \nu} = \frac{\partial v}{\partial \nu} - \frac{\varepsilon \partial z}{\partial \nu} \leqslant \frac{\partial v}{\partial \nu} - \varepsilon v < \frac{\partial v}{\partial \nu} \qquad \text{on } \partial H_1 .$$

Thus, the functions $\tilde{v}$, w satisfy the assumptions of Theorem 3.1, hence $v(t,x) < \tilde{w}(t,x)$ on $\bar{H}$. Taking the limit as $\varepsilon \to 0$ yields the desired result and the proof is complete.

REMARK 3.2. If ∂H_1 is empty so that $\partial H_0 = \partial H$, the assumption (C_0) in Theorem 3.2 can be replaced by a weaker hypothesis, namely a one-sided Lipschitz's condition of the form

$$(C_1)\quad f(t,x,u,p,q,\varphi)\cdot,x) - f(t,x,v,p,q,\psi(\cdot,x)) \leqslant L\Big\{(u-v) + \sup_{s\in[-\tau,0]}\{\varphi(\cdot,x)-\psi(\cdot,x)\}$$

for $u \geqslant v$ and $\varphi(\cdot,x) \geqslant \psi(\cdot,x)$.

In this case, it is enough to set $\tilde{v} = v - \varepsilon e^{2Lt} y_0$, $y_0 \in k^0$ (where $\varepsilon > 0$ is sufficiently small) so that $\tilde{v}(t,x) < w(t,x)$ on ∂H and $\tilde{v}_{t_0}(\cdot,x) < w_{t_0}(\cdot,x)$

$$\begin{aligned}\frac{\partial\tilde{v}}{\partial t} &= f(t,x,v,v_x,v_{xx},v_t(\cdot,x)) - 3\varepsilon L e^{3Lt} y_0 \\ &\leqslant f(t,x,\tilde{v},\tilde{v}_x,\tilde{v}_{xx},\tilde{v}_t(\cdot,x)) - \varepsilon L e^{3Lt} y_0 \\ &< f(t,x,\tilde{v},\tilde{v}_x,\tilde{v}_{xx},\tilde{v}_t(\cdot,x)) \qquad \text{on } [t_0,\infty)\times\Omega\,.\end{aligned}$$

Even when ∂H_1 is not empty, the condition (C_1) is enough provided (ii) (c) is strengthened to $\partial v/\partial\nu + Q(t,x,v) \leqslant \partial w/\partial\nu + Q(t,x,w)$ on ∂H_1 where $Q \in C[\bar{H}\times E, E]$ and $Q(t,x,u)$ is strictly increasing in u. To see this, observe that $\tilde{v} < v$ and hence $Q(t,x,\tilde{v}) < Q(t,x,v)$ which gives the desired strict inequality needed in the proof.

4. In this section we give some comparison theorems related to the system

$$\frac{\partial u}{\partial t} = f(t,x,u,u_x,u_{xx},u_t(\cdot,x)) \tag{4.1}$$

satisfying the initial boundary conditions

$$u_{t_0}(\cdot,x) = \varphi_0(\cdot,x) \qquad \text{for } x\in\bar{\Omega}$$

$$\begin{cases} u(t,x) = u_0(t,x) & \text{on } \partial H_0 \\ \text{such that } u_0(t_0,x) = \varphi_0(0,x) \text{ and} \\ \dfrac{\partial u}{\partial\nu}(t,x) = 0 & \text{on } \partial H_1\,. \end{cases} \tag{4.2}$$

A closed set $F \subset E$ is said to be flow invariant relative to the system (4.1)-(4.2) if for every solution $u(t, x)$ of (4.1)-(4.2) we have $\varphi_0(\cdot, x)$ and $u_0(t, x) \in F$ implies $u(t, x) \in F$ on $\bar{H}$.

The function $f(t, x, u, u_x, u_{xx}, \varphi(\cdot, x))$ is said to be quasi-nonpositive (quasinonnegative) if $u \leqslant 0$ $(u \geqslant 0)$, $c(u) = 0$, $c(u_{x_i}) = 0$, $i = 1, 2, \ldots, n$ and $\sum_{i,j=1}^{n} \lambda_i \lambda_j c(u_{x_i x_j}) \leqslant 0$, $\left(\sum_{i,j=1}^{n} \lambda_i \lambda_j c(u_{x_i x_j}) \geqslant 0\right)$ for $\lambda \in R^n$ and also for $(\varphi(\cdot, x)) \leqslant 0$, $(\varphi(\cdot, x) \geqslant 0)$ for some $c \in k_0^*$ implies

$$c\big(f(t, x, u, u_x, u_{xx}, \varphi(\cdot, x))\big) \leqslant 0\,, \quad \big(c(f(t, x, u, u_x, u_{xx}, \varphi(\cdot, x)) \geqslant 0\big)\,.$$

THEOREM 4.1. Assume that f is quasinonpositive and that the condition $C_0(a)$ holds with $v = u$, where $u = u(t, x)$ is any solution of (4.1)-(4.2). Then the closed set $\bar{Q}$ is flow invariant relative to the system (4.1)-(4.2) where $Q = [u \in E, u < 0]$.

PROOF. We set $m(t, x) = u(t, x) - \varepsilon z(t, x)$ where $u(t, x)$ is any solution of (4.1)-(4.2) such that $u_{t_0}(\cdot, x)$, $u_0(t, x) \in \bar{Q}$ and $\varepsilon > 0$ is sufficiently small and $z \in Z$ is as in $C_0(a)$. We wish to show $m(t, x) < 0$ on $\bar{H}$. If not, there would exist a (t_1, x_1), $t_1 > t_0$, $x_1 \in \Omega$ and $c \in k_0^*$ such that

$$m(t, x) < 0 \quad \text{on} \quad [t_0, t_1) \times \bar{\Omega}\,,$$

$$m(t_1, x_1) \leqslant 0 \quad \text{and} \quad c\big(m(t_1, x_1)\big) = 0\,.$$

It is easy to see that the function $c\big(m(t_1, x)\big)$ has its maximum at x_1 which is equal to zero. Clearly, $(t_1, x_1) \notin \partial H_0$, or $(t_1, x_1) \notin \partial H_1$. Let $(t_1, x_1) \in H$ and $c \in k_0^*$ be such that $m(t_1, x_1) \leqslant 0$, $c\big(m(t_1, x_1)\big) = 0$, $c\big(m_{x_i}(t_1, x_1)\big) = 0$, $i = 1, \ldots, n$, $\sum_{i,j=1}^{n} \lambda_i \lambda_j c\big(m_{x_i x_j}(t_1, x_1)\big) \leqslant 0$, $\lambda \in R^n$. Then by $C_0(a)$ and the fact f is quasinonpositive, we obtain

$$\begin{aligned}
c\left(\frac{\partial m}{\partial t}(t_1, x_1)\right) &= c\left(\frac{\partial u}{\partial t}(t_1, x_1) - \varepsilon \frac{\partial z}{\partial t}(t_1, x_1)\right) \\
&\leqslant c\left(f\left(t_1, x_1, u(t_1, x_1), u_x(t_1, x_1), u_{xx}(t_1, x_1), u_{t_1}(\cdot, x), -\varepsilon \frac{\partial z}{\partial t}(t_1, x_1)\right)\right) \\
&< c\big(f(t_1, x_1, m(t_1, x_1), m_x(t_1, x_1), m_{xx}(t_1, x_1), m_{t_1}(\cdot, x))\big) \\
&\leqslant 0
\end{aligned}$$

but

$$c\left(\frac{\partial m}{\partial t}(t_1, x_1)\right) = \lim_{h\to 0} \frac{m(t_1 - h, x_1) - m(t_1, x_1)}{-h} \geqslant 0 .$$

This leads to a contradiction. Hence, $(t_1, x_1) \notin H$. Thus, $m(t, x) < 0$ on $\bar{H}$ which implies, as $\varepsilon \to 0$, the flow invariance of $\bar{Q}$, which com pletes the proof.

REMARK 4.1. Theorem 3.2 can be obtained as a consequence of Theorem 4.1. For this purpose we set $d = v - w$ so that

$$\begin{aligned}\frac{\partial d}{\partial t} &= F(t, x, d, d_x, d_{xx}, d_t(\cdot, x)) \\ &= f(t, x, (w + d), (w + d)_x, (w + d)_{xx}, (w + d)_t(\cdot, x)) \\ &\quad - f(t, x, w, w_x, w_{xx}, w_t(\cdot, x)) + P(t, x)\end{aligned}$$

where

$$\begin{aligned}P(t, x) = \frac{\partial v}{\partial t} &- f(t, x, v, v_x, v_{xx}, v_t(\cdot, v)) \\ &- \frac{\partial w}{\partial t} + f(t, x, w, w_x, w_{xx}, w_t(\cdot, x)) \leqslant 0 .\end{aligned}$$

Clearly $d_{t_0}(\cdot, x) \leqslant 0$, and $d(t, x) \leqslant 0$ on ∂H_0 showing $d_{t_0}(\cdot, x) \in \bar{Q}$, $d(t, x) \in \bar{Q}$ on ∂H_0. It can easily be seen F is quasinonpositive and that F satisfies condition $C_0(a)$ with f replaced by F and v replaced by d. Hence, the conclusion follows.

COROLLARY 4.1. Assume that f is quasinonnegative and that the condition $C_0(b)$ holds with $w = u(t, x)$. Then the closed set $\bar{Q}$ is flow invariant relative to (4.1)-(4.2) where $Q = [u \in E, u > 0]$ if $u_{t_0}(\cdot, x)$ and $u_0(t, x) \in \bar{Q}$.

COROLLARY 4.2. Suppose that condition (C_0) holds with $v = w = u$. Assume also the following condition holds;

If $u \leqslant b$, $u_t(\cdot, x) \leqslant b$, $c(u) = c(b)$, $c(u_{x_i}) = 0$, for $i = 1, \ldots, n$, and $\sum_{i,j=1}^{n} \lambda_i \lambda_j c(u_{x_i x_j}) \geqslant 0$, $\lambda \in R^n$ for $c \in k_0^*$, then $c\left(f(t, x, u, u_x, u_{xx}, u_t(\cdot, x))\right) \leqslant 0$, and also if $a \leqslant u$, $a \leqslant u_t(\cdot, x)$, $c(u) = c(a)$, $c(u_{x_i}) = 0$, $i = 1, 2, \ldots, n$ and $\sum_{i,j=1}^{n} \lambda_i \lambda_j c(u_{x_i x_j}) \geqslant 0$, $\lambda \in R^n$ for $c \in k_0^*$, then $c(f(t, x, u, u_x, u_{xx},$

$u_t(\cdot, x)) \geqslant 0$, then the closed set $\overline{W}$, where $w = [u \in E, a < u < b, a, b \in E]$ is flow invariant relative to (4.1)-(4.2).

REMARK 4.2. If $\partial H_0 = \{t_0\} \times \partial\Omega$, then the initial boundary conditions can be written as $u_{t_0}(\cdot, x) = \varphi_0(\cdot, x)$ for $x \in \bar{\Omega}$, and $(\partial u/\partial t)(t, x) = 0$ on ∂H_1.

We shall next consider a comparison result which yields upper and lower bounds for solutions of (4.1)-(4.2) in terms of solutions of ordinary delay differential equations.

THEOREM 4.2. Assume that

(i) $u = u(t, x)$ is any solution of (4.1)-(4.2) and the condition (C_0) holds with $v = w = u$;

(ii) $g_1, g_2 \in [[t_0, \infty) \times E \times \mathrm{C}, E]$, $g_1(t, u, \varphi)$, $g_2(t, u, \varphi)$ are quasimonotone nondecreasing in u relative to k and monotone decreasing in φ relative to k and for $c \in k_0^*$, if $c(u_{x_i}) = 0$, $i = 1, 2, \ldots, n$ and $\sum_{i,j=1}^{n} \lambda_i \lambda_j c(u_{x_i x_j}) \leqslant 0$, $\lambda \in R^n$, then $c(f(t, x, u, u_x, u_{xx}, \varphi) \leqslant c(g_1(t, u, \varphi)$, and if $c(u_{x_i}) = 0$, $i = 1, 2, \ldots, n$, $\sum_{i,j=1}^{n} \lambda_i \lambda_j c(u_{x_i x_j}) \geqslant 0$, $\lambda \in R^n$, $c(g_2(t, u, \varphi) \leqslant c(f(t, x, u, u_x, u_{xx}, \varphi)$.

(iii) $r(t)$, $\varrho(t)$ are solutions of

$$r'(t) = g_1(t, r, r_t)\,; \qquad r_{t_0} = \chi_0\,, \qquad r(t) = r_0(t) \qquad \text{for } t \text{ on } \partial H_0$$

such that $r_0(t_0) = \chi_0(0)$ and

$$\varrho'(t) = g_2(t, \varrho, \varrho_t)\,, \qquad \varrho_{t_0} = \psi_0\,, \qquad \varrho(t) = \varrho_0(t) \qquad \text{for } t \text{ on } \partial H_0$$

such that $\varrho_0(t_0) = \psi_0(0)$ respectively existing on $[t_0, \infty)$ such that

$$\psi_0 \leqslant \varphi_0 \leqslant \chi_0\,,$$

$$\varrho_0(t) \leqslant u_0(t, x) \leqslant r_0(t) \qquad \text{on } \partial H_0$$

where $u(t, x)$ is any solution of (4.1)-(4.2), then

$$r(t) \leqslant u(t, x) \leqslant \varrho(t) \qquad \text{on } \overline{H}\,.$$

PROOF. Setting $m(t, x) = u(t, x) - r(t)$, we see that m satisfies

$$\frac{\partial m}{\partial t} = F(t, x, m, m_x, m_{xx}, m_t(\cdot, x)) , \tag{4.3}$$

$$\begin{cases} m_{t_0}(\cdot, x) = \varphi_0(\cdot, x) - \chi_0(\cdot) \quad \text{for } x \in \bar{\Omega} \\ m(t, x) = m_0(t, x) = u_0(t, x) - r_0(r) \\ \text{on } \partial H_0 \text{ and} \\ \dfrac{\partial m}{\partial \nu} = \dfrac{\partial u}{\partial \nu} = 0 \end{cases} \tag{4.4}$$

where

$$F(t, x, m, m_x, m_{xx}, m_t(\cdot, x)) = \\ = f(t, x, (m + r), u_x, u_{xx}, (m + r)_t) - g_1(t, r, r_t) .$$

We shall show that (4.3)-(4.4) satisfies the assumptions of Theorem 4.1. Let $m \leqslant 0$, $c(m) = 0$, $m_t(\cdot, x) \leqslant 0$, $c(m_{x_i}) = 0$, $i = 1, 2, \ldots, n$, and $\sum_{i,j=1}^{n} \lambda_i \lambda_j c(m_{x_i x_j}) \leqslant 0$, $\lambda \in R^n$ for some $c \in k_0^*$. This implies that $u(t, x) \leqslant r(t)$, $u_t(\cdot, x) \leqslant r_t$ and $c(u) = c(r)$ and consequently the quasi-monotonicity of g_1 in u and monotonicity of g_1 in φ yield $c(g_1(t, u, u_t)) \leqslant \leqslant c(g_1(t, r, r_t))$. It now follows from (ii) and

$$m_x = u_x , \quad m_{xx} = u_{xx} , \quad \text{that} \quad c(F(t, x, m, m_x, m_{xx}, m_t(\cdot, x)) \\ \leqslant c\Big(f(t, x, u, u_x, u_t(\cdot, x)) - g_1(t, u, u_t)\Big) \leqslant 0 ,$$

proving F is quasinonpositive.

We have

$$\begin{aligned} \frac{\varepsilon \partial z}{\partial t} &> f(t, x, u, u_x, u_{xx}, u_t(\cdot, x)) \\ &\quad - f(t, x, u - \varepsilon z, u_x - \varepsilon z_x, u_{xx} - \varepsilon z_{xx}, (u - \varepsilon z)_t(\cdot, x)) \\ &= f(t, x, m + r, m_x, m_{xx}, (m + r)_t(\cdot, x)) \\ &\quad - f(t, x, m + r - \varepsilon z, m_x - \varepsilon z_x, m_{xx} - \varepsilon z_{xx}, (m + r)_t - \varepsilon z_t) \\ &= F(t, x, m, m_x, m_{xx}, m_t(\cdot, x)) \\ &\qquad - F(t, x, m - \varepsilon z, m_x - \varepsilon z_x, m_{xx} - \varepsilon z_{xx}, m_t(\cdot, x) - \varepsilon z_t(\cdot, x)) . \end{aligned}$$

This proves F satisfies the condition (C_0) (a) with $v = m$. Thus, by Theorem 4.1 it follows that $m(t, x) \leqslant 0$ on $\bar{H}$ which proves $u(t, x) \leqslant r(t)$ on $\bar{H}$.

On similar lines we can show that $\varrho(t) \leqslant u(t, x)$ by setting $m = u - \varrho$. This proves the theorem.

COROLLARY 4.3. If $\bar{H}$ is flow invariant relative to the system (4.1)-(4.2), there exists functions g_1, g_2 satisfying the assumptions of Theorem 4.2 provided $E = R^n$ and $K = R^n_+$.

PROOF. We construct g_1, g_2 as follows: for each i, $1 \leqslant i \leqslant N$,

$$g_{1i}(t, u, \varphi) =$$
$$= \sup \{f_i(t, x, v, 0, 0, \psi);\ x \in \bar{\Omega},\ a_j \leqslant v_j \leqslant u_j,\ v_i = u_i,\ a_i \leqslant \psi_i(\cdot, x) \leqslant \varphi_i(s)\}$$

$$g_{2i}(t, u, \varphi) =$$
$$= \inf \{f_i(t, x, v, 0, 0, \psi);\ x \in \bar{\Omega},\ u_j \leqslant v_j \leqslant b_j,\ v_i = u_i \text{ and } \varphi_i(s) \leqslant \psi_i(\cdot, x) \leqslant b_i\}$$

and f is elliptic for each i.

Next we give a comparison theorem which is an extension of the classical result of Müller [6] which is valid when $E = R^n$ and $k = R^n_+$.

THEOREM 4.5. Assume that

(i) for each i, $1 \leqslant i \leqslant N$, v, $w \in Z$, $\partial v^i/\partial t \leqslant f_i(t, x, \sigma, v^i_x, v^i_{xx}, \varphi)$ for all σ and φ such that $v^j(t, x) \leqslant \sigma_j \leqslant w^j(t, x)$ $j \neq i$ and $\sigma_i = v_i(t, x)$, also $v_t(\cdot, x) \leqslant \varphi \leqslant w_t(\cdot, x)$, $\partial w^i/\partial t \geqslant f_i(t, x, \sigma, w^i_x, w^i_{xx}, \varphi)$ for all σ such that $v^j(t, x) \leqslant \sigma_j \leqslant w^j(t, x)$, $j \neq i$, and $\sigma_i = w^i(t, x)$, also $v_t(\cdot, x) \leqslant \varphi \leqslant w_t(\cdot, x)$;

(ii) $f_i(t, x, \bar{\sigma}, \bar{\sigma}^i_x, \bar{\sigma}^i_{xx}, \bar{\varphi}) - f_i(t, x, \sigma, \sigma^i_x, \sigma^i_{xx}, \varphi) \leqslant L_i(t, x, |\bar{\sigma}_1 - \sigma_1|, \ldots$ $\ldots, \bar{\sigma}_i - \sigma_i, \ldots |\bar{\sigma}_N - \sigma_N|, (\bar{\sigma}_i - \sigma_i)_x, (\bar{\sigma}_i - \sigma_i)_{xx}, |\bar{\sigma}_1 - \varphi_1|, \ldots, |\bar{\varphi}_N - \varphi_N|)$, whenever $\bar{\sigma}_i \geqslant \sigma_i$, where $L \in C[[t_0, \infty) \times \Omega \times R^N \times R^n \times R^{n^2} \times R^N, R^N]$, L_i is quasimonotone nondecreasing in u and monotone nondecreasing in φ and there exists a $z \in Z$ such that $z > 0$ on $[t_0, \infty) \times \Omega$, $z_{t_0} > 0$ for $x \in \bar{\Omega}$, $(\partial z/\partial \nu)(t, x) \geqslant \gamma > 0$ on ∂H_1, and for all sufficiently small $\varepsilon > 0$,

$$\frac{\varepsilon\, \partial z}{\partial t} > L_i(t, x, \varepsilon z, \varepsilon z^i_x, \varepsilon z^i_{xx}, \varepsilon z_t) \qquad \text{on } [t_0, \infty) \times \Omega\,;$$

(iii) $u(t, x)$ is any solution of (4.1)-(4.2) such that $v_{t_0} \leqslant u_{t_0} \leqslant w_{t_0}$, and $v \leqslant u_0 \leqslant w$ on ∂H_0, $\partial v/\partial \nu \leqslant \partial u/\partial \nu \leqslant \partial w/\partial \nu$ on ∂H_1, then $v(t, x) \leqslant$ $\leqslant u(t, x) \leqslant w(t, x)$ on $G \cup \bar{H}$.

PROOF. We shall first assume that v, w satisfy strict inequalities and prove the conclusion for strict inequalities. We let $m = u - w$ and $n = u - v$ on $[t_0, \infty)\times\bar{\Omega}$. We show $m < 0 < n$. If not, there would exist a (t_1, x_1) and a j such that either $m^i(t_1, x_1)\leqslant 0$, $m^j(t_1, x_1) = 0$, $m^j_{x_i}(t_1, x_1) = 0$, $i = 1, 2, \ldots, n$ and $\sum_{k,l=1}^{n} \lambda_k\lambda_l m^j_{x_k x_l}(t_1, x_1)\leqslant 0$, $\lambda\in R^n$ or $n_i(t_1, x_1)\leqslant 0$, $n_j(t_1, x_1) = 0$, $n^i_{x_j}(t_1, x_1) = 0$, $i = 1, 2, \ldots, n$ and

$$\sum_{k,l=1}^{n} \lambda_k\lambda_l n^j_{x_k x_l}(t_1, x_1)\leqslant 0\,, \qquad \lambda\in R^n\,.$$

Suppose the first alternative holds. Certainly $t_1 > t_0$ by (iii). Hence, at (t_1, x_1) we have $v\leqslant u\leqslant w$, $v_{t_1}\leqslant u_{t_1}\leqslant w_{t_1}$, $u_j(t_1, x_1) = w_j(t_1, x_1)$, $u^j_{x_i}(t_1, x_1) = w^j_{x_i}(t_1, x_1)$ and $\sum_{k,l=1}^{n} \lambda_k\lambda_l(u^i_{x_k x_l} - w^j_{x_k x_l})\leqslant 0$. Hence,

$$\frac{\partial m^j}{\partial t}(t_1, x_1) = \frac{\partial u^j}{\partial t}(t_1, x_1) - \frac{\partial w^j}{\partial t}(t_1, x_1)$$
$$< f_j\big(t_1, x_1, u, u^j_x, u^j_{xx}, u_t(\cdot, x)\big) - f_j\big(t_1, x_1, u, u^j_x, u^j_{xx}, u_t(\cdot, x)\big) = 0$$

but

$$\lim_{h\to 0}\frac{m^j(t_1, x_1) - m^j(t_1 - h_1, x_1)}{h}\geqslant 0\,,$$

which leads to a contradiction. A similar proof holds when the second alternative is true. Thus, we get $m(t, x) < 0 < n(t, x)$ on $G\cup\bar{H}$ and this proves the claim for strict inequalities.

Consider now $\tilde{w} = w + \varepsilon z$, $\tilde{v} = v - \varepsilon z$ on $G\cup\bar{H}$. Let $P^i(t, x, \tilde{\sigma}) = \max\,[v^i(t, x), \min\{\tilde{\sigma}^i, w^i_t(\cdot, x)\}]$. Then it is clear that if $\tilde{\sigma}$ and $\tilde{\varphi}$ is such that replace by $\tilde{v}\leqslant\tilde{\sigma}\leqslant\tilde{w}$, $[\tilde{v}_t\leqslant\tilde{\varphi}\leqslant\tilde{w}_t$ and $\tilde{\sigma}_i = \tilde{w}_i$, it follows that $\sigma = P(t, x, \tilde{\sigma})$, $\varphi = P_t(\cdot, x, \tilde{\varphi})$ satisfy $v\leqslant\sigma\leqslant w$, $v_t\leqslant\varphi\leqslant w_t$ and $\sigma_i = w_i$. Hence, using (i) and (ii) we get

$$\frac{\partial\tilde{w}^i}{\partial t} = \frac{\partial w^i}{\partial t} + \varepsilon\frac{\partial z^i}{\partial t}\geqslant f_i\,(t, x, \sigma, w^i_x, w^i_{xx}, \varphi) + \varepsilon\frac{\partial z^i}{\partial t}$$
$$\geqslant f_i(t, x, \tilde{\sigma}, \tilde{w}^i_x, \tilde{w}^i_{xx}, \tilde{\varphi}) + \varepsilon\frac{\partial z^i}{\partial t}$$
$$- L_i(t, x, |\tilde{\sigma}_1 - \sigma|, \ldots, \varepsilon z^i, \ldots, |\tilde{\sigma}_N - \sigma_N|, \varepsilon z^i_x, \varepsilon z^i_{xx}, |\tilde{\varphi}_1 - \varphi_1|, \ldots, |\tilde{\varphi}_N - \varphi_N|)$$
$$> f_i(t, x, \tilde{\sigma}, \tilde{w}^i_x, \tilde{w}^i_{xx}, \tilde{\varphi})\,.$$

Here we have used that L_i is quasimonotone nondecreasing in u and monotone decreasing in φ and $|\tilde{\sigma}_j - \sigma_j| \leqslant \varepsilon z^i$, $|\tilde{\varphi}_j - \varphi_j| \leqslant \varepsilon z_t^j$ for all j. Since $v_{t_0} < u_{t_0} < \tilde{w}_{t_0}$, also $\tilde{v} < u < \tilde{w}$ on ∂H_0, and $\partial \tilde{v}/\partial \nu < \partial u/\partial \nu < \partial \tilde{w}/\partial \nu$ on ∂H_1. We get immediately $v(t, x) - \varepsilon z(t, x) < u(t, x) < w(t, x) + \varepsilon z(t, x)$ on $G \cup \bar{H}$ for arbitrary $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain the stated result, completing the proof.

COROLLARY 4.4. Let the assumptions (i), (ii) of Theorem 4.2 hold without g_1, g_2 being quasimonotone nondecreasing in u and without being monotone decreasing in φ. Suppose that the conditions (ii), (iii) of Theorem 4.5 are satisfied. Assume further that for each i,

$$\frac{dw^i}{dt}(t) \geqslant g_{1i}(t, \sigma, \varphi) \quad \text{for all } v \leqslant \sigma < w, \quad \text{and} \quad \sigma_i = w_i, \quad v_t \leqslant \varphi \leqslant w_t,$$

and

$$\frac{dv^i}{dt}(t) \leqslant g_{2i}(t, \sigma, \varphi) \quad \text{for all } v \leqslant \sigma \leqslant w, \quad \text{and} \quad \sigma_i = v_i, \quad v_t \leqslant \varphi \leqslant w_t.$$

Then $v(t) \leqslant u(t, x) \leqslant w(t)$ on $G \cup \bar{H}$. Note that the functions v, w do not depend on the space variable x in the foregoing corollary.

5. Consider the reaction-diffusion equation of the special form

$$\frac{\partial u}{\partial t} = A \Delta u + F(t, u, u_t) \tag{5.1}$$

in $R_+ \times \Omega$ with the initial function

$$u_0(\cdot, x) = \varphi_0(s, x) \quad \text{on } \bar{\Omega} \text{ for } s \in [-\tau, 0] \tag{5.2}$$

and the Neumann boundary condition

$$\frac{\partial u}{\partial \nu}(t, x) = 0 \quad \text{on } (0, \infty) \times \Omega. \tag{5.3}$$

In (5.1) Δ denotes the Laplace operator in $x \in R^n$, u, F, $u_0(\cdot, x) \in R^N$ and A is a diagonal matrix.

Consider the standard cone in R^N, namely

$$k = R_+^N = \{u \in R^N, u_i \geq 0, i = 1, 2, \ldots, N\}.$$

Clearly the set

$$S = [c \in k^*, c(u) = u_i, i = 1, 2, \ldots, N] \subset k^*$$

generates the cone k. Let us note that weak coupling of the system (5.1) suggests the choice of this special cone. Thus, the inequality $u \leq v$ implies the componentwise inequalities $u_i \leq v_i$, $i = 1, 2, \ldots, N$.

THEOREM 5.1. Assume that

(i) $A \geq 0$ and $u(t, x)$ is any solution of (5.1) to (5.3) existing on $R_+ \times \bar{\Omega}$,

(ii) $F(t, x, x_t)$ satisfies a Lipschitz condition for a constant $L \geq 0$;

(iii) The boundary $\partial\Omega$ is regular, i.e., there exists $h \in Z$ such that $h(x) \geq 0$ on $\bar{\Omega}$, $(\partial h/\partial \nu)(x) \geq \gamma \geq 0$ on $\partial\Omega$ and h_x, h_{xx} are bounded.

Then the following conclusions are valid.

(*a*) If $u^i = 0$, $u^j \geq 0$, $j \neq i$, $j = 1, 2, \ldots, N$, also $u_t^j(\cdot, x) \geq 0$, $j = 1, 2, \ldots, N$ implies $F_i(t, u, u_t) \geq 0$, then $u(t, x) \geq 0$ on $R_+ \times \bar{\Omega}$ provided $u_0(x) \geq 0$ on $\bar{\Omega}$.

(*b*) If $F(t, u, \varphi)$ is quasimonotone nondecreasing in u and monotone nondecreasing in φ relative to R_+^N, that is for each i, $1 \leq i \leq N$, $F_i(t, u, u_t)$ is nondecreasing in u_j, $j \neq i$, and nondecreasing in $u_t^j(\cdot, x)$ and if the solutions $r(t)$, $\varrho(t)$ of $y' = F(t, y, y_t)$ with $r_0(\cdot) = \tilde{\varphi}_0(s)$ $\varrho_0(\cdot) = \varphi_0(s)$ exist on R_+, then

$$\varrho(t) \leq u(t, x) \leq r(t) \quad \text{on } R_+ \times \bar{\Omega} \tag{5.4}$$

provided that $\varphi_0(s) \leq u_0(\cdot, x) \leq \tilde{\varphi}_0(s)$ on $\bar{\Omega}$.

(*c*) If $F(t, u, \varphi)$ is quasimonotone nondecreasing in u and monotone decreasing in φ, $F(t, 0, 0) \equiv 0$, then

$$0 \leq u_0(\cdot, x) \quad \text{on } [-\tau, 0] \times \bar{\Omega}$$

implies that $u(t, x) \geqslant 0$ on $R_+ \times \bar{\Omega}$,

$$0 < u_0(\cdot, x) \quad \text{on } [-\tau, 0] \times \bar{\Omega}$$

implies that $u(t, x) > 0$ on $R_+ \times \bar{\Omega}$ and

$$0 \leqslant u_0(\cdot, x) \leqslant \bar{\varphi}_0(s) \quad \text{on } [-\tau, 0] \times \bar{\Omega}$$

implies that $0 \leqslant u(t, x) \leqslant r(t)$ on $R_+ \times \bar{\Omega}$, where $r(t)$ is the same function assumed in (b).

(d) If $F(t, u, u_t)$ is not quasimonotone in u, nor monotone in φ and if the closed set $\overline{W} = [u \in R^N; a \leqslant u \leqslant b]$ is flow invariant relative to (5.1) to (5.3), then the estimate (5.4) holds where $r(t)$, $\varrho(t)$ are now being solutions of

$$\frac{dr}{dt} = g_1(t, r, r_t), \quad r_0(\cdot) = \bar{\varphi}_0, \quad \varrho' = g_2(t, \varrho, \varrho_t), \quad \varrho_0(\cdot) = \varphi_0$$

where

$$g_{1i}(t, u, u_t) = \max [F_i(t, v, \varphi); a \leqslant v \leqslant u, v_i = u_i, a \leqslant \varphi \leqslant u_t],$$

$$g_{2i}(t, u, u_t) = \min [F_i(t, v, \varphi); u \leqslant v \leqslant b, v_i = u_i, u_t \leqslant \varphi \leqslant b],$$

$1 \leqslant i \leqslant N$.

(e) If $F(t, u, \varphi)$ is not quasimonotone in u nor monotone in φ and $\overline{W}$ is not known to be flow invariant, then (5.4) holds and if $r(t)$, $\varrho(t)$ satisfy the relations

$$r'_i \geqslant F_i(t, \sigma, \varphi) \text{ for all } \sigma \text{ such that } \varrho \leqslant \sigma \leqslant r, \ \sigma_i = r_i, \ \varrho_t \leqslant \varphi \leqslant r_t$$

$$\varrho'_i \leqslant F_i(t, \sigma, \varphi) \text{ for all } \sigma \text{ such that } \varrho \leqslant \sigma \leqslant r \text{ and } \sigma_i = r_i, \ \varrho_t \leqslant \varphi \leqslant r_t,$$

for $1 \leqslant i \leqslant N$.

Proof. The conclusion (a) follows from Corollary 4.1. Theorem 4.2 yields (b) with the choice $F = g_1 = g_2$. Uniqueness of solutions of $y' = F(t, y, y_t)$ together with the fact $F(t, 0, 0) \equiv 0$ implies (c). Corollary (4.3) gives the conclusion (d) whereas (e) follows from Corollary 4.4.

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