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On the behaviour of the surfaces of equilibrium in the capillary tubes when gravity goes to zero

On the Behaviour of the Surfaces of Equilibrium in the Capillary Tubes when Gravity Goes to Zero.

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Introduction.

In a previous work I have studied by a variational method the problem of the surfaces of equilibrium of a fluid in a capillary tube considering also the gravity force [1].

The functional of the energy is the following:

$$(0.1) \qquad \mathcal{F}(f) = \int_{\Omega} \sqrt{1 + |Df|^2} + \frac{1}{2} \frac{\varrho g}{\sigma} \int_{\Omega} f^2 \, dx - \nu \int_{\partial \Omega} f d \, H_{n-1}$$

where Ω is an open and bounded set in R^n, and

$$(0.2) \qquad k = \frac{\varrho g}{\sigma}$$

is the capillary constant, with $\varrho =$ density difference across the free surface, $g =$ gravitational acceleration, $\sigma =$ surface tension; ν is also a constant depending only on physical conditions. The first integral in (0.1) represents the surface tension, the second the gravity and the third the force of adhesion to the wall of the tube.

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Moreover $f(x)$ is the free surface of the fluid in the tube whose section is Ω.

In [1] I have proved a theorem of existence and unicity for the minimum of the functional $\mathcal{F}(f)$ in the class of $BV(\Omega)$ functions with the following hypotheses:

    i) $\partial\Omega$ lipschitz;

    ii) $|\nu| < 1/\sqrt{1 + L^2}$

where L is the lipschitz constant of $\partial\Omega$.

I have also proved a regularity theorem for the solution of the problem.

I recall these results in the following chapter.

In this work I will consider the behaviour of the surfaces of equilibrium in a capillary tube when the gravity goes to zero.

For each $\varepsilon > 0$ let f_ε minimize the functional

$$(0.3) \qquad \mathcal{F}_\varepsilon(f) = \int\limits_\Omega \sqrt{1 + |Df|^2} - \nu \int\limits_{\partial\Omega} f \, dH_{n-1} + \varepsilon \int\limits_\Omega f^2 \, dx \, .$$

REMARK 0.1. Obviously we have

$$(0.4) \qquad \lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(f) = \mathcal{F}_0(f)$$

where

$$(0.5) \qquad \mathcal{F}_0(f) = \int\limits_\Omega \sqrt{1 + |Df|^2} - \nu \int\limits_{\partial\Omega} f(x) \, dH_{n-1} \, .$$

It easily follows

$$(0.6) \qquad \inf \{\mathcal{F}_0(f); \, f \in BV(\Omega)\} = -\infty \, .$$

Hence it has no meaning to look for a function minimizing the functional $\mathcal{F}_0(f)$ without any other condition. Many authors have studied by different methods this problem for the functional $\mathcal{F}_0(f)$ with a volume constraint, that is with the condition

$$(0.7) \qquad \int\limits_\Omega f(x) \, dx = V \, .$$

(See [2], [3], [4], [5], [6], [7], [8], [9], [10], [11].)

In this work instead of considering a volume constraint I shall examine the behaviour of the free surface when the gravity goes to zero.

The intuitive fact that the surfaces f_ε rise in the tube as the gravity decreases, is first confirmed by the following remark:

$$(0.8) \qquad \int_\Omega f_\varepsilon(x)\, dx = \frac{1}{2\varepsilon}\, \nu H_{n-1}(\partial\Omega)\,.$$

So it obviously follows that

$$(0.9) \qquad \lim_{\varepsilon\to 0^+} \int_\Omega f_\varepsilon(x)\, dx = +\infty$$

that is the volume of the fluid continues to increase.

The proof of the previous well known remark is recalled in the following chapter.

To prove the monotonicity of the family f_ε I will use a strong maximum principle for the minimal surfaces operator, analogous to the results of P. Concus and R. Finn [12], and a comparison theorem between the solutions of the minimum problem for the functional $\mathcal{F}_\varepsilon(f)$ with two different gravities.

Finally I can prove that

$$(0.10) \qquad \lim_{\varepsilon\to 0^+} f_\varepsilon(x) = +\infty \qquad \forall x \in \overline{\Omega}\,.$$

that is the liquid rises at every point of Ω; in other words I can say that when the gravity goes to zero we do not obtain a limit surface.

To prove (0.10) I use a technique introduced by E. Gonzalez, U. Massari and I. Tamanini [13] in the study of sets minimizing perimeter and containing an assigned volume.

In my problem I do not consider a volume constraint but a mean curvature term so the technique is simplified.

I prove the result in various steps; first I suppose that (0.11) is not satisfied, that is it exists a number $L \in R$ such that

$$(0.11) \qquad \inf_{x\in\overline{\Omega}} f_\varepsilon(x) \leqslant L\,, \qquad \forall \varepsilon > 0\,.$$

Then I prove Lemma 3.1 which states

$$(0.12) \qquad\qquad \lim_{\varepsilon \to 0^+} v_\varepsilon = 0$$

where

$$(0.13) \qquad\qquad v_\varepsilon = H_{n+1}\big(F_\varepsilon \cap (-\infty, L+1)\big)$$

and

$$(0.14) \qquad\qquad F_\varepsilon = \big\{(x, f_\varepsilon(x), x \in \bar{\Omega}, x > f_\varepsilon(x))\big\} \,.$$

Then by defining

$$(0.15) \qquad\qquad a_0 = L + \tfrac{1}{2} \,, \qquad b_0 = L + 1$$

it follows that for a sufficiently small $\varepsilon > 0$ there exists a point $\tau^* \in$ $\in [a_0, b_0]$ such that (Proposition 3.2)

$$(0.16) \qquad\qquad \int_{\tau=\tau^*} \varphi_{F_\varepsilon} \, dH_n = 0 \,.$$

To prove Proposition 3.2 I first prove a technical Lemma.

I shall study in a following paper the problem of the behaviour of the surfaces of equilibrium in a capillary tube when the gravity goes to zero together with a volume constraint.

I want to thank M. Miranda and E. Gonzalez for the useful talks on the subject.

1. Let

    i) Ω be an open and bounded set in R^n $(n > 2)$ and $\partial\Omega$ be his boundary;

    ii) $\partial\Omega$ be locally lipschitz, i.e. for every $x \in \partial\Omega$ there exists a sphere $B_\varrho(x)$ such that $\partial\Omega \cap B_\varrho(x)$ is the graph of a function $u \colon R^{n-1} \supset$ $\supset A \to R$ where A is an open set and u a lipschitz function.

    iii) L be the maximum of the lipschitz constants for the functions u.

The functional will be considered in the natural space for variational problems of this kind:

$BV(\Omega)$ i.e. the space of functions $f \in L^1(\Omega)$ whose derivatives $D_1 f, \ldots, D_n f$ in the sense of distributions are finite Radon measures on Ω. For functions $f \in BV(\Omega)$ it is possible by a result of M. Miranda [14] to define the trace on $\partial\Omega$ and $f|_{\partial\Omega}$ is a function in $L^1(\Omega)$.

The symbol $\int_{\Omega} \sqrt{1 + |Df|^2}$ will indicate the total variation of the vector measures whose components are the Lebesgue measure and the measures $D_1 f, \ldots, D_n f$, i.e.

$$(1.1) \qquad \int_{\Omega} \sqrt{1 + |Df|^2} = \sup \left\{ \int_{\mathbf{R}^n} \left(f(x) \sum_{i=1}^{n} D_i g_i(x) + g_{n+1}(x) \right); \right.$$
$$\left. g \in C_0^1(\Omega); \ |g| \leqslant 1 \right\}.$$

Let us suppose the tube be made of an omogeneous material; this fact implies that ν is a constant. Physically ν represents the cosine of the angle between the exterior normal to the walls of the capillary tube and the normal to the free surface of the liquid [15].

Then it is clear that $\nu \in [-1, 1]$.

Let us consider the case $\nu > 0$ and therefore $f(x) > 0$. For $\varepsilon \in R$, $\varepsilon > 0$ let us consider the new functional

$$(1.2) \qquad \mathcal{F}_{\varepsilon}(f) = \int_{\Omega} \sqrt{1 + |Df|^2} - \nu \int_{\partial\Omega} f \, dH_{n-1} + \varepsilon \int_{\Omega} f^2 \, dx$$

where the last integral represents the gravitational potential energy.

In [1] the author has demonstrated, using the following estimate,

$$(1.3) \qquad \int_{\partial\Omega} |f| \, dH_{n-1} \leqslant \sqrt{1 + L^2} \int_{\Omega} |Df| + c(\Omega) \int_{\Omega} |f| \, dx$$

where $c(\Omega)$ is a constant depending on the geometry of Ω, the following result:

THEOREM [1]. i) If $0 < \nu < 1/\sqrt{1 + L^2}$ then

$$(1.4) \qquad \inf_{BV(\Omega)} \mathcal{F}_{\varepsilon}(f) > -\infty .$$

ii) If $0 < \nu < 1/\sqrt{1 + L^2}$, then every minimizing sequence is compact. Moreover the functional $\mathcal{F}_\varepsilon(f)$ is lower semicontinuous.

Hence for every $\varepsilon > 0$ there exists a solution for the problem

$$(1.5) \qquad\qquad \inf_{BV(\Omega)} \mathcal{F}_\varepsilon(f) = \mathcal{F}_\varepsilon(f_\varepsilon) .$$

In the same work [1] it was also demonstrated the uniqueness and regularity of the solution using the regularity methods of U. Massari [16]. L. Pepe [17] has then proved that the solution is analytic in Ω in every dimension (see also [18]).

For other conditions on the domain for the existence of solutions see also I. Tamanini [19], R. Finn-C. Gerhardt [20].

For a discussion of the physical meaning see P. Concus - R. Finn [12]. For an history of capillarity phenomena see [15], [21].

2. Let us now examine the properties of the family $\{f_\varepsilon\}$.

First of all, as I have already said in the Introduction, I notice that

$$(2.1) \qquad\qquad \lim_{\varepsilon \to 0^+} \int_\Omega f_\varepsilon(x)\, dx = +\infty .$$

In fact from the Euler equation of f_ε we obtain

$$(2.2) \qquad\qquad \int_\Omega f_\varepsilon(x)\, dx = \frac{\nu}{2\varepsilon}\, H_{n-1}(\partial\Omega)$$

and (2.1) follows.

REMARK 2.1. The integral (2.2) represents the volume of the liquid which rises in the capillary tube.

I will now prove a strong maximum principle for the minimal surfaces operator; I will use this principle to demonstrate the monotonicity of the sequence $\{f_\varepsilon\}$.

PROP. 2.1. Let M be the minimal surfaces operator. If we have

$$(2.3) \qquad\qquad Mf \geqslant Mg \qquad \text{in } \Omega ,$$

$$(2.4) \qquad\qquad \frac{\nu\, Df}{\sqrt{1 + |Df|^2}} = \frac{\nu\, Dg}{\sqrt{1 + |Dg|^2}} \qquad \text{in } \partial\Omega$$

then we necessarily obtain either

(2.5) $$f(x) \equiv g(x) + c \qquad c > 0 \text{ in } \Omega$$

or

(2.6) $$f(x) \leqslant g(x) \qquad \text{in } \Omega .$$

PROOF. From (2.3) and (2.4) we have, for every function $\phi \geqslant 0$, $\phi \in C_0^1(\Omega)$

(2.7) $$\int_\Omega \frac{Df \, D\psi}{\sqrt{1 + |Df|^2}} \, dx \leqslant \int_\Omega \frac{Dg \, D\psi}{\sqrt{1 + |Dg|^2}} \, dx$$

and then

(2.8) $$\int_\Omega \left(\frac{Df}{\sqrt{1 + |Df|^2}} - \frac{Dg}{\sqrt{1 + |Dg|^2}} \right) D\psi \, dx \leqslant 0 .$$

Let us choose now $\phi = (f - g) \vee 0$. We consider the case $f(x) > > g(x)$; then we have

(2.9) $$\int_\Omega \left(\frac{Df}{\sqrt{1 + |Df|^2}} - \frac{Dg}{\sqrt{1 + |Dg|^2}} \right) D(f - g) \, dx \leqslant 0 .$$

and, by the convexity of the area functional, we obtain

(2.10) $$Df = Dg$$

and then

(2.11) $$f(x) = g(x) + c , \qquad c > 0$$

the connected components of Ω. Moreover

(2.12) $$f(x) \equiv g(x) + c \qquad c > 0 \text{ in } \Omega$$

as the set where (2.11) is valid must be simultaneously closed and open.

Let us now prove that when the gravity decreases the free surfaces of the fluid rise inside the tube, that is that the family of functions $\{f_\varepsilon\}$ is monotone.

PROPOSITION 2.2. Let f_1 be the solution of the problem $\inf \mathcal{F}_{\varepsilon_1}(f)$ and f_2 be the solution of the problem $\inf \mathcal{F}_{\varepsilon_2}(f)$. Let $\varepsilon_1, \varepsilon_2 \in R$ be such that

(2.8) $$\varepsilon_1 > \varepsilon_2 > 0$$

then

(2.9) $$f_1(x) < f_2(x) \qquad \text{in } \Omega .$$

DIM. We know from Proposition 2.1 that we can have

(2.10) $$\begin{cases} f_1(x) = f_2(x) + c, & c > 0 \text{ in } \Omega , \\ Df_1(x) = Df_2(x) & \text{in } \Omega . \end{cases}$$

Now, because $f_1(x)$ minimizes the functional $\mathcal{F}_{\varepsilon_1}(f)$, we obtain

(2.11) $$\int_\Omega \sqrt{1 + |Df_1|^2}\, dx + \varepsilon_1 \int_\Omega f_1^2\, dx - \nu \int_{\partial\Omega} f\, dH_{n-1} \leqslant$$
$$\leqslant \int_\Omega \sqrt{1 + |Df_2|^2}\, dx + \varepsilon_1 \int_\Omega f_2^2\, dx - \nu \int_{\partial\Omega} f\, dH_{n-1} .$$

In the same way we have

(2.12) $$\int_\Omega \sqrt{1 + |Df_2|^2\, dx} + \varepsilon_2 \int_\Omega f_2\, dx - \nu \int_{\partial\Omega} f\, dH_{n-1} \leqslant$$
$$\leqslant \int_\Omega \sqrt{1 + |Df_1|^2\, dx} + \varepsilon_2 \int_\Omega f_1\, dx - \nu \int_{\partial\Omega} f_1\, dH_{n-1} .$$

From (2.10), (2.11) we obtain

(2.13) $$\varepsilon_1 \int_\Omega (f_2 + c)^2\, dx - \nu \int_{\partial\Omega} (f_2 + c)\, dH_{n-1} \leqslant \varepsilon_1 \int_\Omega f_2^2\, dx - \nu \int_{\partial\Omega} f_2\, dH_{n-1}$$

and from (2.10), (2.12) it follows

(2.14) $$\varepsilon_2 \int_\Omega f_2^2\, dx - \nu \int_{\partial\Omega} f_2\, dH_{n-1} \leqslant \varepsilon_2 \int_\Omega (f_2 + c)^2\, dx - \nu \int_{\partial\Omega} (f_2 + c)\, dH_{n-1} .$$

Then by (2.13) we obtain

$$(2.15) \qquad 2\varepsilon_1 C \int_\Omega f_2 \, dx + \varepsilon_1 C^2 \operatorname{mis} \Omega - \nu C H_{n-1}(\partial\Omega) \leqslant 0$$

and by (2.14)

$$(2.16) \qquad 2\varepsilon_2 C \int_\Omega f_2 \, dx + \varepsilon_2 C^2 \operatorname{mis} \Omega - \nu C H_{n-1}(\partial\Omega) \geqslant 0 \,.$$

Then, as $c > 0$, $\varepsilon_1 > \varepsilon_2 > 0$ we have a contradiction. So the only possibility is the following:

$$(2.17) \qquad f_2(x) \geqslant f_1(x) \qquad \text{in } \Omega \,.$$

So we have demonstrated that

$$(2.18) \qquad f_\varepsilon(x) \text{ is increasing as } \varepsilon \to 0^+$$

together with

$$(2.19) \qquad \int_\Omega f_\varepsilon(x) \, dx \to +\infty \qquad \text{as } \varepsilon \to 0^+ \,.$$

3. I can now prove that not only the volume of the liquid rises but also that the free surface of the liquid rises at every point of Ω.

As I have already said in the Introduction, in the proof I will use a technique introduced by E. Gonzalez, U. Massari and I. Tamanini [13].

The main result is the following:

THEOREM 3.1. In the previous hypothesis we obtain the following behaviour for the free surface

$$(3.1) \qquad \lim_{\varepsilon \to 0^+} f_\varepsilon(x) = +\infty \,, \qquad \forall x \in \overline{\Omega} \,.$$

We shall prove the theorem by several steps.

Let us suppose that there exists a number $L \in R$ such that

$$(3.2) \qquad \inf_{x \in \overline{\Omega}} f_\varepsilon(x) \leqslant L \,, \qquad \forall \varepsilon > 0 \,.$$

Let us define

$$(3.3) \qquad \begin{cases} F_\varepsilon = \{(x, f_\varepsilon(x), x \geqslant f_\varepsilon(x), x \in \bar{\Omega}\} \,, \\ E_\varepsilon = \{(x, f_\varepsilon(x), x \leqslant f_\varepsilon(x), x \in \bar{\Omega}\} \,. \end{cases}$$

The first step is to prove a Lemma which gives an estimate about the overgraph of f_ε when ε goes to zero. More precisely

LEMMA 3.1. Let

$$(3.4) \qquad v_\varepsilon = H_{n+1}\big(F_\varepsilon \cap (-\infty, L+1)\big)$$

then

$$(3.5) \qquad \lim_{\varepsilon \to 0^+} v_\varepsilon = 0 \,.$$

DIM.

OBS. 3.1. It is obvious by the hypothesis (3.2) that

$$(3.6) \qquad v_\varepsilon > 0 \,.$$

Let us define

$$(3.7) \qquad \tilde{f}_\varepsilon(x) = \begin{cases} f_\varepsilon(x) & \text{if } f_\varepsilon(x) > L+1 \,, \\ L+1 & \text{if } f_\varepsilon(x) \leqslant L+1 \,. \end{cases}$$

As $f_\varepsilon(x)$ minimizes the functional $\mathcal{F}_\varepsilon(f)$ we obtain

$$(3.8) \qquad \int_\Omega \int_0^{L+1} |D\varphi_{E_\varepsilon}| - \nu \int_{\partial\Omega} \int_0^{L+1} \varphi_{E_\varepsilon} \, dH_n \leqslant \int \varphi_{F_\varepsilon} \, dH_n \Big|_{t=L+1} -$$

$$- \nu\big(|\partial\Omega|(L+1)\big) + \varepsilon \int_\Omega \int_0^{L+1} \varphi_{F_\varepsilon}(x, t) \, t \, dx \, dt$$

where $|\partial\Omega| = H_{n-1}(\partial\Omega)$, $\varphi_A(x)$ is the characteristic function of a set A, $\int_A |D\varphi_E|$ is the perimeter of the set E in the open set A, $P(E, A)$ [22].

Now as

$$(3.9) \qquad \nu\Big(|\partial\Omega|(L+1) - \int_{\partial\Omega} \int_0^{L+1} \varphi_{E_\varepsilon} \, dH_n\Big) = \nu \int_{\partial\Omega} \int_0^{L+1} \varphi_{E_\varepsilon} \, dH_n$$

and also

(3.10)
$$\int_{\Omega}\int_{0}^{L+1}\varphi_{F_{\varepsilon}}(x,\,t)\,t\,dx\,dt\leqslant (L+1)\,v_{\varepsilon}$$

we have

(3.11)
$$\int_{\Omega}\int_{0}^{L+1}|D\varphi_{E_{\varepsilon}}|+v\int_{\partial\Omega}\int_{0}^{L+1}\varphi_{F_{\varepsilon}}\,dH_{n}\leqslant\int_{t=L+1}\varphi_{F_{\varepsilon}}\,dH_{n}+\varepsilon(L+1)\,v_{\varepsilon}\,.$$

Let now $B(R,\,c)$ be a sphere whose radius is R and the center is $(0,\,\dots,\,c)$, $c>0$ and such that

(3.12)
$$\begin{cases} H_{n}\big(B(R,\,c)\cap\{t=L+1\}\big)=\int\limits_{t=L+1}\varphi_{F_{\varepsilon}}(x)\,dH_{n}\,, \\ H_{n+1}\big(B(R,\,c)\cap\{t<L+1\}\big)=v_{\varepsilon}\,. \end{cases}$$

REMARK 3.2.

(3.13)
$$\int\limits_{t=L+1}\varphi_{F_{\varepsilon}}(x)\,dH_{n}\leqslant H_{n}(Q)\,.$$

By conditions (3.12) on the sphere we have

(3.14)
$$\int_{\Omega}\int_{0}^{L+1}|D\varphi_{E_{\varepsilon}}|+\int_{\partial\Omega}\int_{0}^{L+1}\varphi_{F_{\varepsilon}}\,dH_{n}=\int_{\bar{\Omega}\times(0,L+1)}|D\varphi_{F_{\varepsilon}}|\geqslant\int_{t<L+1}|D\varphi_{B(R,c)}|\,.$$

Moreover it is obvious that

(3.15)
$$\int_{\Omega\times(0,L+1)}|D\varphi_{E_{\varepsilon}}|\geqslant\int_{t=L+1}\varphi_{F_{\varepsilon}}\,dH_{n}$$

then, by (3.12)

(3.16)
$$\int_{\Omega\times(0,L+1)}|D\varphi_{E_{\varepsilon}}|\geqslant H_{n}\big(B(R,\,c)\cap\{t=L+1\}\big)\,.$$

Moreover by (3.11), (3.14) we obtain

$$(3.17) \quad \nu\int_{\partial\Omega}\int_0^{L+1}\varphi_{F_\varepsilon}\,dH_n + \nu\int_\Omega\int_0^{L+1}|D\varphi_{E_\varepsilon}| + (1-\nu)\int_\Omega\int_0^{L+1}|D\varphi_{E_\varepsilon}| \geqslant$$

$$\geqslant \nu\int_{t<L+1}|D\varphi_{B(R,c)}| + (1-\nu)\int_\Omega\int_0^{L+1}|D\varphi_{E_\varepsilon}| \;.$$

By (3.15) we also obtain, recalling (3.12),

$$(3.18) \quad \nu\int_{t<L+1}|D\varphi_{B(R,c)}| - H_n\big(B(R,\,c)\cap\{t=L+1\}\big) +$$

$$+ H_n\big(B(R,\,c)\cap\{t=L+1\}\big) \leqslant$$

$$\gtrless H_n\big(B(R,\,c)\cap\{t=L+1\}\big) + \varepsilon(L+1)v_\varepsilon \;.$$

Then finally

$$(3.19) \quad 0 \leqslant \frac{\displaystyle\int_{t<L+1}|D\varphi_{B(R,c)}| - H_n\big(B(R,\,c)\cap\{t=L+1\}\big)}{v_\varepsilon} \leqslant \varepsilon\,\frac{L+1}{\nu}\;.$$

In the end we obtain

$$(3.20) \quad \lim_{\varepsilon\to 0^+}\frac{\displaystyle\int_{t<L+1}|D\varphi_{B(R,c)}| - H_n\big(B(R,\,c)\cap\{t=L+1\}\big)}{v_\varepsilon} = 0 \;.$$

By the definition of v_ε (3.12) we have

$$(3.21) \qquad\qquad\qquad \lim_{\varepsilon\to 0^+}v_\varepsilon = 0 \;.$$

So Lemma 3.1 is proved.

We must now prove a technical Lemma. Let us first define

$$(3.22) \qquad\qquad a_0 = L+\tfrac{1}{2}\;, \qquad b_0 = L+1\;.$$

Let us consider three points $t_1,\,t_2,\,t_3$ s.t.

$$(3.23) \qquad\qquad a_0 < t_1 < t_2 < t_3 < b_0$$

and let us define

$$(3.24) \qquad \begin{cases} v_1 = |F_\varepsilon \cap (t_1, t_2)| \,, \\ v_2 = |F_\varepsilon \cap (t_2, t_3)| \,, \\ v = v_1 + v_2 \,, \end{cases}$$

where

$$(3.25) \qquad \begin{cases} (t_1, t_2) = \bar{\Omega} \times (t_1, t_2) \,, \\ (t_2, t_3) = \bar{\Omega} \times (t_2, t_3) \,. \end{cases}$$

Let

$$(3.26) \qquad m = \max_{i=1,2,3} \int_{t=t_i} \varphi_{F_\varepsilon} dH_n \,.$$

We now demonstrate the following:

LEMMA 3.2. There exists a constant c_1 such that

$$(3.27) \qquad v_1 \wedge v_2 < c_1 (m + b_0 v)^N$$

where $N = (n+1)/n$.

PROOF. Let

$$(3.28) \qquad \begin{cases} F_1 = F_\varepsilon \cap (t_1, t_2) \,, \\ F_2 = F_\varepsilon \cap (t_2, t_3) \,. \end{cases}$$

We now consider two spheres $B_1, B_2 \subset R^{n+1}$ s.t.

$$(3.29) \qquad |B_i| = |F_i| = v_i \,, \qquad i = 1, 2 \,.$$

By the isoperimetric property of the sphere we obtain

$$(3.30) \qquad \int_{R^{n+1}} |D\varphi_{B_i}| \leqslant \int_{R^{n+1}} |D\varphi_{F_i}| \,, \qquad i = 1, 2$$

i.e.

$$(3.31) \quad (n+1)\omega_{n+1}^{1/(n+1)} v_i^{n/(n+1)} \leqslant \int_{(t_i, t_{i+1})} |D\varphi_{F_\varepsilon}| + \int_{t=t_i} \varphi_{F_\varepsilon} dH_n + \int_{t=t_{i+1}} \varphi_{F_\varepsilon} dH_n \,, \quad i = 1, 2 \,.$$

By adding the two relationships (3.31) for $i = 1, 2$, we have

$$(3.32) \quad \int\limits_{(t_1, t_2)} |D\varphi_{F_\varepsilon}| \geq (n+1)\omega_{n+1}^{1/(n+1)}(v_1^{n/(n+1)} + v_2^{n/(n+1)}) -$$

$$- \int\limits_{t=t_1} \varphi_{F_\varepsilon}\, dH_n - \int\limits_{t=t_2} \varphi_{F_\varepsilon}\, dH_n - 2\int\limits_{t=t_2} \varphi_{F_\varepsilon}\, dH_n \, .$$

REMARK 3.3. The set F_ε is the complementary set of E_ε, then by the minimizing property of E_ε we obtain that F_ε realizes the minimum for the following functional:

$$(3.33) \quad \int\limits_{\Omega \times (0, +\infty)} |D\varphi_F| + \nu\int\limits_{\partial\Omega} \varphi_F\, dH_{n-1} - \varepsilon\int\limits_{\Omega \times (0, +\infty)} t\varphi_F\, dx\, dt$$

among all the overgraphs F such that $F \Delta F_\varepsilon \subset\subset \Omega \times (0, +\infty)$. Then by Remark 3.3 we obtain

$$(3.34) \quad \int\limits_{\Omega \times (t_1, t_2)} |D\varphi_{F_\varepsilon}| - \varepsilon\int\limits_{t=t_2} t\varphi_{F_\varepsilon}\, dx\, dt + \nu\int\limits_{\partial\Omega \times (t_1, t_2)} \varphi_{F_\varepsilon}\, dH_n \leq \int\limits_{t=t_1} \varphi_{F_\varepsilon}\, dH_n + \int\limits_{t=t_2} \varphi_{F_\varepsilon}\, dH_n$$

and then

$$(3.55) \quad \int\limits_{\Omega \times (t_1, t_2)} |D\varphi_{F_\varepsilon}| + \nu\int\limits_{\partial\Omega \times (t_1, t_2)} \varphi_{F_\varepsilon}\, dH_n \leq \int\limits_{t=t_1} \varphi_{F_\varepsilon}\, dH_n + \int\limits_{t=t_2} \varphi_{F_\varepsilon}\, dH_n + \varepsilon\int\limits_{(t_1, t_2)} t\varphi_{F_\varepsilon}\, dx\, dt \, .$$

Now by the fact that

$$(3.36) \quad \int\limits_{\Omega \times (t_1, t_2)} |D\varphi_{F_\varepsilon}| + \nu\int\limits_{\partial\Omega \times (t_1, t_2)} \varphi_{F_\varepsilon}\, dH_n = \int\limits_{\overline{\Omega} \times (t_1, t_2)} |D\varphi_{F_\varepsilon}|$$

we also have

$$(3.37) \quad \int\limits_{(t_1, t_2)} |D\varphi_{F_\varepsilon}| \leq \int\limits_{t=t_1} \varphi_{F_\varepsilon}\, dH_n + \int\limits_{t=t_2} \varphi_{F_\varepsilon}\, dH_n + \varepsilon\int\limits_{(t_1, t_2)} t\varphi_{F_\varepsilon}\, dx\, dt \, .$$

From (3.31), (3.37), and also considering that $0 < \nu < 1$, we

obtain

(3.38)
$$\int_{t=t_1}\varphi_{F_\varepsilon}\,dH_n+\int_{t=t_3}\varphi_{F_\varepsilon}\,dH_n+\varepsilon\int_{(t_1,t_3)}t\varphi_{F_\varepsilon}\,dx\,dt\geqslant$$

$$\geqslant\nu(n+1)\omega_{n+1}^{1/(n+1)}(v_1^{n/(n+1)}+v_2^{n/(n+1)})-\nu\int_{t=t_1}\varphi_{F_\varepsilon}\,dH_n-2\nu\int_{t=t_2}\varphi_{F_\varepsilon}\,dH_n-\nu\int_{t=t_3}\varphi_{F_\varepsilon}\,dH_n$$

and then

(3.39)
$$(1+\nu)\left\{\int_{t=t_1}\varphi_{F_\varepsilon}\,dH_n+\int_{t=t_3}\varphi_{F_\varepsilon}\,dH_n\right\}+$$

$$+2\nu\int_{t=t_2}\varphi_{F_\varepsilon}\,dH_n\geqslant\nu(n+1)\omega_{n+1}^{1/(n+1)}(v_1^{n/(n+1)}+v_2^{n/(n+1)})-\varepsilon\int_{(t_1,t_3)}t\varphi_{F_\varepsilon}\,dx\,dt\,.$$

Obviously we have

(3.40) $$(n+1)\omega_{n+1}^{1/(n+1)}(v_1^{n/(n+1)}+v_2^{n/(n+1)})\geqslant(n+1)\omega_{n+1}^{1/(n+1)}2(v_1\wedge v_2)^{n/(n+1)}\,.$$

Moreover $t<L+1$; then, by considering (3.29), (3.40) and the definition of v, we obtain

(3.41) $$2\nu(n+1)\omega_{n+1}^{1/(n+1)}(v_1\wedge v_2)^{n/(n+1)}\leqslant(L+1)v+(1+\nu)4m$$

and finally

(3.42) $$(v_1\wedge v_2)\leqslant c_1\big((L+1)v+m\big)^{n/(n+1)}$$

where

(3.43) $$c_1=c_1(n,\nu)\,.$$

So we have proved Lemma 3.3.

I can now prove Theorem 3.1; by the regularity of the functions f_ε it is sufficient to prove the following:

PROPOSITION 3.1. Let $\varepsilon>0$ be sufficiently small, then there exists $t^*\in[a_0,b_0]$ such that

(3.44) $$\int_{t=t^*}\varphi_{F_\varepsilon}\,dH_n=0\,.$$

PROOF. We shall construct the point t^* as the limit of two sequences $\{a_j\}$ and $\{b_j\}$, the first one increasing and the second one decreasing towards t^* and also such that

$$(3.45) \qquad \lim_{j \to +\infty} \int_{t=a_j} \varphi_{F_\varepsilon} dH_n = \lim_{j \to +\infty} \int_{t=b_j} \varphi_{F_\varepsilon} dH_n = 0 \,.$$

In order to construct the two sequences $\{a_j\}$ and $\{b_j\}$ we shall use an iterative method starting with $a_0 = L + \frac{1}{2}$, $b_0 = L + 1$.

Let us now suppose to have constructed a_j and b_j with $j \geqslant 0$ and $a_0 \leqslant a_j < b_j \leqslant b_0$; let us define

$$(3.46) \qquad \begin{cases} l_j = b_j - a_j \,, \\ v_j = |F_\varepsilon \cap (a_j, b_j)| \,. \end{cases}$$

Let now be $h_j \in (0, l_j/3)$; we can find three points t_i^j $(i = 1, 2, 3)$ s.t.

$$(3.47) \qquad \begin{cases} t_1^j \in (a_j, a_j + h_j) \,, \\ t_2^j \in \left(a_j + \dfrac{l_j}{2} - \dfrac{h_j}{2}, \; a_j + \dfrac{l_j}{2} + \dfrac{h_j}{2} \right), \\ t_3^j \in (b_j - h_j, b_j) \,, \end{cases}$$

and moreover

$$(3.48) \qquad \int_{t=t_i^j} \varphi_{F_\varepsilon} dH_n \leqslant \frac{v_j}{h_j} \qquad (i = 1, 2, 3) \,.$$

Let us define

$$(3.49) \qquad m_j = \max_{i=1,2,3} \int_{t=t_i^j} \varphi_{F_\varepsilon} dH_n \,,$$

$$(3.50) \qquad v_i^j = |F_\varepsilon \cap (t_i^j, t_{i+1}^j)| \qquad (i = 1, 2) \,.$$

Obviously from (3.48) we have

$$(3.51) \qquad m_j \leqslant \frac{v_j}{h_j} \,.$$

Let

$$(3\cdot52) \qquad\qquad v_{j+1} = v_1^j \wedge v_2^j \,.$$

By Lemma 3.2 we have

$$(3.53) \qquad\qquad v_{j+1} \leqslant C_1 (m_j + b_0\, v_j)^N \leqslant C_2 \left(\frac{v_j}{h_j}\right)^N$$

as $h_j < 1$. Let us now define the points $a_{j+1},\, b_{j+1}$

$$(3.54) \qquad\qquad (a_{j+1},\, b_{j+1}) = \begin{cases} (t_1^j,\, t_2^j) & \text{if } v_1^j \leqslant v_2^j, \\ (t_2^j,\, t_2^j) & \text{otherwise} \,. \end{cases}$$

Now we want to estimate the quantities (3.46), (3.49), (3.50). We have

$$(3.55) \qquad\qquad 2^j l_j \geqslant l_0 - 3 \sum_{i=0}^{j-1} 2^i h_i \,.$$

Moreover by (3.53) we obtain

$$(3.56) \qquad\qquad v_{j+1} \leqslant C_2^{\sum_{i=0}^{j} N^i} \frac{v_0^{N^{j+1}}}{\prod_{i=0}^{j} h_i^{N^{j+1-i}}}$$

then by (3.51), (3.56) we obtain

$$(3.57) \qquad\qquad m_{j+1} \leqslant \frac{\widetilde{C}\, v_0^{N^{j+1}}}{\prod_{i=0}^{j+1} h_i^{N^{j+1-i}}} = \left(\frac{\widetilde{\widetilde{C}}\, v_0}{\prod_{i=0}^{j+1} h_i^{N^{-i}}}\right)^{N^{j+1}} \,.$$

Let us now choose

$$(3.58) \qquad\qquad h_j = \frac{l_0}{k} \left(\left(\tfrac{1}{4}\right)^j\right) \,.$$

It remains now to fix k such that

$$(3.59) \qquad\qquad 0 < h_j < l_j/3 \,.$$

By the estimate (3.55) we have

$$(3.60) \qquad 3h_j \leqslant \frac{l_0 - 3\sum\limits_{i=0}^{j-1} 2^i h_i}{2^j}$$

then, remembering (3.58)

$$(3.61) \qquad k \geqslant 3 + 6\left(\frac{2^j - 1}{2^j}\right), \qquad \forall j$$

$$(3.62) \qquad k \geqslant 9 \ .$$

Let us put $k = 9$ and then

$$(3.63) \qquad h_i = \frac{l_0}{k}\left(\tfrac{1}{4}\right)^i \ .$$

We have now to prove that

$$(3.64) \qquad \lim_{j \to +\infty} m_j = 0 \ .$$

Now by (3.57), (3.63) we have

$$(3.65) \qquad m_{j+1} \leqslant \left(\frac{\tilde{C} v_0}{(l_0/9)^{n+1}} \cdot \frac{1}{(4)^{n(n+1)}}\right)^{N^{j+1}}$$

and then

$$(3.66) \qquad m_{j+1} \leqslant \left(\overline{C}\,\frac{v_0}{l_0^{n+1}}\right)^{N^{j+1}}$$

where

$$(3.67) \qquad v_0 = |F_\varepsilon \cap (a_0, b_0)| = |F_\varepsilon \cap (L + \tfrac{1}{2}, L + 1)|$$

then

$$(3.68) \qquad v_0 \leqslant v_\varepsilon \ .$$

Remembering the Lemma 3.1 we have

$$(3.69) \qquad \lim_{\varepsilon \to 0^+} v_0 = 0$$

and then

$$(3.70) \qquad\qquad \lim_{j \to +\infty} m_j = 0 \ .$$

So Theorem 3.1 is proved.

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