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A Subspace of Spec(A) and Its Connexions with the Maximal Ring of Quotients.

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Introduction.

The results exposed in this paper have originated from a notion which arises in a natural way in the study of the rings of real-valued continuous functions and which can be generalized to arbitrary rings: we mean to refer to ζ-ideals, which we define in 2.1 (see also 4.1); we use a notation which clearly recalls the familiar one of z-ideal [GJ], because of the likeness of the two concepts.

The second section and part of the fourth one are devoted to the description of the properties of ζ-ideals and to their relations with z-ideals: for instance we show that there exist some analogies (2.4), that in the semisimple rings the notion of ζ-ideal is finer (2.3) and we discuss some conditions under which the two concepts coincide.

Furthermore we prove that there is a connection between the ζ-ideals of a ring A and the prime spectrum of the maximal ring of quotients $Q(A)$: in fact if we consider the map $\eta\colon \mathrm{Max}\,(Q(A)) \to \mathrm{Spec}\,(A)$ so defined: $\eta(M) = M \cap A$, it turns out (3.3) that the subspace of the ζ-ideals coincides with the image of the map η; the same theorem provides other conditions equivalent to being a ζ-ideal. The third section ends with a study of the injectivity of the map η.

Finally, in the theorems 4.5, 4.5 *bis* we give some necessary and sufficient conditions in order that the universal regular ring is a ring of quotients: one of them is that every prime ideal is a ζ-ideal;

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moreover we show that this situation takes place in $C(X)$ if and only if X is a P-space.

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1. Preliminary results.

1.0. Throughout the present paper the word «ring» will always mean commutative semiprime (i.e. without non-zero nilpotents) ring with 1. Let A be a ring, $Q(A)$ the maximal ring of quotients of A, $K(A)$ the total ring of fractions, $\operatorname{Spec}(A)$, $\operatorname{Max}(A)$ and $\operatorname{Min}(A)$ respectively the prime, maximal and minimal spectra of A equipped with the Zariski topology. For every a belonging to A, we put: $\mathcal{V}(a) = \{P \in \operatorname{Spec}(A): a \in P\}$, $\mathcal{D}(a) = \operatorname{Spec}(A) \setminus \mathcal{V}(a)$, $V(a) = \mathcal{V}(a) \cap \cap \operatorname{Max}(A)$, $\mathcal{V}_0(a) = \operatorname{Min}(A) \cap \mathcal{V}(a)$ and analogously for $D(a)$ and $\mathcal{D}_0(a)$. If we replace a with any subset S of A, the previous symbols get the obvious significance; if Y is a subspace of $\operatorname{Spec}(A)$, we denote $\mathcal{V}_Y(S)$ the space $\mathcal{V}(S) \cap Y$, analogously for $\mathcal{D}_Y(S)$. Then we put $S^\perp = \{a \in A: sa = 0, \forall s \in S\}$ the annihilator of S in A.

For $a \in A$ we say that $a^{\dashv}$ is the quasi-inverse of a if it is the (unique) element of A which satisfies the system:

$$\begin{cases} x^2 \cdot a = x \\ a^2 \cdot x = a; \end{cases}$$

if every element of A has a quasi-inverse we say that the ring A is regular: this is equivalent to say $\operatorname{Max}(A) = \operatorname{Spec}(A)$.

We call singular an ideal whose elements are all zero-divisors.

We recall a characterization of the minimal prime ideals ([HJ]) which we shall need in the following: a prime ideal P is minimal if and only if the annihilator of every element of P is not contained in P.

1.1 Proposition. *Let $S \subseteq A$, Y a subspace of* $\operatorname{Spec}(A)$ *containing* $\operatorname{Min}(A)$. *Then* $\operatorname{int}_Y(\mathcal{V}_Y(S)) = \mathcal{D}_Y(S^\perp)$.

Proof. Clearly $\mathcal{V}_Y(S) \supseteq \mathcal{D}_Y(S^\perp)$ which is open. For the converse inclusion take $P \in \operatorname{int}_Y(\mathcal{V}(S))$: there exists $a \notin P$ such that $\mathcal{D}_Y(a) \subseteq \subseteq \mathcal{V}_Y(S)$. We claim that a belongs to $S^\perp$: otherwise there would exist $s \in S$ such that $\{(s \cdot a)^n: n \in \mathbb{N}\}$ is a multiplicatively closed set which does not contain zero, hence there would be a minimal prime Q not containing $s\cdot a$, therefore $Q \in \mathcal{D}_Y(a)\setminus\mathcal{V}_Y(S)$ against the assumption. Then P belongs to $\mathcal{D}_Y(S^\perp)$. ■

It is now easy to prove the following proposition, stated in [HJ] for $Y = \text{Min}(A)$.

1.2 PROPOSITION. *Consider the following*:

i) Y *is extremally disconnected*;

ii) *for every ideal* I *of* A, $\mathcal{V}_Y(I^\perp)$ *is open*;

iii) *for every ideal* I *of* A, $\mathcal{V}_Y(I^\perp)$ *and* $\mathcal{V}_Y(I^{\perp\perp})$ *are complementary in* Y;

iv) *for every ideal* I *of* A, $\mathcal{V}_Y(I^\perp + I^{\perp\perp})$ *is empty*;

v) *for every ideal* I *of* A, $I^\perp$ *is a direct summand, that is* $I^\perp = e\cdot A$ *for an idempotent* e *of* A.

Conditions i)-iv) *are equivalent and are implied by* v), for *every* Y *containing* $Min(A)$; *all the conditions are equivalent if* $Y = Spec(A)$.

PROOF. i) $\Leftrightarrow$ ii) $\Leftrightarrow$ iii): take the complements in proposition 1.1.

iii) $\Leftrightarrow$ iv): obvious since $\mathcal{V}_Y(I^\perp + I^{\perp\perp}) = \mathcal{V}_Y(I^\perp) \cap \mathcal{V}_Y(I^{\perp\perp})$ and $I^\perp \cap I^{\perp\perp} = (0)$, A being a semiprime ring.

v) $\Rightarrow$ iv): trivial.

iv) $\Rightarrow$ v): $\mathcal{V}(I^\perp + I^{\perp\perp}) = \emptyset$ implies $I^\perp + I^{\perp\perp} = A$. ■

1.3 THEOREM. ([HJ] 2.3) *For every* $a \in A$, $\mathcal{D}_0(a) = \mathcal{V}_0(a^\perp)$; *hence* Min (A) *is Hausdorff and has a basis of clopen sets.* ■

One can equip the set of prime ideals of A with the patch-topology ([H], [G]) which has as a basis the clopen subsets of the form $\mathcal{D}(a) \cap \mathcal{V}(a_1) \cap \ldots \cap \mathcal{V}(a_n)$ $a, a_i \in A$. We denote by $\mathcal{T}(A)$ this topological space; $\mathcal{T}(A)$ is Hausdorff, compact, totally disconnected and Min (A) is a subspace of $\mathcal{T}(A)$. Furthermore if A, B are rings and $\varphi\colon A \to B$ is a ring homomorphism, then the adjoint map ${}^a\varphi\colon \mathcal{T}(B) \to \mathcal{T}(A)$ given by ${}^a\varphi(P) = \varphi^{\leftarrow}[P]$ is continuous.

We indicate by $j\colon A \to \hat{A}$ the canonical embedding of A into the universal regular ring associated to A ([W], [G]). It is known that every morphism of A into a regular ring R extends uniquely to a morphism of $\hat{A}$ into R. Moreover the adjoint map ${}^aj\colon \text{Max}(\hat{A}) \to \mathcal{T}(A)$ is a homeomorphism. For every $P \in \mathcal{T}(A)$ we shall denote by $\hat{P}$ the maximal ideal such that ${}^aj(\hat{P}) = P = \hat{P} \cap A$.

We recall that if A is rationally complete (i.e. $A = Q(A)$) then A is self-injective and regular.

1.4 THEOREM. *If A is a self-injective ring, then* $\operatorname{Max}(A) = \operatorname{Spec}(A)$ *is extremally disconnected.*

PROOF. Let I be an ideal of A. Since A is semiprime, $I^\perp \cap I^{\perp\perp} = (0)$. Let π_1, π_2 be the projections of $I^\perp \oplus I^{\perp\perp}$ onto $I^\perp, I^{\perp\perp}$ respectively; as A is self-injective, $\pi_1, \pi_2, \pi_1 + \pi_2$, extend to endomorphisms π_1^*, π_2^*, $(\pi_1 + \pi_2)^*$ of the A-module A. Since $M = I^\perp \oplus I^{\perp\perp}$ is dense in A (i.e. its annihilator is (0)) and $\pi_1 + \pi_2$ is the identity of M, then $(\pi_1 + \pi_2)^*$ is the identity of A. On the other hand $(\pi_1 + \pi_2)^* = \pi_1^* + \pi_2^*$ because the equality holds in M. At last π_1^*, π_2^* are orthogonal idempotents because they are such in $\operatorname{Hom}_A(M, M)$. Hence $A = \operatorname{Im}(\pi_1^*) \oplus \operatorname{Im}(\pi_2^*)$; since $\operatorname{Im}(\pi_1^*) \supseteq I^\perp$ and $\operatorname{Im}(\pi_2^*) \supseteq I^{\perp\perp}$ we get that $\operatorname{Im}(\pi_2^*) = I^{\perp\perp}$ and $\operatorname{Im}(\pi_1^*) = I^\perp$, that is $A = I^\perp \oplus I^{\perp\perp}$. The conclusion follows from 1.2. ■

The proof of theorem 1.4 implies:

1.5 COROLLARY. *For any subset S of A: $S^\perp = A \cap e \cdot Q(A)$ for e idempotent of $Q(A)$; hence $S^{\perp\perp} = A \cap (1 - e) \cdot Q(A)$.* ■

2. ζ-ideals.

2.1. Let $i: A \to Q(A)$ be the canonical embedding, $\eta: \operatorname{Max}(Q(A)) \to \mathfrak{T}(A)$ the adjoint map; our purpose is to investigate the properties of η. The notion of z-ideal used in the study of $C(X)$ has been extended to arbitrary rings by Mason [M]; for our aim it will be helpful to introduce the concept of ζ-ideal which arises quite naturally in $C(X)$ (see section 4) and extends to arbitrary rings.

DEFINITIONS. Let I be an ideal of A.

a) I is a z-ideal if $a \in A$, $b \in I$, $V(a) = V(b)$ implies $a \in I$;

b) I is a ζ-ideal if $a \in A$, $b_1, \ldots, b_n \in I$, $\mathfrak{V}(a) \supseteq \mathfrak{D}((b_1, \ldots, b_n)^\perp)$ implies $a \in I$.

Notice that by 1.1 $\mathfrak{D}((b_1, \ldots, b_n)^\perp) = \operatorname{int}(\mathfrak{V}(b_1, \ldots, b_n))$.

2.2 LEMMA. *Every minimal prime ideal P is a ζ-ideal.*

PROOF. Let $b_1, \ldots, b_n \in P$, $a \in A$, $\mathfrak{V}(a) \supseteq \mathfrak{D}((b_1, \ldots, b_n)^\perp)$. By the characterization of the minimal prime ideals cited in 1.0, P belongs to $\mathfrak{D}((b_1, \ldots, b_n)^\perp)$, hence a belongs to P. ∎

2.3 THEOREM. *Let A be a ring, $J(A)$ its Jacobson radical. The following are equivalent*:

i) $J(A) = 0$;

ii) $\mathfrak{V}(a) \supseteq \mathfrak{D}((b_1, \ldots, b_n)^\perp)$ *if and only if* $V(a) \supseteq D((b_1, \ldots, b_n)^\perp)$ *for* $a, b_i \in A$;

iii) *every ζ-ideal is a z-ideal.*

PROOF. i) $\Rightarrow$ ii): $\mathfrak{V}(a) \supseteq \mathfrak{D}((b_1, \ldots, b_n)^\perp)$ implies trivially $V(a) \supseteq D((b_1, \ldots, b_n)^\perp)$; conversely if there exists $P \in \mathfrak{D}((b_1, \ldots, b_n)^\perp) \setminus \mathfrak{V}(a)$, there is $s \in (b_1, \ldots, b_n)^\perp$ such that $s \cdot a \notin P$. By i) there exists a maximal ideal M which does not contain $s \cdot a$, hence $M \in D((b_1, \ldots, b_n)^\perp) \setminus V(a)$.

ii) $\Rightarrow$ iii): trivial.

iii) $\Rightarrow$ i): ab absurdo, suppose $a \neq 0$ belongs to $J(A)$ and take a minimal prime ideal $P \in \mathfrak{D}(a)$. Then P is a ζ-ideal and it is not a z-ideal for $V(a) = V(0) = \operatorname{Max}(A)$ and a does not belong to P. ∎

If $J(A) = 0$, theorem 2.3 allows us to state definition 2.1 *b*) looking only at the maximal spectrum, that is replacing $\mathfrak{V}, \mathfrak{D}$ by V, D respectively.

Clearly by the definition every proper ζ-ideal is singular; and if an ideal I is intersection or (set theoretic) union of ζ-ideals, it is a ζ-ideal.

We indicate by $\mathfrak{Z}^0(A)$ the open sets of the form $\mathfrak{D}((a_1, \ldots, a_n)^\perp)$, $a_i \in A$. The proof of next proposition is routine.

2.4 PROPOSITION. i) *If I is an ideal then $\zeta(I) = \{\mathfrak{D}((a_1, \ldots, a_n)^\perp) : a_i \in I\}$ is a basis for a filter (possibly non-proper) in $\mathfrak{Z}^0(A)$; if I is a proper ζ-ideal, $\zeta(I)$ is a proper filter.*

ii) *If $\mathcal{F}$ is a filter basis in $\mathfrak{Z}^0(A)$ then $\zeta^{\leftarrow}(\mathcal{F}) = \{a \in A : \exists F \in \mathcal{F}, \mathfrak{D}(a^\perp) \supseteq F\}$ is a ζ-ideal, proper if $\mathcal{F}$ is proper.*

iii) *The assignation $I \to \zeta(I)$ is a $1-1$ correspondence between ζ-ideals and filters on $\mathfrak{Z}^0(A)$, whose inverse is $\zeta^{\leftarrow}$.* ∎

As an easy consequence we obtain that $\zeta^{\leftarrow}(\zeta(I))$ is the smallest ζ-ideal containing I, not necessarily proper even if I is a singular ideal, as the following theorem shows:

2.5 THEOREM. *The following are equivalent:*

i) *every singular ideal I is contained in a proper ζ-ideal;*

ii) *every finitely generated singular ideal I is contained in a minimal prime;*

iii) *every ideal maximal among the singular ones is a (prime) ζ-ideal.*

PROOF. i) $\Leftrightarrow$ ii): observe that $\zeta^{\leftarrow}(\zeta(I))$ is proper if and only if $\zeta(I) \not\ni \emptyset$ if and only if for every $b_1, \ldots, b_n \in I$, $\mathfrak{D}((b_1, \ldots, b_n)^{\perp}) \neq \emptyset$ if and only if $\mathfrak{V}_0(b_1, \ldots, b_n) = \mathfrak{D}_0((b_1, \ldots, b_n)^{\perp}) \neq \emptyset$; the conclusion follows.

i) $\Leftrightarrow$ iii): trivial. ■

It is easy to see that the condition ii) in the previous theorem is equivalent to the condition « C » of Quentel [Q]:

C Every finitely generated ideal such that its annihilator is (0) contains a non-zero–divisor.

In the general case a singular prime z-ideal is not necessarily a ζ-ideal even if $J(A) = 0$ and A satisfies the condition « C » as we shall show in 4.3.

3. **Relations between** Spec(A) **and** Max$(Q(A))$.

We are now ready to begin the study of the map η introduced in the previous section.

3.1 LEMMA. *Let A be a subring of the regular ring R and R be a ring of quotients of A. For every finite subset $a_1, \ldots, a_n$ of A, one has* $(a_1, \ldots, a_n)^{\perp\perp} = \left(\sum_{i=1}^{n} R\cdot a_i\right) \cap A$.

PROOF. We have $\sum_{i=1}^{n} R\cdot a_i = R\cdot e$, for an idempotent e of R. Observe that $R\cdot e \cap A \subseteq (a_1, \ldots, a_n)^{\perp\perp}$ since $R\cdot(1-e) \supseteq (a_1, \ldots, a_n)^{\perp}$. Now let x belong to $A \setminus R\cdot e$. So $x\cdot(1-e) \neq 0$ and as R is a ring of quotients of A, there exists t belonging to A such that $x\cdot t(1-e) \in A$, $x\cdot t(1-e) \neq 0$. Hence $x\cdot xt(1-e) \neq 0$ otherwise $(xt(1-e))^2 = 0$. It follows that $x \in (a_1, \ldots, a_n)^{\perp\perp}$ since $xt(1-e) \in (a_1, \ldots, a_n)^{\perp} = A \cap \cap R\cdot(1-e)$. ■

3.2 LEMMA. *Let R and A be as above. An ideal I of A is of the form $J \cap A$ for an ideal J of R if and only if for every $a_1, \dots, a_n \in I$ we have $(a_1, \dots, a_n)^{\perp\perp} \subseteq I$.*

PROOF. Necessity: let $I = J \cap A$, $a_1, \dots, a_n \in I$. The ideal K spanned by $a_1, \dots, a_n$ in R is contained in J; furthermore $K \cap A = (a_1, \dots, a_n)^{\perp\perp}$ by lemma 3.1.

Sufficiency: consider the ideal J spanned by I in R. Clearly $J \cap A \supseteq I$. Now take x belonging to $J \cap A$; then $x = \sum_{i=1}^{n} r_i a_i$ for certain $a_i \in A$, $r_i \in R$. By lemma 3.1 x belongs to $(a_1, \dots, a_n)^{\perp\perp}$ which is contained in I by hypothesis. ■

3.3 THEOREM. *Let P belong to* Spec (A). *The following are equivalent:*

i) *P is a ζ-ideal;*

ii) *P belongs to* $\mathrm{cl}_{\mathcal{T}(A)}(\mathrm{Min}\,(A))$;

iii) *for every $b_1, \dots, b_n \in P$, $a \in A \setminus P$, there exists $s \in (b_1, \dots, b_n)^{\perp}$ such that $s \cdot a \neq 0$;*

iv) *for every $b_1, \dots, b_n \in P$, $(b_1, \dots, b_n)^{\perp\perp} \subseteq P$;*

v) *P belongs to $\eta(\mathrm{Max}\,(Q(A)))$;*

vi) *P belongs to $^a i(\mathrm{Max}\,(R))$ if R is a regular ring of quotients of A.*

PROOF. i) $\Rightarrow$ iv): take $b_1, \dots, b_n \in P$, $s \in (b_1, \dots, b_n)^{\perp\perp}$; then $\mathfrak{V}(s) \supseteq \mathfrak{D}((b_1, \dots, b_n)^{\perp})$, hence s belongs to P.

iv) $\Rightarrow$ iii): trivial.

iii) $\Rightarrow$ ii): a neighborhood of P contains a set of the form $\mathfrak{D}(a) \cap \mathfrak{V}(b_1) \cap \dots \cap \mathfrak{V}(b_n)$, $a \notin P$, $b_1, \dots, b_n \in P$. If s is given as in iii), take a minimal prime Q which does not contain $s \cdot a$; hence $Q \in \mathfrak{D}(a) \cap \mathfrak{V}(b_1) \cap \dots \cap \mathfrak{V}(b_n)$.

ii) $\Rightarrow$ i): let $b_1, \dots, b_n$ belong to P, a belong to A, $\mathfrak{V}(a) \supseteq \mathfrak{D}((b_1, \dots, b_n)^{\perp})$. If a does not belong to P, $\mathfrak{D}(a) \cap \mathfrak{V}(b_1) \cap \dots \cap \mathfrak{V}(b_n)$ is a neighborhood of P, and therefore it contains a minimal prime ideal Q. By lemma 2.2 Q is a ζ-ideal, so it needs to contain a.

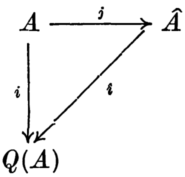
iv) $\Rightarrow$ vi): by lemma 3.2 there exists an ideal I of R such that $I \cap A = P$; then there exists a prime one which contains I and does not meet $A \setminus P$.

vi) $\Rightarrow$ v): trivial.

v) $\Rightarrow$ iv): by lemma 3.2. ■

3.4 Corollary ([Me]). Min (A) *is compact if and only if* Min $(A) = \eta(\mathrm{Max}\,(Q(A)))$. ■

Consider the following commutative diagram:



where $\hat{A}$ is the universal regular ring associated to A, i and j are the canonical embeddings.

Put $\tilde{A} = \hat{\imath}(\hat{A})$. Plainly $\tilde{A}$ is the smallest regular subring of $Q(A)$ containing A, and it coincides with $A[\{a^{-1}\colon a \in A\}]$ ([G]). Put $\mathcal{C}(A) = \mathrm{cl}_{\mathcal{S}(A)}(\mathrm{Min}\,(A))$.

3.5 Corollary ([Me]). Max $(\tilde{A})$ *is homeomorphic to* $\mathcal{C}(A)$.

Proof. In the above diagram replace $Q(A)$ with $\tilde{A}$. Consider ${}^{a}j \circ {}^{a}\hat{\imath} = {}^{a}i\colon \mathrm{Max}\,(\tilde{A}) \to \mathcal{C}(A)$: ${}^{a}i$ is onto by 3.3 vi), ${}^{a}\hat{\imath}$ is $1-1$ and ${}^{a}j$ is $1-1$ onto; hence ${}^{a}i$, being $1-1$, onto and continuous between Hausdorff compact spaces, is a homeomorphism. ■

The following theorem will be useful later on:

3.6 Theorem. *Using the above notations, we have:*

i) $\ker(\hat{\imath}) = \cap\,\{\hat{P}\colon P \in \mathcal{C}(A)\}$;

ii) $\ker(\hat{\imath}) = \cap\,\{\hat{P}\colon P \in \mathrm{Min}\,(A)\}$;

iii) $\hat{\imath}$ *is* $1-1$ *if and only if* $\mathcal{C}(A) = \mathcal{S}(A)$.

Proof. i): $\ker(\hat{\imath}) = \hat{\imath}^{\leftarrow}[0] = \cap\,\{\hat{\imath}^{\leftarrow}[M]\colon M \in \mathrm{Max}\,(Q(A))\}$. The commutativity of the previous diagram ensures that $\hat{\imath}^{\leftarrow}[M] = (M \cap A)\hat{}$.

ii): follows from i) since Min (A) is dense in $\mathcal{C}(A)$.

iii): if $\hat{\imath}$ is $1-1$ we have $\mathrm{Max}\,(\tilde{A}) \simeq \mathrm{Max}\,(\hat{A})$; on the other hand $\mathrm{Max}\,(\tilde{A}) \simeq \mathcal{C}(A)$ by 3.5 and $\mathrm{Max}\,(\hat{A}) \simeq \mathcal{S}(A)$ (section 1). Conversely: $\ker(\hat{\imath}) = \cap\,\{\hat{P}\colon P \in \mathcal{S}(A)\} = \cap\,\{P\colon P \in \mathrm{Max}\,(\hat{A})\} = 0$. ■

For every idempotent e of $Q(A)$ put $I_e = A \cap eQ(A)$. We recall that the ideals I_e are exactly the annihilator ideals in A (1.5).

3.7 LEMMA. *For every ideal P belonging to* Min (A) *the following are equivalent:*

i) *there exist $M, N \in \operatorname{Max}(Q(A))$, $M \neq N$ such that $P = M \cap \cap A = N \cap A$;*

ii) *there exists an idempotent $e \in Q(A)$ such that $I_e + I_{1-e} \subseteq P$;*

iii) *there exist two idempotents $e, f \in Q(A)$ such that $I_e + I_f \subseteq P$ and $eQ(A) + fQ(A) = Q(A)$.*

PROOF. i) $\Rightarrow$ ii): let e be an idempotent of $Q(A)$ belonging to $M \setminus N$; then $1 - e$ belongs to N, hence $I_e + I_{1-e} \subseteq P$.

ii) $\Rightarrow$ iii): trivial.

iii) $\Rightarrow$ i): let $S = A \setminus P$; S is a multiplicatively closed subset of $Q(A)$ which does not meet the ideals $eQ(A), fQ(A)$. Let M, N be prime ideals of $Q(A)$ containing respectively e and f, which do not meet S. M and N do not coincide since $eQ(A) + fQ(A) = Q(A)$, and $P = M \cap A = N \cap A$ since P is minimal. ■

Let B be a ring containing A, i the inclusion map of A in B; put $Y = {}^a i^{\leftarrow}\{\operatorname{Min}(A)\} = \{M \in \operatorname{Spec}(B) : M \cap A \in \operatorname{Min}(A)\}$. Observe that ${}^a i(Y) = \operatorname{Min}(A)$ and if B is a ring of quotients of A, Y is a dense subspace of Spec (B). Using the notations introduced in 1.0:

3.8 LEMMA. *Let I be an ideal of B. We have:*

$$\mathcal{V}_0(I \cap A) = {}^a i\{\mathcal{V}_Y(I)\}.$$

PROOF. Clearly $\mathcal{V}_0(I \cap A)$ contains ${}^a i\{\mathcal{V}_Y(I)\}$; for the converse take P belonging to $\mathcal{V}_0(I \cap A)$. There exists a prime $P' \in \operatorname{Spec}(B)$ which contains I and does not meet $A \setminus P$: then $P' \cap A = P$ since P is minimal. ■

Remark that lemma 3.8 says that ${}^a i|_Y$ is a closed map. This fact becomes rather interesting in the case $B = Q(A)$ (hence ${}^a i = \eta$) since it is used to get some conditions equivalent to the injectivity of $\eta|_Y$.

3.9 THEOREM. *Let A be a ring, $Y = \eta^{\leftarrow}\{\operatorname{Min}(A)\}$. The following are equivalent:*

i) Min (A) *is extremally disconnected*;

ii) *for every idempotent* $e \in Q(A)$, $\mathcal{V}_0(I_e)$ *is open*;

iii) *if* e, f *are idempotents of* $Q(A)$ *such that* $eQ(A) + fQ(A) = Q(A)$, *then* $\mathcal{V}_0(I_e + I_f) = \emptyset$;

iv) $\eta|_Y$ *is* $1-1$;

v) $\eta|_Y$ *is a homeomorphism of* Y *onto* Min (A);

vi) Max $(Q(A))$ *is a compactification of* Min (A);

vii) Max $(Q(A))$ *is the Stone-Čech compactification of* Min (A).

PROOF. i) $\Leftrightarrow$ ii): by 1.2.

ii) $\Leftrightarrow$ iii): by lemma 3.7, condition iii) is true if and only if it is verified with $f = 1 - e$; the conclusion follows by 1.2.

iii) $\Leftrightarrow$ iv): by 3.7.

iv) $\Leftrightarrow$ v): by 3.8.

v) $\Rightarrow$ vi): trivial since Y is dense in Max $(Q(A))$.

vi) $\Rightarrow$ vii): Max $(Q(A))$ is compact and extremally disconnected, hence it is the Stone-Čech compactification of any dense subset ([GJ] 6M).

vii) $\Rightarrow$ i): [GJ] 6M. ∎

We conclude with a theorem which gives a condition equivalent to the injectivity of η.

3.10 LEMMA. *Let* R, S *be regular rings,* $R \subseteq S$, $E(R)$ *and* $E(S)$ *the sets of the idempotents of* R *and* S *respectively. Then* $E(R) = E(S)$ *if and only if the natural map* ${}^a i$: Max $(S) \to$ Max (R) *is a homeomorphism.*

PROOF. Necessity: it is enough to show that ${}^a i$ is $1-1$. Let M, N belong to Max (S), $M \neq N$. There exists an idempotent $e \in M \setminus N$, hence e belongs to $(M \cap R) \setminus (N \cap R)$.

Sufficiency: Let e be an idempotent of S. $V_S(e)$ is a clopen set in Max (S), hence ${}^a i[V_S(e)]$ is a clopen in Max (R); therefore ${}^a i[V_S(e)] = V_R(f)$, $f \in R$ so that $V_S(e) = V_S(f)$ which implies $e = f$. ∎

3.11 PROPOSITION. *The following are equivalent*:

i) η *is* $1-1$;

ii) η: Max $(Q(A)) \to \mathcal{C}(A)$ *is a homeomorphism*;

iii) $\tilde{A}$ *contains* $E(Q(A))$.

PROOF. i) $\Leftrightarrow$ ii): trivial.

ii) $\Leftrightarrow$ iii): use 3.5 and 3.10. ∎

4. Applications to $C(X)$.

Let X be a Tychonoff space, $C(X)$ its ring of real-valued continuous functions. The notion of ζ-ideal applied to $C(X)$ may be formulated by means of the topology of X. Hence this notion turns out to be similar to the one of z-ideal ([GJ]).

For f belonging to $C(X)$, call $Z(f)$ the zero-set of f, Coz (f) its complement; then putting $M_x = \{f \in C(X): f(x) = 0\}$ we have that $\vartheta[X] = \{M_x: x \in X\}$ is a dense subspace of Max $(C(X))$. Observe that for $f_1, \ldots, f_n$ belonging to $C(X)$, $(f_1, \ldots, f_n)^\perp = (f_1^2 + \ldots + f_n^2)^\perp$; furthermore $J(C(X)) = 0$, hence by proposition 2.3 I is a ζ-ideal if (and only if) for any g belonging to I and f belonging to $C(X)$ such that $V(f) \supseteq D(g^\perp)$ we have that f belongs to I; moreover every ζ-ideal is a z-ideal.

4.1 PROPOSITION. *For every ideal I of $C(X)$ the following are equivalent*:

i) *I is a ζ-ideal*;

ii) *if $Z(f) \supseteq \operatorname{int}(Z(g))$ and $g \in I$ then $f \in I$.*

PROOF. For every $f, g \in C(X)$ we have: $V(f) \supseteq D(g^\perp)$ if and only if $V(f) \cap \vartheta[X] \supseteq D(g^\perp) \cap \vartheta[X]$; then observe that $Z(f) = \{x \in X: M_x \in V(f) \cap \vartheta[X]\}$ and $\operatorname{int}(Z(g)) = \{x \in X: M_x \in D(g^\perp) \cap \vartheta[X]\}$. The conclusion follows. ∎

We notice that, in view of the previous proposition, 2.4 may be expressed in a more significant way replacing $\mathfrak{Z}^0(C(X))$ with the family of open sets of X: $\{\operatorname{int}(Z(f)): f \in C(X)\}$.

It is easily seen that $C(X)$ satisfies the condition « C » of Quentel (2.5); so that every singular ideal is contained in a proper (prime) ζ-ideal. Moreover if P is a prime ideal of $C(X)$, the smallest ζ-ideal containing P is prime (see [GJ] 2.9).

One may wonder when ζ-ideals and z-ideals coincide; furthermore, since we have already remarked that a proper ζ-ideal must be singular, it is reasonable to ask when singular z-ideals and ζ-ideals coincide; we have:

4.2 THEOREM. *The following are equivalent:*

i) *every z-ideal is a ζ-ideal;*

ii) *every singular z-ideal is a ζ-ideal;*

iii) *every non-empty zero-set has non-empty interior;*

iv) $C(X) = K(C(X))$.

PROOF. i) $\Rightarrow$ ii): trivial. ii) $\Rightarrow$ iii): suppose there exists $f \in C(X)$ such that $Z(f) \neq \emptyset$, $\operatorname{int}(Z(f)) = \emptyset$ and take $x \in X \setminus Z(f)$. Consider the singular z-ideal I of the functions which vanish on $Z(f)$ and on a neighborhood of x: I is not a ζ-ideal since there exists a function g such that $g[Z(f)] = \{1\}$ and $Z(g)$ is a neighborhood of x: $g \notin I$ while $\operatorname{int}(Z(g)) = \operatorname{int}(Z(f \cdot g))$ and $f \cdot g$ belongs to I.

iii) $\Rightarrow$ i): let f belong to the z-ideal I and $Z(g) \supseteq \operatorname{int}(Z(f))$. We claim that $Z(g)$ contains $Z(f)$: otherwise there would exist a point $x \in Z(f) \setminus Z(g)$ and a function h such that $h[Z(g)] = \{1\}$, $h(x) = 0$; hence the interior of $Z(f^2 + h^2)$ (non-empty by hypothesis) is contained in $Z(g)$; but $Z(f^2 + h^2) \cap Z(g) = \emptyset$. Since I is a z-ideal, I contains g.

iii) $\Leftrightarrow$ iv): trivial; however see [FGL]. ■

4.3. Theorem 4.2 does not explain the situation among the prime ideals: one may still think that the singular prime z-ideals are always ζ-ideals. The following example provides a negative answer.

EXAMPLE. Let $W^* = \{\sigma\colon \sigma \leqslant \omega_1\}$, ω_1 denoting the first uncountable ordinal, $I = [0, 1] \subseteq \mathbb{R}$; define $X = \{\mathbb{N} \times I\} \dot{\cup} W^*$ where $\mathbb{N} \times I$ has the product topology, $W^* \setminus \{\omega_1\}$ has its order topology and the filter $\mathcal{F}$ of neighborhoods of the point ω_1 is defined as follows: let $\mathcal{A}$ be a free ultrafilter on $\mathbb{N}$; U belongs to $\mathcal{F}$ if $U \cap W^*$ contains a tail of W^* and $\{n\colon U \cap \{\mathbb{N} \times I\}$ is a neighborhood of $(n, 0)\}$ belongs to $\mathcal{A}$. X is a σ-compact space.

Take $P = \{f \in C(X)\colon \{n\colon f(n, 0) = 0\} \in \mathcal{A}\}$; we observe that any f belonging to P vanishes on a tail of W^*; hence, using the fact that an ultrafilter is prime, we can say that P is a prime ideal. Clearly P is a singular z-ideal but it is not a ζ-ideal: in fact if f, g vanish on W^* and f takes the value x and g takes the value $1/n$ at the point (n, x), we have that f, g belong to $C(X)$, $\operatorname{int}(Z(f)) = \operatorname{int}(Z(g)) = W^* \setminus \{\omega_1\}$, and P contains f but it does not contain g. Finally we may remark that $\zeta^{\leftarrow}(\zeta(P))$ is the maximal ideal M_{ω_1} which is therefore the unique ζ-ideal containing P.

4.4 Remark. A positive answer to the previous question may be given in a certain class of spaces (larger than the class of perfectly normal spaces). In view of [HJ] 5.5, using corollary 3.4 and the characterization of minimal prime ideals, we may observe that the following are equivalent:

i) $\operatorname{Min}(C(X))$ is compact;

ii) for every $f \in C(X)$ there exists $f' \in C(X)$ such that $Z(f) \cup Z(f') = X$ and $\operatorname{int}(Z(f) \cap Z(f')) = \emptyset$;

iii) every singular prime ideal is minimal.

In particular, singular prime z-ideals, prime ζ-ideals and singular prime ideals coincide if $\operatorname{Min}(C(X))$ is compact: e.g. if the support of each function of $C(X)$ is a zero-set (hence if X is basically disconnected) or if the closure of each open set in X is the support of a continuous function (see [HJ]); observe also that X is basically disconnected if and only if $C(X)$ is a Baer ring [AM]; on the other hand every Baer ring A satisfies condition iii), hence if $J(A) = 0$ singular prime z-ideals and prime ζ-ideals coincide.

In [W] R. Wiegand has provided necessary and sufficient conditions in order that $\hat{A}$ is a ring of quotients of A. We can enrich his theorem with some more conditions and we shall show that this situation is never verified in $C(X)$ unless $C(X)$ is regular itself.

4.5 Theorem. *The following are equivalent:*

i) *$\hat{A}$ is a ring of quotients of A;*

ii) *there exists a homomorphism $\varphi\colon \tilde{A} \to \hat{A}$ which is the identity on A;*

iii) *$\hat{A}$ is isomorphic to $\tilde{A}$;*

iv) $\mathcal{C}(A) = \mathcal{T}(A)$;

v) *every prime ideal is a ζ-ideal;*

vi) *for every finitely generated ideal I, the radical of I is the intersection of the minimal prime ideals containing I.*

PROOF. i) $\Rightarrow$ ii): we have $\tilde{A} \subseteq \hat{A} \subseteq Q(A)$.

ii) $\Rightarrow$ iii): let ψ be a homomorphism from A into a regular ring R, $\hat{\psi}$ its extension to $\hat{A}$; $\hat{\psi}\circ\varphi$ extends ψ to $\tilde{A}$, therefore $\tilde{A}$ satisfies the universal property, hence it is isomorphic to $\hat{A}$.

iii) $\Rightarrow$ i): trivial.

iii) $\Leftrightarrow$ iv): by theorem 3.6.

iv) $\Leftrightarrow$ v): by theorem 3.3.

v) $\Rightarrow$ vi): take $I = (b_1, \ldots, b_n)$, $a \notin \sqrt{I}$; there exists a prime P containing I and not containing a. Since P is a ζ-ideal, given s as in 3.3 iii), there exists a minimal prime which does not contain $s \cdot a$.

vi) $\Rightarrow$ v): let P be a prime, $b_1, \ldots, b_n \in P$, $a \in A$, $\mathfrak{V}(a) \supseteq \mathfrak{D}((b_1, \ldots, b_n)^{\perp})$. Since the minimal prime ideals are ζ-ideals, a belongs to every minimal prime ideal containing $b_1, \ldots, b_n$ hence to the radical of the ideal generated by $b_1, \ldots, b_n$. ■

4.5 *bis* THEOREM. *Let X be a Tychonoff space. If $A = C(X)$ the following conditions are equivalent to the conditions* i)-vi) *of theorem* 4.5:

vii) *every prime is a z-ideal*;

viii) *X is a P-space.*

PROOF. v) $\Rightarrow$ vii): by theorem 2.3.

vii) $\Rightarrow$ viii): let f belong to $C(X)$, $Z(f) \neq \emptyset$. Define

$$g(x) = \exp\left(-1/f^2(x)\right) \quad \text{for} \quad x \notin Z(f), \quad g(x) = 0 \text{ for } x \in Z(f)\,.$$

Clearly g belongs to $C(X)$. Since every prime is a z-ideal, there exist $n \in \mathbb{N}$, $h \in C(X)$ such that $f^n = h \cdot g$. If $Z(f)$ is not open, there exists a net of points of $\operatorname{Coz}(f)$, say x_λ, $\lambda \in \Lambda$, convergent to a point $x \in Z(f)$. But

$$\lim_{\lambda \in \Lambda} h(x_\lambda) = \lim_{\lambda \in \Lambda} \frac{f^n(x_\lambda)}{g(x_\lambda)} = \infty\,, \qquad \text{absurd}\,.$$

viii) $\Rightarrow$ i): $C(X)$ is regular. ■

The condition vii) is not equivalent to the ones of theorem 4.5 in the general case, as one may easily verify by looking at the ring **Z**. At last we notice that R. Wiegand has provided an example of a ring which satisfies the conditions of theorem 4.5 without being regular.

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