

RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

ALBERTO ZANARDO

A completeness theorem for the general interpreted modal calculus MC^v of A. Bressan

Rendiconti del Seminario Matematico della Università di Padova,
tome 64 (1981), p. 39-57

http://www.numdam.org/item?id=RSMUP_1981__64__39_0

© Rendiconti del Seminario Matematico della Università di Padova, 1981, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (<http://rendiconti.math.unipd.it/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

A Completeness Theorem for the General Interpreted Modal Calculus MC^ν of A. Bressan.

ALBERTO ZANARDO (*)

SUMMARY - A new semantics is introduced for Bressan's modal calculus MC^ν based on types of all finite levels. By this semantics we extend a completeness theorem of Zane Parks concerning the first order segment of MC^ν deprived of the description operator, to a completeness theorem for (possibly contingent) modal theories based on the full calculus MC^ν .

1. Introduction.

In [11] Zane Parks gives a completeness theorem for the first order part of Bressan's calculus MC^ν deprived of descriptions. In this paper we extend Z. Parks' theorem to MC^ν itself (and every theory based on it, i.e. every MC^ν -theory) treating types and descriptions too. Unlike [11] this paper deals also with contingent (i.e. not modally closed) theories based on MC^ν . This is achieved by identifying particular semantical structures which are sound for the definition of \mathcal{F} -validity, where \mathcal{F} is an arbitrary MC^ν -theory.

The calculus MC^ν is based on the modal language ML^ν . In [1] a semantics for ML^ν is introduced: starting from ν sets D_1 to D_ν of (typed) individuals and a set Γ of elementary possible cases (elsewhere called worlds, or points), for every type t it is defined the set QI_t of quasi intensions—briefly QI s—of type t on which variables and constants of this type can be valued.

More precisely, QI_i , the set (of individual concepts) on which individual variables (of type i) run, is $(\Gamma \rightarrow D_i)$ ($i = 1, \dots, \nu$); $QI_{(t_1, \dots, t_n)}$,

(*) Indirizzo dell'A.: Seminario Matematico, Università di Padova - Via Belzoni 7 - 35100 Padova.

on which relational variables (of type (t_1, \dots, t_n)) run, is $(\Gamma \rightarrow \mathcal{P}(QI_{t_1} \times \dots \times QI_{t_n}))$; and $QI_{(t_1, \dots, t_n; t_0)}$, on which functional variables (of type $(t_1, \dots, t_n; t_0)$) run, is $(QI_{t_1} \times \dots \times QI_{t_n} \rightarrow QI_{t_0})$ ⁽¹⁾.

Those interpretations for ML^v which are based on a structure of quasi intensions as the above one will be called *standard interpretations*; in these interpretations every QI_t is uniquely determined by the choice of the sets D_1 to D_v and Γ .

Following [7], our definition of \mathcal{I} -validity shall refer to a wider class of interpretations, the so-called general interpretations. In general interpretations the set of objects with a certain type is not uniquely determined by the sets of objects of lower type-level; for instance, if O_t and $O_{t'}$ are the sets of the objects of type t and t' respectively, then $O_{(t; t')}$ will be an arbitrary subset of $(O_t \rightarrow O_{t'})$ (closed with respect to definable functions). In particular, the set of individual concepts of type r will be an arbitrary subset of $(\Gamma \rightarrow D_r)$, like in [11].

The proof of the completeness theorem for MC^v -theories is an Henkin type proof; i.e., starting from a consistent set K of formulas, a general interpretation defined by means of linguistic entities is constructed in which K is satisfiable. Such a general interpretation is denumerable and hence a form of the Löwenheim-Skolem theorem holds.

Let us remark that the semantics introduced in this way is essentially non-extensional, i.e. the extension of a $QI \xi$ in a possible case γ , in general, does not depend only on the extension (in γ) of the parts of ξ , but on the whole intension of them (for more details, see N. Belnap's foreword to [1], or [3]).

At the end of N. 9 some hints are briefly given for the proof of a completeness theorem for contingent theories based on MC^v , i.e. theories whose proper axioms are arbitrary (possibly not modally closed). Of course, in the interpretations of such theories a particular elementary case is privileged, the so-called real case. Contingent theories are very important in view of axiomatizations of physical theories. Indeed, only some of these admit modally closed axiomatization in MC^v ; in astronomy, for instance, contingent axioms are needed—see [2].

Let us remark that in [1] NN. 52, 53 the (modally closed) calculus MC^v_0 is introduced by which contingent theories can be investigated. In view of this possibility, the completeness theorem for modally

⁽¹⁾ If A_0 to A_n are sets, we denote their cartesian product by $A_0 \times A_1 \times \dots \times A_n$, the class of subset of A_0 by $\mathcal{P}A_0$, and the set of mappings of $A_1 \times \dots \times A_n$ into A_0 by $A_1 \times \dots \times A_n \rightarrow A_0$. Furthermore, we denote the empty set by \emptyset .

closed theories covers all possibilities. However we shall consider contingent theories independently because they seem interesting at least from a formal point of view.

2. The modal language ML^v .

The modal language ML^v is based on a type system τ^v that contains v *individual types*: $1, \dots, v$. We denote n -tuples by $\langle \dots \rangle$ and define τ^v recursively by

- (a) $\{1, \dots, v\} \subset \tau^v$ and
- (b) if $t_1, \dots, t_n \in \tau^v$ and $t_0 \in \tau^v \cup \{0\}$, then $\langle t_1, \dots, t_n, t_0 \rangle \in \tau^v$.

We call *types* the elements of $\tau^v \cup \{0\}$ ($= \bar{\tau}^v$). A type t of the form $\langle t_1, \dots, t_n, t_0 \rangle$ with $t_0 = 0$ [$t_0 \neq 0$] is called (and used as) a *relation* [*function*] *type* and, following Carnap, is denoted by (t_1, \dots, t_n) [$(t_1, \dots, t_n; t_0)$].

The symbols of the modal language ML^v are the variables v_{tn} and the constants c_{tn} (where $n \in Z^+$ ($= \{1, 2, \dots\}$) and $t \in \tau^v$), « \sim » (not), « \wedge » (and), « \square » (necessarily), « $=$ » (identity), reversed iota « ι » for descriptions, the comma, and the parentheses.

The class \mathcal{E}_t of the designators or wfes (*well formed expressions*) of type t ($\in \bar{\tau}^v$) for ML^v is defined recursively by the following (formation) rules (f_1) to (f_8) where n [t_0] runs over Z^+ [$\bar{\tau}^v$] and t, t_1, \dots, t_n run over τ^v .

- (f_1) $v_{tn}, c_{tn} \in \mathcal{E}_t$;
- (f_2) if $\Delta, \Delta' \in \mathcal{E}_t$, then $\Delta = \Delta' \in \mathcal{E}_0$;
- (f_r) if $\Delta_1 \in \mathcal{E}_{t_1}, \dots, \Delta_n \in \mathcal{E}_{t_n}$ and $\Delta \in \mathcal{E}_{\langle t_1, \dots, t_n, t_0 \rangle}$, then $(\Delta(\Delta_1, \dots, \Delta_n)) \in \mathcal{E}_{t_0}$;
- (f_4) if $\Delta \in \mathcal{E}_0$, then $((v_{tn})\Delta) \in \mathcal{E}_t$;
- (f_{5-8}) if $\Delta, \Delta' \in \mathcal{E}_0$, then $(\sim \Delta), (\Delta \wedge \Delta'), ((v_{tn})\Delta), (\square \Delta) \in \mathcal{E}_0$.

The connectives \vee, \supset , and \equiv, \exists , and \diamond (it is possible) are understood to be introduced in the usual way; furthermore we use $(\forall x_1, \dots, x_n)p$ and $(\exists x_1, \dots, x_n)p$ as metalinguistic abbreviations of $(x_1) \dots (x_n)p$ and $\sim (\forall x_1, \dots, x_n) \sim p$ respectively. In order to drop parentheses we consider $(\iota x), (x), \sim, \wedge, \vee, \supset$, and \equiv as having decreasing cohesive powers and we use also dots to devide expressions.

A wff (*well formed formula*) p is said to be *modally closed* if it is constructed starting out from some wffs $\Box p_1, \dots, \Box p_n$ by means of $\sim, \wedge, (v_{in}),$ and \Box . The *modal closure* of p is p or $\Box p$ according to whether p is modally closed or not. The modal closure of the (*extensional*) closure $(\forall x_1, \dots, x_n)p$ of p is called the *total closure* of p .

If a and x are respectively a term and a variable of the same type, then we denote by $\Delta[x/a]$ the result of substituting occurrences of a for free occurrences of x in the wff Δ after having performed changes of bound variables in it in order to obtain a free for x in (an « equivalent » of) Δ .

Using the above convention we put

$$(2.1) \quad (\exists x)p \equiv_D (\exists x)(p \wedge (y)(p[x/y] \supset x = y)).$$

Furthermore we assume that every expression used in what follows has a type, i.e. it is well formed. This will make several explanations unnecessary. For instance, if we speak of the term $\Delta(\Delta')$, where $\Delta \in \mathcal{E}_{(v':i)}$, this implies $\Delta' \in \mathcal{E}_{v'}$.

3. ML^v -interpretations.

In [1] N. 6 a semantical system is introduced for ML^v : $v + 1$ non-empty sets $D_1, \dots, D_v,$ and $D_{v+1} = \Gamma$ are fixed (for $i = 1, \dots, v,$ $D_i[\Gamma]$ is called the i -th individual domain [the set of elementary possible cases or Γ -cases]) and the class QI_t of the quasi-intensions—briefly QI s—of type t ($\in \bar{\tau}^v$) based on D_1 to D_{v+1} is defined recursively by the conditions (3.1-3) below ($n \in \mathbb{Z}^+; t_0, t_1, \dots, t_n \in \tau^v$);

$$(3.1) \quad QI_0 = \mathcal{P}(\Gamma), \quad QI_r = (\Gamma \rightarrow D_r) \quad (r = 1, \dots, v);$$

$$(3.2) \quad QI_{(t_1, \dots, t_n)} = \mathcal{P}(QI_{t_1} \times \dots \times QI_{t_n} \times \Gamma) (= (\Gamma \rightarrow \mathcal{P}(QI_{t_1} \times \dots \times QI_{t_n})));$$

$$(3.3) \quad QI_{(t_1, \dots, t_n; t_0)} = (QI_{t_1} \times \dots \times QI_{t_n} \rightarrow QI_{t_0}).$$

Furthermore a function a^v , of domain τ^v , is considered such that $a^v =_D a^v(t) \in QI_t$; a^v_t is called the *non-existing object* of type t ($\in \tau^v$) because it serves to give a designation to descriptions in the Γ -cases in which they do not fulfil their conditions of exact uniqueness. In [1] (p. 19) $a^v_{(t_1, \dots, t_n)}$ is assumed to be the empty set, and in addition the counterdomain of $a^v_{(t_1, \dots, t_n; t_0)}$ is assumed to be $\{a^v_{t_0}\}$. These (natural) conditions are conventional and we can omit them.

DEF. 3.1. In connection with the above sets D_1 to D_{v+1} and function α^v we assume $\mathcal{Q}\mathcal{I}_t$ ($t \in \bar{\tau}^v$) to fulfil (3.1'-3') where (3.i') is what (3.i) becomes if we substitute QI_t with $\mathcal{Q}\mathcal{I}_t$ and the equality sign « = » with the inclusion sign « \subseteq ». If $\alpha_t^v \in \mathcal{Q}\mathcal{I}_t$ for every $t \in \tau^v$, then we put

$$(3.4) \quad D = \{\mathcal{Q}\mathcal{I}_t: t \in \bar{\tau}^v\}, \quad \mathcal{D} = \langle D, \alpha^v \rangle,$$

and we say that D is a QI -structure and \mathcal{D} a QI -system.

An interpretation for ML^v (briefly, an ML^v -interpretation) is then an ordered pair $\mathcal{M} = \langle \mathcal{D}, M \rangle$ where \mathcal{D} is a QI -system and M is a valuation of the constants of ML^v , that is a function which, for all $t \in \tau^v$, assigns an element of $\mathcal{Q}\mathcal{I}_t$ to every constant of type t . If V is any valuation of the variables of ML^v on the ML^v -interpretation \mathcal{M} (briefly, V is an \mathcal{M} -valuation), then the ordered pair $\mathcal{S} = \langle \mathcal{M}, V \rangle$ will be said an ML^v -system. As usual, if V and V' are two valuation such that $V(x) = V'(x)$ for $x \notin \{x_1, \dots, x_n\}$ and $V'(x_i) = \xi_i$, then we denote V' by $V \begin{pmatrix} x_1, \dots, x_n \\ \xi_1, \dots, \xi_n \end{pmatrix}$.

Every QI -system, or ML^v -interpretation, or ML^v -system, based on a QI -structure in which $\mathcal{Q}\mathcal{I}_t = QI_t$ for all $t \in \bar{\tau}^v$, will be called *standard* ⁽²⁾.

Furthermore, in order to introduce a semantics for theories having some contingent axioms, we consider *contingent* ML^v -interpretations and systems; these are ordered pairs $\langle \mathcal{M}, \gamma_R \rangle$ and $\langle \mathcal{S}, \gamma_R \rangle$ respectively, where γ_R is an arbitrarily fixed possible case, the so-called *real case*.

DEF. 3.2. For $\gamma \in \Gamma$ and $\xi, \eta \in \mathcal{Q}\mathcal{I}_t$ with $t \in \bar{\tau}^v$ we say that ξ and η are equivalent QI s of type t in the case γ (with respect to the QI -structure D), and we write

$$(3.5) \quad \xi =_{D, \gamma}^t \eta \quad (\text{or } \xi =_{\gamma}^t \eta),$$

if one of the following conditions holds:

- (a) $t \in \{1, \dots, v\}$ or $t = (t_1, \dots, t_n)$, and $\xi(\gamma) = \eta(\gamma)$;
- (b) $t = (t_1, \dots, t_n: t_0)$ and $\xi(\xi_1, \dots, \xi_n) =_{\gamma}^{t_0} \eta(\xi_1, \dots, \xi_n)$ for all n -tuples $\langle \xi_1, \dots, \xi_n \rangle \in (Q\mathcal{I}_{t_1} \times \dots \times Q\mathcal{I}_{t_n})$;
- (c) $t = 0$ and $\xi \cap \{\gamma\} = \eta \cap \{\gamma\}$.

⁽²⁾ If $D (= \{QI_t: t \in \bar{\tau}^v\})$ and $D' (= \{\mathcal{Q}\mathcal{I}_t: t \in \bar{\tau}^v\})$ are two QI -structures based on the same sets D_1, \dots, D_v , and Γ , and D is standard, then for $t \in \{0, 1, \dots, v\}$, $\mathcal{Q}\mathcal{I}_t$ is a subset of QI_t and, in general, $\mathcal{Q}\mathcal{I}_t$ can be embedded in QI_t . However, for the sake of simplicity, we shall write $\mathcal{Q}\mathcal{I}_t \subseteq QI_t$ for all t .

The proof of Theor. 10.2 in [1], concerning standard QI -structures, can be trivially adapted to demonstrate the following theorem in which D is an arbitrary QI -structure.

THEOR. 3.1. *Let $t \in \bar{v}^r$ and $\xi, \eta \in Q\mathcal{S}_t$ ($\in D$); then $\xi =_{D,\gamma} \eta$ for every γ iff $\xi = \eta$.*

4. Designation rules for ML^r in connection with ML^r -systems.

For every ML^r -system \mathcal{S} ($= \langle \mathcal{M}, V \rangle$) or contingent ML^r -system $\langle \mathcal{S}, \gamma_R \rangle$ we associate every $\Delta \in \mathcal{E}_t$ with an *intensional designatum* $\tilde{\Delta}$ in QI_t . This designatum is unique, as will appear from the nature of the rules. Hence we denote it by $\text{des}_{\mathcal{S}}(\Delta)$ or $\text{des}_{\mathcal{M}V}(\Delta)$. We define it recursively by the rules (\tilde{d}_1) to (\tilde{d}_9) below which are extensions to ML^r -systems of the rules (δ_1) to (δ_9) in [1], NN. 8, 11. For the sake of simplicity the equalities $\tilde{\Delta}_i = \text{des}_{\mathcal{S}}(\Delta_i)$ ($i = 0, 1, \dots, n$), and $\tilde{R} = \text{des}_{\mathcal{S}}(R)$, are assumed in the following table; furthermore, the afore-mentioned recursion consists of an induction on the number ν_Δ of occurrences of ι in Δ and, in connection with a given value of ν_Δ , of an induction on the length l_Δ of Δ .

Rule	If Δ is	Then $\tilde{\Delta} (= \text{des}_{\mathcal{S}}(\Delta))$ is
(\tilde{d}_1)	v_{tn} or c_{tn}	$V(v_{tn})$ or $M(c_{tn})$, respectively;
(\tilde{d}_2)	$\Delta_1 = \Delta_2 (\Delta_1, \Delta_2 \in \mathcal{E}_t)$	$\{\gamma \in \Gamma: \tilde{\Delta}_1 = \tilde{\Delta}_2\}$;
(\tilde{d}_3)	$R(\Delta_1, \dots, \Delta_n) (\in \mathcal{E}_0)$	$\{\gamma \in \Gamma: \langle \tilde{\Delta}_1, \dots, \tilde{\Delta}_n, \gamma \rangle \in \tilde{R}\}$;
(\tilde{d}_4)	the term $\Delta_0(\Delta_1, \dots, \Delta_n)$	$\tilde{\Delta}_0(\tilde{\Delta}_1, \dots, \tilde{\Delta}_n)$;
(\tilde{d}_{5-6})	$\sim \Delta_1$ or $\Delta_1 \wedge \Delta_2$	$\Gamma - \tilde{\Delta}_1$ or $\tilde{\Delta}_1 \cap \tilde{\Delta}_2$, respectively;
(\tilde{d}_7)	$(x)\Delta_1$ where x is v_{tn}	$\bigcap_{\xi \in \mathcal{L}\mathcal{S}_t} \text{des}_{\mathcal{M}V'}(\Delta_1)$, where $V' = V \begin{pmatrix} x \\ \xi \end{pmatrix}$;
(\tilde{d}_8)	$\square \Delta_1$	Γ if $\tilde{\Delta}_1 = \Gamma$, \emptyset otherwise;
(\tilde{d}_9)	$(\iota x)\Delta_1$ where x is v_{tn}	the element η of QI_t (possibly not in $\mathcal{L}\mathcal{S}_t$) that fulfils conditions (α) and (β) below.

(α) If $\gamma \in \text{des}_{\mathcal{S}}(\sim (\exists_1 x)\Delta_1)$, then $\eta = \tilde{\iota}_{D,\gamma} a_i^r$.

(β) If $\gamma \in \text{des}_{\mathcal{S}}((\exists_1 x)\Delta_1)$, $\xi \in \mathcal{L}\mathcal{S}_t$, and $\gamma \in \text{des}_{\mathcal{M}V'}(\Delta_1)$ for $V' = V \begin{pmatrix} x \\ \xi \end{pmatrix}$, then $\eta = \tilde{\iota}_{D,\gamma} \xi$.

Remark first that in (d_7) the intersection is considered for $\xi \in \mathcal{Q}\mathcal{I}_t$ and not for $\xi \in \mathcal{Q}I_t$. Second, remark that $\text{des}_{\mathcal{S}}(\Delta)$ may fail to be in $\mathcal{Q}\mathcal{I}_t$, which is unsatisfactory; however we are almost exclusively interested in general interpretations—see N. 6—in which, as we shall show, $\text{des}_{\mathcal{S}}(\Delta)$ is always in $\mathcal{Q}\mathcal{I}_t$ and hence the fact that $\text{des}_{\mathcal{S}}(\Delta)$ does not necessarily belong to $\mathcal{Q}\mathcal{I}_t$ constitutes no trouble for us.

For \mathcal{S} standard, $\text{des}_{\mathcal{S}}(\Delta)$ is the quasi intensional designatum of Δ according to NN. 8, 11 in [1]; in particular Theor. 11.1 in [1] holds. It is straightforward to check that this theorem can be extended to every ML^v -system:

THEOR. 4.1. *For every choice of \mathcal{M} and V , conditions (α) and (β) are fulfilled by exactly one $\eta \in \mathcal{Q}I_t$.*

DEF. 4.1. *Let $\langle \mathcal{M}, \gamma_R \rangle$ be a contingent ML^v -interpretation, p a wff, and K a class of wffs. We say that*

- (a) $p [K]$ is γ -satisfiable in $\langle \mathcal{M}, \gamma_R \rangle$ (or \mathcal{M}) if, for some \mathcal{M} -valuation V , $\gamma \in \text{des}_{\mathcal{M}V}(p) \left[\gamma \in \bigcap_{q \in K} \text{des}_{\mathcal{M}V}(q) \right]$;
- (b) p is γ -true in $\langle \mathcal{M}, \gamma_R \rangle$ (or \mathcal{M}) if, for every \mathcal{M} -valuation V , $\gamma \in \text{des}_{\mathcal{M}V}(p)$; (b') $\langle \mathcal{M}, \gamma_R \rangle$ (or \mathcal{M}) is a γ -model for K if every formula in K is γ -true in $\langle \mathcal{M}, \gamma_R \rangle$;
- (c) p is true in $\langle \mathcal{M}, \gamma_R \rangle [\mathcal{M}]$ if it is γ_R -true [γ -true for every $\gamma \in \Gamma$]; (c') $\langle \mathcal{M}, \gamma_R \rangle [\mathcal{M}]$ is a model for K if every p in K is true in $\langle \mathcal{M}, \gamma_R \rangle [\mathcal{M}]$.

The following theorem is obviously true.

THEOR. 4.2. *If (1) x is v_{in} , Δ is a wfe, and a is a term of type t , (2) V is an \mathcal{M} -valuation, and (3) $\xi = \text{des}_{\mathcal{M}V}(a)$ and $V' = V \begin{pmatrix} x \\ \xi \end{pmatrix}$, then $\text{des}_{\mathcal{M}V}(\Delta[x/a]) = \text{des}_{\mathcal{M}V'}(\Delta)$.*

5. An axiom system for the modal calculus MC^v based on ML^v .

The axiom schemes A5.1-17 below for MC^v are written following more [4] than [1]. For them we assume that (1) p and q are wffs, (2) Δ is a term, and (3) x, y, z, x_1 to x_n, F, G, f , and g are distinct

variables.

A5.1-3 AS12.1-3 in [B] that are equivalent to tautologies.

A5.4 $(x)(p \supset q) \supset (x)p \supset (x)q$.

A5.5 $\Box(p \supset q) \supset \Box p \supset \Box q$.

A5.6 $p \supset (x)p$, where x is not free in p .

A5.7 $p \supset \Box p$, where p is modally closed.

A5.8 $(x)p \supset p[x/\Delta]$.

A5.9 $\Box p \supset p$ ^(*).

A5.10, 11 $x = x$; $x = y \wedge y = z \supset x = z$.

A5.12 $\Box x = y \supset \Delta[z/x] = \Delta[z/y]$.

A5.13 $F = G \equiv (\forall x_1, \dots, x_n).F(x_1, \dots, x_n) \equiv G(x_1, \dots, x_n)$.

A5.14 $f = g \equiv (\forall x_1, \dots, x_n)f(x_1, \dots, x_n) = g(x_1, \dots, x_n)$.

A5.15 $(\exists F)(\forall x_1, \dots, x_n).p \equiv F(x_1, \dots, x_n)$.

A5.16 $(\exists f)(\forall x_1, \dots, x_n)\Delta = f(x_1, \dots, x_n)$.

A5.17 (a) $(\exists_1 x)p \wedge p[x/y] \supset y = (1x)p$;
 (b) $\sim (\exists_1 x)p \supset (1x)p = a_i^*$, where x is v_{in} and
 $a_i^* =_D (1v_{in})v_{in} \neq v_{in}$.

In addition to A5.1-17, we assume:

(*) if p is an axiom of MC^v , then $(x)p$ and $\Box p$ are axioms of MC^v , for every variable x of ML^v .

E. Omodeo has proved in [10] that descriptions can be eliminated from MC^v according to the Russel method, only provided we replace MC^v with the calculus MC^v , which has those among the axioms above that do not contain ι , and has an additional axiom (no such axioms

^(*) A5.5, A5.7, and A5.9 tell us that the modal calculus MC^v is based on the Lewis system S5—see [9]. The semantical counterpart of this features is in the designation rule (d_8) in N4.

are wanted in extensional logic). The last axiom can be

$$A5.17' \quad (z)(\exists y)\Box [(\exists x)p \wedge (x)(p \supset x = y) \cdot \vee \cdot \sim (\exists_1 x)p \wedge y = z].$$

In [1] further axioms are considered for MC^v ; there are, for instance, the axiom of choice, an axiom asserting the existence of a contingent attribute, and two conventional axioms which are the syntactical counterpart of the conventional conditions on the non-existing object. However we prefer to consider the «minimal» version of MC^v (i.e. that based on A5.1-17) and hence to enclose other versions, like that considered in [1], in the wider concept of MC^v -theory.

We say that the theory \mathcal{T} is an MC^v -theory if \mathcal{T} has the symbols of ML^v except some (perhaps all) constants, and the axioms of \mathcal{T} are those of MC^v —to be called *logical axioms*—and other wffs to be called *proper axioms*. An MC^v -theory \mathcal{T} is said to be *modally closed* if such are its proper axioms; otherwise \mathcal{T} is said to be *contingent*.

The only deduction rule in MC^v (and MC^v -theories) is the Modus Ponens. The definitions of wffs *deducible from K in \mathcal{T}* ($K \vdash_{\mathcal{T}}$), and *theorems of \mathcal{T}* ($\vdash_{\mathcal{T}}$) are as usual; furthermore we will omit the subscript \mathcal{T} in $\vdash_{\mathcal{T}}$ when no confusion can arise.

It is very easy to realize that, if s_1, \dots, s_n is any string of modal quantifiers (that is, s_i is \Box or \Diamond) and \mathcal{T} [\mathcal{T}'] is an arbitrary [a modally closed] MC^v -theory, then

$$(5.1) \quad \vdash_{\mathcal{T}} s_1, \dots, s_n p \equiv s_n p \quad \text{and} \quad K \vdash_{\mathcal{T}} p \Rightarrow \{(x)q : q \in K\} \vdash_{\mathcal{T}} (x)p;$$

$$(5.2) \quad K \vdash_{\mathcal{T}} p \Rightarrow \{\Box q : q \in K\} \vdash_{\mathcal{T}} \Box p \quad \text{and} \quad \vdash_{\mathcal{T}} (x)\Box p \equiv \Box(x)p.$$

Some contingent assertions concerning the real world—such as «at the instant t the earth has angular velocity w »—constitute some postulate of e.g. astronomy. The easiest way of treating such postulates is to give them modally closed forms by use of the calculus MC^v_c —see NN. 52, 53 in [1]—which is also an MC^v -theory. On the basis of this remark first a completeness theorem will be proved in connection with modally closed theories; then it will be extended to contingent theories, for greater (admittely formal) generality.

A (contingent or not) ML^v -interpretation in which the axioms of the MC^v -theory \mathcal{T} are true is said to be a *model* of \mathcal{T} (briefly, a \mathcal{T} -model). It is straightforward to check that the theorems of an MC^v -theory \mathcal{T} are true in every \mathcal{T} -model.

6. General ML^v -interpretations.

DEF. 6.1. Assume that (1) $\mathcal{S} = \langle \mathcal{M}, V \rangle$ is an ML^v -system, (2) $n \geq 0$, x_1 to x_n are distinct variables of the respective types t_1 to t_n , and $X = \{x_1, \dots, x_n\}$, (3) $\Delta \in \mathcal{E}_{t_0}$ and either

(4) $t_0 = 0$ and ξ is the set of the $(n+1)$ -tuples $\langle \xi_1, \dots, \xi_n, \gamma \rangle$ such that $\xi_i \in \mathcal{L}_{\mathcal{S}_{t_i}}$ ($i = 1, \dots, n$), $\gamma \in \Gamma$, and $\gamma \in \text{des}_{\mathcal{M}V'}(\Delta)$ with $V' = V \left(\begin{smallmatrix} x_1, \dots, x_n \\ \xi_1, \dots, \xi_n \end{smallmatrix} \right)$, so that $\xi \in QI_{(t_1, \dots, t_n)}$; or

(4') $t_0 \in \tau^v$ and ξ is the element of $QI_{(t_1, \dots, t_n; t_0)}$ such that, for every choice of $\xi_i \in \mathcal{L}_{\mathcal{S}_{t_i}}$ ($i = 1, \dots, n$), $\xi(\xi_1, \dots, \xi_n) = \text{des}_{\mathcal{M}V'}(\Delta)$ where $V' = V \left(\begin{smallmatrix} x_1, \dots, x_n \\ \xi_1, \dots, \xi_n \end{smallmatrix} \right)$.

Then ξ is said to be the QI denoted by Δ with respect to X and \mathcal{S} , and this fact is expressed by $\xi = d(\Delta, X, \mathcal{S}) = d(\Delta, X, \mathcal{M}, V)$.

DEF. 6.2. The QI ξ is said to be definable with respect to the ML^v -interpretation \mathcal{M} if there is a wfe Δ of ML^v , a finite set X of variables, and an \mathcal{M} -valuation V for which $\xi = d(\Delta, X, \mathcal{M}, V)$.

DEF. 6.3. (a) The ML^v -interpretation \mathcal{M} is said to be general if, for $t \in \bar{\tau}^v$, every QI of type t , definable with respect to \mathcal{M} , is in $\mathcal{L}_{\mathcal{S}_t}$;

(b) the ML^v -interpretation \mathcal{M} is said to be weakly general if it becomes general by adding $\mathcal{L}_{\mathcal{S}_0}$ with the set $D_0 = \{d(p, \emptyset, \mathcal{M}, V) : p \in \mathcal{E}_0 \text{ and } V \text{ is an } \mathcal{M}\text{-valuation}\}$ ⁽⁴⁾.

In the sequel we shall say that the ML^v -system \mathcal{S} ($= \langle \mathcal{M}, V \rangle$) is general or weakly general if \mathcal{M} is general or weakly general respectively.

Recalling rules (d_1) to (d_9) in N. 4, we easily see that $\text{des}_{\mathcal{M}V}(\Delta) = d(\Delta, \emptyset, \mathcal{M}, V)$, and hence, if \mathcal{M} is a general interpretation, then

⁽⁴⁾ In [7] an interpretation is said to be general if its domain contains the designatum of every expression. This simple definition is equivalent (over the system in [7]) to ours; indeed, in the calculus investigated in [7], the $d(\Delta, \{x_1, \dots, x_n\}, \mathcal{M}, V)$ is nothing else than $\text{des}_{\mathcal{M}V}(\lambda x_1, \dots, x_n)\Delta$. In [1] the operator λ is defined by means of the operator ι , thus the non-existing object may appear and hence the more elaborated definition 6.3 is needed.

the designatum of every expression is in the domain of \mathcal{M} ; so that the function $\text{des}_{\mathcal{M}V}$ is satisfactory.

THEOR. 6.1. (a) AA5.1-14 are true in every ML^v -interpretation;
 (b) \mathcal{M} is a weakly general ML^v -interpretation iff \mathcal{M} is an MC^v -model.

PROOF. To prove part (a) is a matter of routine. Furthermore, we see that, if \mathcal{M} is weakly general, then it is an MC^v -model; indeed the truth of AA5.15-17 follows directly from Defs. 6.1-3 and rule (d_0) in N. 4. Now, in order to prove the other half of (b), first remark that, as is substantially shown in [1] (Theor. 40.1),

$$(6.1) \quad \begin{aligned} & \vdash_{MC^v} (\exists F)(\forall x_1, \dots, x_n) \Box .F(x_1, \dots, x_n) \equiv p, \quad \text{and} \\ & \vdash_{MC^v} (\exists f) (\forall x_1, \dots, x_n) \Box f(x_1, \dots, x_n) = \Delta \text{ }^{(5)} \end{aligned}$$

for every choice of the wff p , term $\Delta \in \mathcal{E}_{t_0}$, and variables x_1, \dots, x_n (of the respective types t_1, \dots, t_n), F , and f , with $F [f]$ not free in $p [\Delta]$. In addition \mathcal{M} is an MC^v -model by an hypothesis; hence for every p, x_1, \dots, x_n , and F as above, and \mathcal{M} -valuation V , $\text{des}_{\mathcal{M}V}(q) = \Gamma$, where q is $(\exists F)(\forall x_1, \dots, x_n) \Box .F(x_1, \dots, x_n) \equiv p$; but this is equivalent to the existence of a $\xi \in \mathcal{Q}_{\mathcal{I}(t_1, \dots, t_n)}$ such that, for all n -tuples $\langle \xi_1, \dots, \xi_n \rangle$ with $\xi_i \in \mathcal{Q}_{\mathcal{I}_i}^*$, $\langle \xi_1, \dots, \xi_n, \gamma \rangle \in \xi$ iff $\gamma \in \text{des}_{\mathcal{M}V'}(p)$ with $V' = V \begin{pmatrix} x_1, \dots, x_n \\ \xi_1, \dots, \xi_n \end{pmatrix}$, hence $d(p, \{x_1, \dots, x_n\}, \mathcal{M}, V) = \xi \in \mathcal{Q}_{\mathcal{I}(t_1, \dots, t_n)}$. By similar reasonings (using (6.1)₂) one can easily see that $d(\Delta, \{x_1, \dots, x_n\}, \mathcal{M}, V) \in \mathcal{Q}_{\mathcal{I}(t_1, \dots, t_n; t_0)}$ for every term $\Delta \in \mathcal{E}_{t_0}$, variables x_1, \dots, x_n , and \mathcal{M} -valuation V . Q.E.D.

The following theorem refers, through its assumption (2), to countable QI -structures.

THEOR. 6.2. Assume that (1) \mathcal{M} is an ML^v -interpretation, (2) V_0 is an \mathcal{M} -valuation such that, for every $\xi \in \mathcal{Q}_{\mathcal{I}_t}$ ($t \in \bar{\tau}^v$) $\xi = \text{des}_{\mathcal{M}V_0}(\Delta')$ for some wfe Δ' , and (3) $\eta = d(\Delta, X, \mathcal{M}, V)$; then there exists a wfe Δ_0 such that $\eta = d(\Delta_0, X, \mathcal{M}, V_0)$.

PROOF. Let us first remark that $d(\Delta, X, \mathcal{M}, V) = d(\Delta, X, \mathcal{M}, V')$ if $V'(x) = V(x)$ for every variable x free in Δ and not belonging to X ,

⁽⁵⁾ Remark that in the proof of (6.1) in [1], A5.17 is effectively needed and hence it is not possible to strengthen this half of part (b) by requiring \mathcal{M} to be only a model of the part of MC^v based on A5.1-16.

and hence we can suppose $V_0(x)$ arbitrary, for x as above, and equal to $V(x)$ otherwise. Let y_1, \dots, y_n be the variables not in X and free in Δ , and let $\eta_i = V(y_i)$ ($i = 1, \dots, n$). By hypothesis (2) there exist n wffs a_1, \dots, a_n such that $\eta_i = \text{des}_{\mathcal{M}_{V_0}}(a_i)$. Let $\Delta_0 = \Delta[y_1/a_1, \dots, y_n/a_n]$. Then the equality $d(\Delta, X, \mathcal{M}, V) = d(\Delta_0, X, \mathcal{M}, V_0)$ is a straightforward consequence of Theor. 4.2. Q.E.D.

7. Statement of the completeness theorem. Saturated sets.

If p is a wff of the MC^v -theory \mathcal{T} , then p is said to be \mathcal{T} -valid —briefly $\models_{\mathcal{T}} p$ —iff p is true in every general \mathcal{T} -model.

Now we can write our completeness theorem for modally closed MC^v -theories, which will be proved in the next sections.

THEOR. 7.1. *For every wff p of the modally closed MC^v -theory \mathcal{T} , $\models_{\mathcal{T}} p$ iff $\vdash_{\mathcal{T}} p$.*

Let us first recall some standard definitions. A set K of (closed or open) formulas is said to be \mathcal{T} -consistent if there is a formula which is not deducible from K in \mathcal{T} . Of course, a maximal \mathcal{T} -consistent set K contains every formula deducible from it in \mathcal{T} and, in particular, every theorem of \mathcal{T} .

A language \mathcal{L}' is called an *extension* of the language \mathcal{L} if it is obtained from \mathcal{L} by adding a (possibly empty) set of new constants for each type (if \mathcal{L} and \mathcal{L}' are based on a type system). An ω -*extension* is an extension in which for each type the set of added constants is denumerable.

DEF. 7.1. *Let the theory \mathcal{T} be based on the language \mathcal{L} and let \mathcal{L}' be an extension of \mathcal{L} . Then the set H of wffs of \mathcal{L}' is said to be \mathcal{T} - \mathcal{L}' -saturated provided conditions (i) and (ii) below hold:*

- (i) H is maximal \mathcal{T}' -consistent, where \mathcal{T}' is the extension of \mathcal{T} obtained by adding \mathcal{T} with the logical axioms involving all constants of \mathcal{L}' ,
- (ii) if $(\exists x)p \in H$, then for some constant a of \mathcal{L}' $p[x/a] \in H$.

Remark that, by the maximality of H , (ii) is equivalent to

- (ii') if $p[x/a] \in H$ for all constants a of \mathcal{L}' , then $(x)p \in H$.

LEMMA 7.1. Assume that (1) H is a \mathcal{F} - ML^r -saturated set of wffs, (2) Γ_H is defined by means of

$$(7.1) \quad \Gamma_H = \{ \gamma : \gamma \text{ is a } \mathcal{F}\text{-}ML^r\text{-saturated set of wffs and} \\ \{ p : \Box p \in H \} \subseteq \gamma \},$$

(3) $\gamma_1 \in \Gamma_H$, and (4) \mathcal{F} is modally closed. Then $\Box p \in \gamma_1$ iff, for all $\gamma_2 \in \Gamma_H$, $p \in \gamma_2$.

PROOF. First we assume (α) $\Box p \in \gamma_1$, $p \notin \gamma_2$ ($\in \Gamma_H$). By assumption (1), (7.1), and Def. 7.1, H , γ_1 , and γ_2 are maximal \mathcal{F} -consistent. Hence $\sim p \in \gamma_2$ so that $\Box p \notin H$. Then $\sim \Box p \in H$, hence $\Box \sim \Box p \in H$ and $\sim \Box p \in \gamma_1$ which contrasts to $\Box p \in \gamma_1$. We conclude that, if $\Box p \in \gamma_1$, then (β) $p \in \gamma_2$ for all $\gamma_2 \in \Gamma_H$. We now conversely assume (β) and $\Box p \notin \gamma_1$. Let $K_1 = \{ q : \Box q \in H \}$. K_1 satisfies condition (ii') of Def. 7.1; indeed, if $r[x/a] \in K_1$ for ever constant a (of the same type of x), then $\Box r[x/a] \in H$ for every constant a and, by the \mathcal{F} - ML^r -saturation of H , $(x)\Box r \in H$; but this is equivalent to $\Box(x)r \in H$, and hence $(x)r \in K_1$. The closure K_2 of K_1 through $\vdash_{\mathcal{F}}$ (i.e. $\{ q : K_1 \vdash_{\mathcal{F}} q \}$) satisfies condition (ii') of Def. 7.1 too; indeed, if $K_1 \vdash_{\mathcal{F}} r[x/a]$ for every constant a , then $H \vdash_{\mathcal{F}} \Box r[x/a]$ for every a and $\Box r[x/a] \in H$ for every a , so that $(x)r \in K_1$ and $(x)r \in K_2$. Now, the closure K_3 of $K_1 \cup \{ \sim p \}$ through $\vdash_{\mathcal{F}}$ can be shown to satisfy condition (ii') of Def. 7.1. Indeed, $K_1 \cup \{ \sim p \} \vdash_{\mathcal{F}} r[x/a]$ for every a , implies $K_1 \vdash_{\mathcal{F}} \sim p \supset r[x/a]$ for every a , and $K_1 \vdash_{\mathcal{F}} (y)(\sim p \supset r[x/y])$ for a suitable variable y not free in p ; hence, $K_1 \vdash_{\mathcal{F}} \sim p \supset (y)r[x/y]$ and $K_1 \cup \{ \sim p \} \vdash_{\mathcal{F}} (y)r[x/y]$, which is equivalent to $K_1 \cup \{ \sim p \} \vdash_{\mathcal{F}} (x)r$. K_3 is also \mathcal{F} -consistent; otherwise $K_1 \vdash_{\mathcal{F}} p$, $H \vdash_{\mathcal{F}} \Box p$, $\Box \Box p \in H$, and $\Box p \in \gamma_1$, which contrasts to an hypothesis. Using the proof of Theor. 3 in [8], a \mathcal{F} - ML^r -saturated extension of $K_1 \cup \{ \sim p \}$ can be constructed. Of course this contradicts the hypothesis (β). Q.E.D.

8. The Henkin construction on which the proof of the completeness theorem is based.

Let \mathcal{F} be a modally closed MC^r -theory, and let K be a \mathcal{F} -consistent set of formulas given arbitrarily. In this section we construct an ML^r -interpretation \mathcal{M}_0 in which the set K will be proved to be sat-

isfiable—see N. 9; in N. 9 we also prove that \mathcal{M}_0 is a general \mathcal{F} -model, so that the completeness theorem will follow in a standard way.

The first step is to prove that the set K has a \mathcal{F} - \mathcal{L} -saturated extension H , for some extension \mathcal{L} of ML^ν . This can be easily achieved following, for instance, [9] (pp. 160, 161); furthermore, by replacing every constant c_{i_n} in \mathcal{F} by $c_{i,2n}$, we can identify \mathcal{L} with ML^ν ⁽⁶⁾.

Now, in order to construct an interpretation \mathcal{M}_0 , we first identify D_r with \mathcal{E}_r —see N. 3—($r = 1, \dots, \nu$) and Γ with the set Γ_H defined in (7.1), i.e. we identify individuals with individual expressions and possible cases with \mathcal{F} - ML^ν -saturated sets of formulas.

For $\gamma \in \Gamma$ we consider the equivalence relation \approx_γ in $D_1 \cup \dots \cup D_\nu$, such that $\Delta_1 \approx_\gamma \Delta_2$ iff $\Delta_1 = \Delta_2 \in \gamma$. For $\Delta \in \mathcal{E}_r$ and $\gamma \in \Gamma$ let $Q_\Delta(\gamma)$ be a particular term Δ' in \mathcal{E}_r such that $\Delta' \approx_\gamma \Delta$. Thus we have associated every $\Delta \in \mathcal{E}_r$ with a function Q_Δ to be dealt with as the QI of Δ . We now give Q_Δ a meaning also for an arbitrary non-individual Δ in ML^ν , recursively, by means of the conditions (8.1-3) below.

$$(8.1) \quad \text{If } \Delta \in \mathcal{E}_{(t_1, \dots, t_n)}, \quad Q_\Delta = \{ \langle Q_{\Delta_1}, \dots, Q_{\Delta_n}, \gamma \rangle : \Delta(\Delta_1, \dots, \Delta_n) \in \gamma \}.$$

$$(8.2) \quad \text{If } \Delta \in \mathcal{E}_{(t_1, \dots, t_n; t_0)}, \quad Q_\Delta \text{ is the mapping of} \\ \{Q_{\Delta_1} : \Delta_1 \in \mathcal{E}_{t_1}\} \times \dots \times \{Q_{\Delta_n} : \Delta_n \in \mathcal{E}_{t_n}\} \text{ into } \{Q_{\Delta_0} : \Delta_0 \in \mathcal{E}_{t_0}\} \\ \text{for which } Q_\Delta(Q_{\Delta_1}, \dots, Q_{\Delta_n}) = Q_{\Delta(\Delta_1, \dots, \Delta_n)}.$$

$$(8.3) \quad \text{If } \Delta \in \mathcal{E}_0, \text{ then } Q_\Delta = \{ \gamma : \Delta \in \gamma \}.$$

Now the ML^ν -interpretation \mathcal{M}_0 and the ML^ν -system $\mathcal{S}_0 (= \langle \mathcal{M}_0, V_0 \rangle)$ can be defined by means of

$$(8.4) \quad \mathcal{I}_i = \{Q_\Delta : \Delta \in \mathcal{E}_i\} \quad (t \in \bar{\tau}^\nu), \\ a_i^\tau = Q_{a_i^\tau} \quad (t \in \tau^\nu), \quad M_0(c_{i_n}) = Q_{c_{i_n}}, \quad V_0(v_{i_n}) = Q_{v_{i_n}}.$$

⁽⁶⁾ In [7] the (correspondent of the) set H is required only to be maximal \mathcal{F} -consistent and, in general, saturated sets are not considered. Our departure from [7] is necessary because of the different uses of the operator ι ; in [7] (which follows [5]), ι is a choice operator and, in particular, the formula $(\exists x)A(x) \supset A(\iota x)A(x)$ is a valid formula. From the designation rule (d_0) we see that the above formula may fail to be true if we refer to an ML^ν -interpretation.

THEOR. 8.1. *For $t \in \tau^v$ and $\Delta, \Delta' \in \mathcal{E}_t$ we have $\Delta = \Delta' \in \gamma$ iff $Q_\Delta = {}^t_\gamma Q_{\Delta'}$ —see (3.5).*

PROOF. In the case $t \in \{1, \dots, \nu\}$ the thesis is trivial.

Case $t = (t_1, \dots, t_n)$. Since γ is maximal consistent, by A5.13 $\Delta = \Delta' \in \gamma$ iff $[\Delta(\Delta_1, \dots, \Delta_n) \equiv \Delta'(\Delta_1, \dots, \Delta_n)] \in \gamma$ for all $\langle \Delta_1, \dots, \Delta_n \rangle \in \mathcal{E}_{t_1} \times \dots \times \mathcal{E}_{t_n}$. This holds iff for everyone of these n -tuples the conditions $\Delta(\Delta_1, \dots, \Delta_n) \in \gamma$ and $\Delta'(\Delta_1, \dots, \Delta_n) \in \gamma$ are equivalent, and this holds iff $Q_\Delta = {}^t_\gamma Q_{\Delta'}$ —cf. (8.1).

Case $t = (t_1, \dots, t_n : t_0)$. Let the thesis hold for $t = t_0$ as an inductive hypothesis. By A5.14 $\Delta = \Delta' \in \gamma$ iff $\Delta(\Delta_1, \dots, \Delta_n) = \Delta'(\Delta_1, \dots, \Delta_n) \in \gamma$ for every n -tuple $\langle \Delta_1, \dots, \Delta_n \rangle \in \mathcal{E}_{t_1} \times \dots \times \mathcal{E}_{t_n}$. By the inductive hypothesis this holds iff $Q_{\Delta(\Delta_1, \dots, \Delta_n)} = {}^{t_0}_\gamma Q_{\Delta'(\Delta_1, \dots, \Delta_n)}$ for all n -tuples above. By (8.2) this holds in turn iff $Q_\Delta = {}^t_\gamma Q_{\Delta'}$. **Q.E.D.**

Theors. 8.1 and 3.1 obviously imply the following

THEOR. 8.2. *For $t \in \tau^v$ and $\Delta, \Delta' \in \mathcal{E}_t$, we have $\Delta = \Delta' \in \gamma$ for every $\gamma \in \Gamma$ iff $Q_\Delta = Q_{\Delta'}$.*

COROLLARY 8.1. *Every $\xi \in \mathcal{L}_{\mathcal{S}_t}$ ($t \in \tau^v$) has the form Q_c for some constant c of type t .*

PROOF. By (8.4), ξ is Q_Δ for some term Δ ; furthermore, $(\exists x) \square (\Delta = x)$ is provable in MC^v ; hence by the \mathcal{S} - ML^v -saturation of H , there is a constant c such that $\square (\Delta = c) \in H$, that is, by (7.1), $\Delta = c \in \gamma$ for all $\gamma \in \Gamma$. The thesis follows now from Theor. 8.2. **Q.E.D.**

THEOR. 8.3. *For every wfe Δ of ML^v , $\text{des}_{\mathcal{S}_0}(\Delta) = Q_\Delta$; and hence, by (8.3),*

$$(8.5) \quad p \in \gamma \text{ iff } \gamma \in \text{des}_{\mathcal{S}_0}(p), \text{ for every wff } p.$$

PROOF. We use an induction on the number ν_Δ of occurrences of ι in Δ , and for every n (≥ 0) we treat the wfes Δ with $\nu_\Delta = n$ by induction on their lengths l_Δ . For $l_\Delta = 1$, Δ is c_{in} or v_{in} , hence the thesis follows by (8.4)₄.

Case 1: Δ is the term $\Delta'(\Delta_1, \dots, \Delta_n)$. By the inductive hypothesis and (8.2), $\text{des}_{\mathcal{S}_0}(\Delta) = \text{des}_{\mathcal{S}_0}(\Delta')(\text{des}_{\mathcal{S}_0}(\Delta_1), \dots, \text{des}_{\mathcal{S}_0}(\Delta_n)) = Q_{\Delta'}(Q_{\Delta_1}, \dots, Q_{\Delta_n}) = Q_\Delta$.

Case 2a: Δ is the wff $R(\Delta_1, \dots, \Delta_n)$. Then by the designation rule (d_3), the inductive hypothesis, and (8.1)–(8.3), $\gamma \in \text{des}_{\mathcal{S}_0}(\Delta) \Leftrightarrow$

$\Leftrightarrow \langle \text{des}_{\mathcal{F}_0}(\Delta_1), \dots, \text{des}_{\mathcal{F}_0}(\Delta_n), \gamma \rangle \in \text{des}_{\mathcal{F}_0}(R) \Leftrightarrow \langle Q_{\Delta_1}, \dots, Q_{\Delta_n}, \gamma \rangle \in Q_R \Leftrightarrow R(\Delta_1, \dots, \Delta_n) \in \gamma \Leftrightarrow \gamma \in Q_{\Delta}$.

Case 2b: Δ is $\Delta_1 = \Delta_2$ with $\Delta_1, \Delta_2 \in \mathcal{E}_t$. Then by rule (d_2) , the inductive hypothesis, Theor. 8.1, and (8.3), $\gamma \in \text{des}_{\mathcal{F}_0}(\Delta) \Leftrightarrow \text{des}_{\mathcal{F}_0}(\Delta_1) =^t_{\gamma} =^t_{\gamma} \text{des}_{\mathcal{F}_0}(\Delta_2) \Leftrightarrow Q_{\Delta_1} =^t_{\gamma} Q_{\Delta_2} \Leftrightarrow \Delta_1 = \Delta_2 \in \gamma \Leftrightarrow \gamma \in Q_{\Delta}$.

Case 2c: Δ is $\sim p$. Then $\text{des}_{\mathcal{F}_0}(\Delta) = \Gamma - \text{des}_{\mathcal{F}_0}(p)$ by rule (d_5) . Furthermore, since every $\gamma \in \Gamma$ is maximal consistent, by (8.3), $Q_{\Delta} = \Gamma - Q_p$. The thesis now follows by the inductive hypothesis.

Case 2d: Δ is $p \wedge q$. The proof is similar to the above one.

Case 2e: Δ is $\Box p$. Then by rule (d_8) , the inductive hypothesis, Lemma 7.1, and (8.3), $\gamma \in \text{des}_{\mathcal{F}_0}(\Delta) \Leftrightarrow \gamma_1 \in \text{des}_{\mathcal{F}_0}(p)$ for all $\gamma_1 (\in \Gamma) \Leftrightarrow \gamma_1 \in Q_p$ for all $\gamma_1 \Leftrightarrow p \in \gamma_1$ for all $\gamma_1 \Leftrightarrow \Delta \in \gamma \Leftrightarrow \gamma \in Q_{\Delta}$.

Case 2f: Δ is $(x)p$ where x is v_{tn} . Then, by rule (d_7) , $\gamma \in \text{des}_{\mathcal{F}_0}(\Delta)$ iff for all $\xi \in \mathcal{Q}_{\mathcal{F}_t}$, $\gamma \in \text{des}_{\mathcal{M}_0 V'}(p)$ with $V' = V \left(\begin{smallmatrix} x \\ \xi \end{smallmatrix} \right)$. Since, by Corollary 8.1, every $\xi \in \mathcal{Q}_{\mathcal{F}_t}$ is a Q_b (that is $\text{des}_{\mathcal{F}_0}(b)$) for some constant b , and Theor. 4.2 holds, the last condition holds iff for every constant $b \in \mathcal{E}_t$, $\gamma \in \text{des}_{\mathcal{F}_0}(p[x/b])$, which by the inductive hypothesis is equivalent to $\gamma \in Q_{p[x/b]}$ for all constants $b \in \mathcal{E}_t$ and hence, by (8.3), to $p[x/b] \in \gamma$ for the same constants. Since γ is \mathcal{F} - ML^r -saturated, this holds iff $(x)p \in \gamma$ and hence, by (8.3), iff $\gamma \in Q_{\Delta}$.

We conclude that the thesis holds for $\nu_{\Delta} = 0$. Now fix $n > 0$ and let the thesis hold for $\nu_{\Delta} < n$; and assume $\nu_{\Delta} = n$.

Case 3: Δ has the form $(ix)p$ where x is v_{tn} . By the inductive hypothesis the thesis holds for p and every wff q that contains p and has no occurrences of \imath outside p . Let q be $(\exists_1 x)p$. Remark that by Theor. 4.1, the transitivity of the relation $=^t_{\gamma}$, and rule (d_9) , it suffices to prove that Q_{Δ} is equivalent to a_i^* in the cases $\gamma_1 \notin \text{des}_{\mathcal{F}_0}(q)$, and to some $QI \xi$ such that $\gamma_2 \in \text{des}_{\mathcal{M}_0 V'}(p)$, for $V' = V \left(\begin{smallmatrix} x \\ \xi \end{smallmatrix} \right)$, in the cases $\gamma_2 \in \text{des}_{\mathcal{F}_0}(q)$.

Let $\gamma_1 \in \text{des}_{\mathcal{F}_0}(\sim q)$. By the inductive hypothesis, $\gamma_1 \in Q_{\sim q}$, i.e. $\sim q \in \gamma_1$; hence, by A5.17 (b), $(ix)p = a_i^* \in \gamma_1$. Then, by Theor. 8.1,

$$(8.6) \quad Q_{\Delta} =^t_{\gamma_1} Q_{a_i^*} (= a_i^*) \quad \text{for all } \gamma_1 \in \text{des}_{\mathcal{F}_0}(\sim (\exists_1 x)p).$$

Now let $\gamma_2 \in \text{des}_{\mathcal{F}_0}(q)$. By the inductive hypothesis $\gamma_2 \in Q_q$, i.e. $(\exists_1 x)p \in \gamma_2$. Then by A5.17 (a), (2.1), and (5.1)₂,

$$(8.7) \quad (y)(p[x/y] \supset y = (ix)p) \in \gamma_2 \quad \text{and} \quad (\exists x)p \in \gamma_2.$$

Since γ_2 is \mathcal{F} - ML^v -saturated, by (8.7)₂ $p[x/a] \in \gamma_2$ for some constant a ; furthermore by (8.7)₁ $(p[x/a] \supset a = (ix)p) \in \gamma_2$: Hence $(a = (ix)p) \in \gamma_2$ which is equivalent to

$$(8.8) \quad Q_A =^t_{\gamma_2} Q_a \quad \text{for some } a \text{ and } \gamma_2 \in \text{des}_{\mathcal{S}_0}((\exists_1 x)p).$$

By the inductive hypothesis, $\text{des}_{\mathcal{S}_0}(p[x/a]) = Q_{p[x/a]}$, so that by (8.3), $\gamma_2 \in \text{des}_{\mathcal{S}_0}(p[x/a])$ which by Theor. 4.2 is equivalent to

$$(8.9) \quad \gamma_2 \in \text{des}_{\mathcal{M}_0 V'}(p) \quad \text{for} \quad V' = V \left(\begin{array}{c} x \\ \text{des}_{\mathcal{S}_0}(a) \end{array} \right) = V \left(\begin{array}{c} x \\ Q_a \end{array} \right).$$

The thesis follows now by (8.6), (8.8), and (8.9). Q.E.D.

9. Accomplishment of the proof of the completeness theorems.

Let us return now to our completeness theorem. It remains to prove that H (and hence K) is γ -satisfiable—see Def. 4.1—in \mathcal{M}_0 for some $\gamma \in \Gamma$. Recalling how the set Γ_H of the elementary possible cases was constructed—i.e. (7.1)—we note that H itself is an element, $\bar{\gamma}$, of Γ_H . Hence, by (8.5) applied to $\bar{\gamma}$, we have

$$(9.1) \quad p \in H \quad \text{iff} \quad \bar{\gamma} \in \text{des}_{\mathcal{M}_0 V_0}(p), \quad \text{for every wff } p,$$

that is, H is $\bar{\gamma}$ -satisfiable in \mathcal{M}_0 .

We now prove that \mathcal{M}_0 is a general \mathcal{F} -model. \mathcal{M}_0 is a \mathcal{F} -model; indeed, every $\gamma \in \Gamma_H$ (being maximal \mathcal{F} -consistent) contains the axioms of \mathcal{F} and their extensional closure, and hence, by (8.5), $\text{des}_{\mathcal{M}_0 V}(p) = \Gamma$ for every \mathcal{M}_0 -valuation V and every axiom p of \mathcal{F} . Since \mathcal{F} is an MC^v -theory, by Theor. 6.1, \mathcal{M}_0 is weakly general. Furthermore, let $\xi = d(p, \emptyset, \mathcal{M}_0, V)$ for some wff p and \mathcal{M}_0 -valuation V . Since, by (8.4) and Theor. 8.3, every QI (in the domain of \mathcal{M}_0) has the form $\text{des}_{\mathcal{S}_0}(\Delta)$ for some Δ , we may use Theor. 6.2 to derive $\xi = d(p', \emptyset, \mathcal{M}_0, V_0)$, for some wff p' . But $d(p', \emptyset, \mathcal{M}_0, V_0) = \text{des}_{\mathcal{S}_0}(p')$. Therefore by Theor. 8.3, $\xi = Q_{p'}$ and $\xi \in \mathcal{L}_{\mathcal{S}_0}$; that is, \mathcal{M}_0 is general. Thus Theor. 7.1 has been proved.

We can now briefly show how a completeness theorem for contingent MC^v -theories can be proved.

The proof of such a theorem can be wholly analogue to the one relative to modally closed MC^v -theories; that is, it consists in the construction (from a given contingent MC^v -theory \mathcal{T}_c) of a contingent \mathcal{T}_c -model $\langle \mathcal{M}, \gamma_R \rangle$ in which a given \mathcal{T}_c -consistent set K_0 of formulas is γ_R -satisfiable. However, in this case, we cannot use every result relative to modally closed theories; for instance, (5.2) does not hold when \mathcal{T} is (properly) contingent and hence the proof of Lemma 7.1—in which (5.2) is applied—fails to be valid.

In any case, it is not necessary to repeat the whole proof of the completeness theorem; indeed, by an easy device, we can use the preceding proof for our present goals.

Let \mathcal{T}_c be a contingent MC^v -theory and let K_0 be a \mathcal{T}_c -consistent set of formulas. Let us denote by C the set of contingent proper axioms of \mathcal{T}_c (that is, the proper axioms of \mathcal{T}_c that are not modally closed) and by \mathcal{T} the modally closed part of \mathcal{T}_c (that is the MC^v -theory obtained from \mathcal{T}_c by subtracting C from the set of its axioms). Furthermore, let

$$(9.2) \quad K = K_0 \cup \{(\dots)p : p \in C\}, \quad \text{where } (\dots)p \text{ denotes the} \\ \text{extensional closure of } p.$$

Obviously, K is \mathcal{T}_c -consistent.

We may now build up a \mathcal{T} - ML^v -saturated extension H of K and the general ML^v -system \mathcal{S}_0 just as in N. 8. Of course, Theor. 8.3 holds and, in particular, (9.1) holds too.

If we consider the contingent ML^v -interpretation $\mathcal{M}_c = \langle \mathcal{M}_0, \gamma_R \rangle$ where γ_R is H , then, by (9.1), H is γ_R -satisfiable in \mathcal{M}_c .

It remains to prove that \mathcal{M}_c is a \mathcal{T}_c -model. We already know that \mathcal{M}_c is a \mathcal{T} -model, then let p be a contingent axiom of \mathcal{T}_c ; by (9.2) the extensional closure p' of p belongs to H and hence, by (9.1), $\gamma_R \in \text{des}_{\mathcal{M}_0 V}(p')$ and $\gamma_R \in \text{des}_{\mathcal{M}_0 V}(p)$ for all \mathcal{M}_0 -valuation V . That is, \mathcal{M}_c is a general \mathcal{T}_c -model.

REFERENCES

- [1] A. BRESSAN, *A General Interpreted Modal Calculus*, New Haven, Yale University Press, 1972.
- [2] A. BRESSAN, *On the usefulness of modal logic in axiomatizations of physics*, Conference at PSA meeting in Lansing (September 1972), PSA 1972, pp. 285-303.

- [3] A. BRESSAN, *On the semantics for the language ML' based on a type system, and those for the type-free language ML^∞* , Journal of Philosophical Logic, **3** (1974), pp. 171-194.
- [4] A. BRESSAN, *Extension of the modal calculi MC' and MC^∞ . Comparison of them with similar calculi endowed with different semantics. Application to probability theory*, to be printed in the Proceedings of the workshop on modal logic held in Tübingen, Dec. 1977.
- [5] A. CHURCH, *A formulation of the simple theory of types*, Journal of Symbolic Logic, **5** (1940), pp. 56-68.
- [6] L. HENKIN, *The completeness of the first-order functional calculus*, Journal of Symbolic Logic, **14** (1949), pp. 159-166.
- [7] L. HENKIN, *Completeness in the theory of types*, Journal of Symbolic Logic, **15** (1950), pp. 81-91.
- [8] L. HENKIN, *A generalization of the concept of ω -completeness*, Journal of Symbolic Logic, **22** (1957), pp. 1-14.
- [9] G. E. HUGHES - M. J. CRESSWELL, *An Introduction to Modal Logic*, London, Methuen and Co., Ltd., 1968.
- [10] E. G. OMODEO, *The elimination of descriptions from A. Bressan's modal language ML' on which the logical calculus MC' is based*, Rend. Sem. Mat. Univ. Padova, **56** (1977), pp. 269-292.
- [11] Z. PARKS, *Investigations into quantified modal logic - I*, Studia Logica, **35** (1976), pp. 109-125.

Manoscritto pervenuto in redazione l'11 dicembre 1979.