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A Characterization of Discrete Linearly Compact Rings by Means of a Duality.

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1. Introduction.

All rings considered in this paper have a non zero identity and all modules are unitary.

A ring A is said to have a right Morita duality if there exists a faithfully balanced bimodule ${}_RK_A$ such that ${}_RK$ and K_A are injective cogenerators of R-Mod and Mod-A respectively. This means that the subcategories of R-Mod and Mod-A consisting of K-reflexive modules are both finitely closed and contain all finitely generated modules.

It is well known (see Müller [4]) that if A has a right Morita duality then A is right linearly compact (in the discrete topology). The converse of this result is false for non commutative rings (see Sandomierski [5]) while for commutative rings the question is still open and seems to be hard to solve (see Müller [4], Vamos [6], [7]).

The purpose of this paper is to show that a ring A is right linearly compact if and only if A has a *good duality.*

This means that there exists a faithfully balanced bimodule ${}_RK_A$ such that K_A is a cogenerator of Mod-A and ${}_RK$ is strongly quasi-injective. This means also that there exists a duality between Mod-A and the category of K-compact left R-modules (see section 2 below).

In particular it is shown that, if A is linearly compact, then such a duality may be induced by the minimal cogenerator of Mod-A.

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Furthermore we prove that if a ring A is Morita equivalent to a right linearly compact ring then A is such.

Finally we give a description of the basic ring of a linearly compact ring (which is semiperfect) by means of a representation property.

2. Strongly quasi-injective modules and good dualities.

In this section we recall some known facts which will be useful later.

2.1. Let A, R be two rings and ${}_RK_A$ a faithfully balanced bimodule (right on A and left on R). This means that $A \cong \mathrm{End}\,({}_RK)$ and $R \cong \mathrm{End}\,(K_A)$ canonically.

Denote by Mod-A the category of right A-modules and by R-LT the category of linearly topologized Hausdorff left R-modules over the ring R endowed with the ${}_RK$-topology. This ring topology on R is obtained by taking as a basis of neighbourhoods of zero in R the annihilators of the finite subsets of K.

In the following ${}_RK$ will have the discrete topology. Then ${}_RK \in R$-LT.

Let M be a module belonging to Mod-A (to R-LT). A *character* of M is a morphism of M in K_A (a continuous morphism of M in ${}_RK$).

Let $M \in$ Mod-A; we define the *character module* M^* of M as the left R-module $\mathrm{Hom}_A\,(M, K_A)$ endowed with the *finite topology*. This topology has as a basis of neighbourhoods of 0 in M^* the submodules

$$W(F) = \{\xi \in \mathrm{Hom}_A\,(M, K_A) \colon \xi(F) = 0\}$$

where F is a finite subset of M. Then $M^* \in R$-LT and it is K-*compact.*

Recall that a module $M \in R$-LT is K-compact if it is topologically isomorphic to a closed submodule of a topological product of copies of ${}_RK$. Let $\mathcal{C}({}_RK)$ be the subcategory of R-LT consisting of K-compact modules. Clearly M is K-compact if and only if M is complete and its topology coincides with the weak topology of characters.

Let $M \in R$-LT. The character module M^* of M is simply the abstract A-module $\mathrm{Chom}_R\,(M, {}_RK)$. M^* is K-*discrete.* Recall that a right A-module is K-discrete if it is isomorphic to a submodule of a product of copies of K_A. Denote by $\mathfrak{D}(K_A)$ the category of K-discrete modules. A module $M \in$ Mod-A is K-discrete if and only if $\mathrm{Hom}_A(M, K_A)$ separates points of M.

Let $\Delta_1 \colon \mathfrak{D}(K_A) \to \mathcal{C}({}_RK)$ be the contravariant functor that associates to each K-discrete module M its character module M^* and to each morphism in $\mathfrak{D}(K_A)$ its transposed morphism. The functor $\Delta_2\colon \mathcal{C}({}_RK) \to \mathfrak{D}(K_A)$ is defined in a similar way.

We say that $\Delta_K = (\Delta_1, \Delta_2)$ is a *good duality* if:

1) Δ_K is a duality in the sense that for every K-discrete and every K-compact module M the canonical morphism $\omega_M\colon M \to M^{**}$ is an isomorphism in the corresponding category.
2) The category $\mathcal{C}({}_RK)$ has the extension property of characters, *i.e.* for every topological submodule L of a module $M \in \mathcal{C}({}_RK)$, any character of L extends to a character of M.

If Δ_K is a duality and $\mathfrak{D}(K_A) = \text{Mod-}A$ then Δ_K is necessarily good (cf. [3], Prop. 1.11).

Looking for conditions in order that Δ_K be a good duality, leads us to consider strongly quasi-injective modules.

The module $M \in R\text{-Mod}$ is said *strongly quasi-injective* (s.q.i. for short) if for every submodule $L \leqslant {}_RM$ and every $x_0 \in M \setminus L$, any morphism $\xi\colon L \to {}_RM$ extends to an endomorphism $\bar{\xi}$ of ${}_RM$ such that $(x_0)\bar{\xi} \neq 0$. In particular ${}_RM$ is quasi-injective.

Recall that a module $M \in R\text{-Mod}$ is a *selfcogenerator* if for every $n \in \mathbb{N}$, given a submodule L of M^n and an element $x_0 \in M^n \setminus L$, there exists $f \in \operatorname{Hom}_R(M^n, M)$ such that $(L)f = 0$, $(x_0)f \neq 0$.

2.2 PROPOSITION ([2], Lemmata 2.1 and 2.5). *A module $M \in R$-Mod is strongly quasi-injective if and only if M is a quasi-injective selfcogenerator.*

Let $\mathcal{F}$ be the filter of open left ideals in the ${}_RK$-topology of R. Put

$$\mathcal{T}_{\mathcal{F}} = \{M \in R\text{-Mod}\colon \operatorname{Ann}_R(x) \in \mathcal{F},\ \forall x \in M\}.$$

The modules belonging to $\mathcal{T}_{\mathcal{F}}$ will be called $\mathcal{F}$-torsion modules. The $\mathcal{F}$-torsion submodule of a module $M \in R\text{-Mod}$ will be denoted by $t_{\mathcal{F}}(M)$. For every $M \in R\text{-Mod}$ $E(M)$ is the injective envelope of M.

Let $(S_\lambda)_{\lambda \in \Lambda}$ be a system of representatives of left $\mathcal{F}$-torsion simple modules and set $S_{\mathcal{F}} = \bigoplus_{\lambda \in \Lambda} S_\lambda$.

2.3. THEOREM ([3], Theorem 6.7).

Let ${}_RK_A$ be a faithfully balanced bimodule. The following statements are equivalent.

(*a*) Δ_K *is a good duality between* $\mathcal{D}(K_A)$ *and* $\mathcal{C}({}_RK)$.

(*b*) ${}_RK$ *is strongly quasi-injective.*

(*c*) ${}_RK$ *is quasi-injective and contains a copy of* $S_{\mathcal{F}}$.

(*d*) ${}_RK$ *is quasi-injective and contains a copy of* $\bigoplus_{\lambda\in\Lambda} t_{\mathcal{F}}(E(S_\lambda))$.

(*e*) ${}_RK$ *is an injective cogenerator of* $\mathcal{C}_{\mathcal{F}}$.

(*f*) *For every* $M\in R\text{-}LT$, *for every closed submodule* L *of* M *and for every* $x_0\in M\setminus L$, *any character* ξ *of* L *extends to a character* $\bar{\xi}$ *of* M *such that* $(x_0)\bar{\xi}\neq 0$.

Recall that the socle Soc $({}_RM)$ of the module ${}_RM$ is the sum of the simple submodules of ${}_RM$.

Observe that Soc $({}_RK)$ is the sum of the annihilators in ${}_RK$ of the maximal left ideals of R. Then Soc $({}_RK)$, being fully invariant, is a submodule of K_A.

2.4. PROPOSITION ([3], Proposition 6.10).

Let ${}_RK$ *be a s.q.i. left* R*-module and let* $A=\text{End}\,({}_RK)$. *Then*

a) $\text{Soc}\,({}_RK)=\text{Soc}\,(K_A)$.

b) $\text{Soc}\,(K_A)$ *is an essential submodule of* K_A.

2.5. PROPOSITION ([3], Corollary 7.4).

Let ${}_RK$ *be a selfcogenerator and* $A=\text{End}\,({}_RK)$. *Then* $\text{End}\,(K_A)$ *is naturally isomorphic to the Hausdorff completion of* R *in its* ${}_RK$*-topology.*

2.6. REMARK. The theory of s.q.i. modules may be developed in the more general setting ${}_RK\in R$-Mod and $A=\text{End}\,({}_RK)$.

Let $\tilde{R}$ be the Hausdorff completion of R in the ${}_RK$-topology. Then ${}_RK$ is in a natural way a left $\tilde{R}$-module and the R-submodules of ${}_RK$ are $\tilde{R}$-submodules. Moreover $A=\text{End}\,({}_{\tilde{R}}K)$ and ${}_RK$ is s.q.i. iff ${}_{\tilde{R}}K$ is s.q.i. In this case $\text{End}\,(K_A)=\tilde{R}$ by Proposition 2.5 and thus ${}_{\tilde{R}}K_A$ is faithfully balanced.

Finally ${}_RK$-compact modules and ${}_{\tilde{R}}K$-compact modules are essentially the same.

For more information about s.q.i. modules and good dualities see [3].

3. Some useful results.

3.1. Let M be a linearly topologized Hausdorff left module over the discrete ring R. M is said to be *linearly compact* if any finitely solvable system of congruences $x \equiv x_i \bmod X_i$, where the X_i are closed submodules of M, is solvable.

R is left linearly compact if ${}_R R$ is such and multiplication is continuous.

We write d.l.c. for linearly compact in the discrete topology.

The following result is essentially due to Müller ([4], Lemma 4) and Sandomierski ([5], Corollary 2, pag. 342).

3.2. PROPOSITION. *Let ${}_R K$ be a selfcogenerator and let $A = \mathrm{End}\,({}_R K)$. Then:*

K_A is injective if and only if ${}_R K$ is linearly compact in the discrete topology.

(For a proof see [3], Theorem 9.4).

3.3. LEMMA. *Let ${}_R K$ be a selfcogenerator and let $A = \mathrm{End}\,({}_R K)$. Let L be a finitely generated submodule of a module $M \in \mathfrak{D}(K_A)$. Then every morphism of L in K_A extends to a morphism of M in K_A.*

PROOF. Let $\{x_1, \ldots, x_n\}$ be a set of generators of L and $f \in \mathrm{Hom}_A(L, K_A)$. Consider the subset B of K^n defined by:

$$B = \{(g(x_1), \ldots, g(x_n)) : g \in \mathrm{Hom}_A(M, K_A)\}\,.$$

Since $\mathrm{Hom}_A(M, K_A)$ is a left R-module, B is a submodule of ${}_R K^n$. Put $y = (f(x_1), \ldots, f(x_n))$.

We claim that $y \in B$. Suppose $y \notin B$.

Then there exists $\alpha \in \mathrm{Hom}_R(K^n, K)$ such that

$$B\alpha = 0\,, \qquad y\alpha \neq 0\,.$$

Then $\alpha = (a_1, \ldots, a_n)$ where $a_i \in A$, $i = 1, \ldots, n$.

For every $g \in \mathrm{Hom}_A(M, K_A)$ we have:

$$\sum_{i=1}^{n} g(x_i)a_i = \sum_{i=1}^{n} g(x_i a_i) = g\Big(\sum_{i=1}^{n} x_i a_i\Big) = 0\,,$$

thus $\sum_{i=1}^{n} x_i a_i = 0$ since $M \in \mathfrak{D}(K_A)$.

Therefore $y\alpha = \sum_{i=1}^{n} f(x_i)a_i = f\Big(\sum_{i=1}^{n} x_i a_i\Big) = 0$, contradiction.

3.4. Proposition. *Let ${}_RK_A$ be a faithfully balanced bimodule.*

a) If ${}_RK$ is a selfcogenerator and R is linearly compact in the ${}_RK$-topology, then K_A is quasi-injective.

b) If ${}_RK$ is a cogenerator, then ${}_RR$ is linearly compact in the discrete topology if and only if K_A is quasi-injective.

Proof. *a*) Let L be a submodule of K_A and $g \in \mathrm{Hom}_A(L, K_A)$. We have to show that g coincides with the left multiplication by an element of R.

Let $(L_i)_{i\in I}$ be the family of all finitely generated submodules of L. By Lemma 3.3 $g|L_i$ coincides with the left multiplication by an element $r_i \in R$. Consider the following system of congruences

$$r \equiv r_i \bmod \mathrm{Ann}_R(L_i)\,. \tag{1}$$

Obviously $\mathrm{Ann}_R(L_i)$ are closed left ideals in the ${}_RK$-topology of R and (1) is finitely solvable. Let r be a solution of (1). Then for every $i \in I$ and $x \in L_i$ we have $rx = r_i x = g(x)$.

b) Suppose that ${}_RR$ is linearly compact in the discrete topology. Then ${}_RR$ is linearly compact in any Hausdorff linear topology. Therefore K_A is quasi-injective. Suppose that K_A is quasi-injective and consider the finitely solvable system of congruences

$$r \equiv r_i \bmod J_i \qquad i \in I \tag{2}$$

where the J_i, $i \in I$, are left ideals of R. $L = \sum_{i\in I} \mathrm{Ann}_K(J_i)$ is a submodule of K_A. Define the A-morphism $g\colon L \to K_A$ by putting $g\Big(\sum_{i\in F} x_i\Big) =$

$= \sum_{i \in F} r_i x_i$ where F is a finite subset of I and, for every $i \in F$, $x_i \in \mathrm{Ann}_K(J_i)$.

Since (2) is finitely solvable, g is well defined. Indeed suppose $\sum_{i \in F} x_i = \sum_{i \in F} x_i'$. Then there exists $u \in R$ such that $r_i - u \in J_i$, $i \in F$. $\sum_{i \in F} (r_i - u) x_i = 0$ thus $\sum_{i \in F} r_i x_i = u \left(\sum_{i \in F} x_i\right)$ and similarly $\sum_{i \in F} r_i x_i' = u \sum_{i \in F} x_i'$.

Since K_A is quasi-injective g extends to an endomorphism $\bar{g}$ of K_A. $\bar{g}$ is the left multiplication by an element $r \in R$ so that we have for every $i \in I$ and $x \in \mathrm{Ann}_K(J_i)$, $g(x) = rx = r_i x$.

Therefore $r - r_i \in \mathrm{Ann}_R \mathrm{Ann}_K(J_i) = J_i$ since ${}_RK$ is a cogenerator.

Remark. The proof of the above proposition closely follows the methods of Müller [4].

4. The main theorem.

4.1. We say that a ring A has a (right) good duality if there exists a faithfully balanced bimodule ${}_RK_A$ such that K_A is a cogenerator of Mod-A and ${}_RK$ is strongly quasi-injective. This means that Δ_K is a good duality between Mod-A and $\mathcal{C}({}_RK)$.

We will prove that A is right d.l.c. if and only if A has a good duality.

By Proposition 3.4 *b*) we get the following

4.2. Lemma. *If A has a good duality then A is right d.l.c.*

When ${}_RK$ is s.q.i. Proposition 3.2 may be sharpened in the following way.

4.3. Proposition. *Let ${}_RK$ be a s.q.i. module and let $A = $* End $({}_RK)$. *Let $\mathcal{F}$ be the filter of open left ideals in the ${}_RK$-topology of R. Then the following conditions are equivalent.*

(*a*) *${}_RK$ is linearly compact in the discrete topology and* Soc $({}_RK)$ *is essential in ${}_RK$.*

(*b*) *K_A is an injective cogenerator of* Mod-A.

If these conditions are fulfilled then:

1) *${}_RK$ is a finite direct sum ${}_RK = \bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))$ where S_i are $\mathcal{F}$-torsion simple left modules.*

2) *A_A is linearly compact in the discrete topology.*

PROOF. By Remark 2.6 we may suppose that the bimodule $_RK_A$ is faithfully balanced.

$(a) \Rightarrow (b)$. K_A is injective by Propositions 2.2 and 3.2. Let S be a simple module in the category Mod-A and let us prove that $\mathrm{Hom}_A(S, K_A) \neq 0$. Consider the exact sequence

$$0 \to P \xrightarrow{i} A \to S \to 0$$

where P is a right maximal ideal of A and i is the canonical inclusion. Since K_A is injective we have the exact sequence

$$0 \to \mathrm{Hom}_A(S, K_A) \to {}_RK \xrightarrow{i^*} \mathrm{Hom}_A(P, K_A) \to 0 .$$

Suppose $\mathrm{Hom}_A(S, K_A) = 0$. Then i^* is a continuous isomorphism of the K-compact module $_RK$ onto the K-compact module P^*. Since $_RK$ is linearly compact $\mathrm{Soc}(_RK)$ is a direct sum of a finite number of simple modules and moreover $\mathrm{Soc}(_RK)$ is essential in $_RK$. It is well known, and easily checked, that in this case the unique Hausdorff linear topology on $_RK$ (which is algebraically isomorphic to P^*) is the discrete one. Thus i^* is a topological isomorphism. Since the functor $\Delta_1\colon \mathfrak{D}(K_A) \to \mathcal{C}(_RK)$ is a good duality, i is an isomorphism. Contradiction.

$(b) \Rightarrow (a)$. Since $_RK$ and K_A are both s.q.i. and by Propositions 2.4 and 3.2 the conclusion is reached.

Suppose now that conditions (a) and (b) are fulfilled.

1) $\mathrm{Soc}(_RK)$ is d.l.c. thus it is the direct sum of a finite family $\{S_1, \dots, S_n\}$ of left $\mathcal{F}$-torsion simple modules. For every $i = 1, \dots, n$ $_RK$ contains a copy of $t_{\mathcal{F}}(E(S_i))$ since $_RK$ is an injective object in $\mathcal{C}_{\mathcal{F}}$ (see Theorem 2.3). Thus $_RK$ contains a copy of $\bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))$. Put $K_0 = \bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))$ and let $E(K_0)$ be the injective envelope of K_0. Then $E(K_0) = \bigoplus_{i=1}^{n} E(S_i)$. The identity map on K_0 extends to a morphism $j\colon {}_RK \to E(K_0)$. Since $_RK$ is $\mathcal{F}$-torsion, $j(_RK) \leqslant t_{\mathcal{F}}(E(K_0))$ and since $t_{\mathcal{F}}(E(K_0)) = K_0$, $j(_RK) = K_0$. Thus K_0 is a direct summand of $_RK$ and contains the socle of $_RK$. Hence $_RK = K_0$.

Statement 2) follows from Proposition 3.4 b) since K_A is a cogenerator and $_RK$ is quasi-injective.

4.4. PROPOSITION. *Let A be a right d.l.c. ring, $J(A)$ the Jacobson radical of A, U_A the minimal cogenerator of* Mod-A, $R = \mathrm{End}\,(U_A)$. *Then*:

a) *$A/J(A)$ is a semisimple artinian ring, and thus* Mod-A *has only a finite number of non isomorphic simple modules, so that U_A is injective.*

b) *The bimodule ${}_RU_A$ is faithfully balanced.*

PROOF. *a*) By a well known result of Zelinski (cf. [8]), $A/J(A)$ is semisimple artinian, so that A has only a finite number of right maximal ideals. Since U_A is the direct sum of one copy of the injective envelope of each simple module, U_A is the direct sum of a finite number of injective modules, thus U_A is injective.

b) Since U_A is a selfcogenerator the endomorphism ring of ${}_RU$ is the Hausdorff completion of A in the K_A-topology by Proposition 2.5. On the other hand A is right d.l.c. so that A is complete in any right linear Hausdorff topology. Thus $A = \mathrm{End}\,({}_RU)$.

Recall that a module M is *finitely embedded* if M is an essential submodule of a finite direct sum of injective envelopes of simple modules.

4.5. LEMMA ([6], Lemma 1.3; [4], Lemma 2). *Let $\{M_i\}_{i\in I}$ and H be submodules of a d.l.c. module M. Suppose that $\bigcap_{i\in I} M_i \leqslant H$ and that M/H is finitely embedded. Then there exists a finite subset F of I such that $\bigcap_{i\in F} M_i \leqslant H$.*

4.6. LEMMA. *Let ${}_RK$ be quasi-injective and $A = \mathrm{End}\,({}_RK)$. Then ${}_RK$ is s.q.i. if and only if for every submodule L of ${}_RK$ it is* $\mathrm{Ann}_K\,\mathrm{Ann}_A\,(L) = L$.

4.7. THEOREM. *Let A be a ring, U_A the minimal cogenerator of* Mod-A, $R = \mathrm{End}\,(U_A)$. *The following conditions are equivalent*:

(*a*) *A is right linearly compact in the discrete topology.*

(*b*) *${}_RU$ is strongly quasi-injective and* $\mathrm{End}\,({}_RU) = A$.

(*c*) *Δ_U is a good duality between* Mod-A *and* $\mathcal{C}({}_RU)$.

(*d*) *A has a good duality on the right.*

(*e*) *For every faithfully balanced bimodule* ${}_TK_A$, *if* K_A *is a cogenerator then* ${}_TK$ *is quasi-injective.*

(*f*) $A = \mathrm{End}\,({}_TK)$ *where* ${}_TK$ *is a discrete linearly compact and strongly quasi-injective module with essential socle.*

Moreover:

1) *If condition* (*a*) *is fulfilled, then* A *is semiperfect,* U_A *is an injective cogenerator and* ${}_RU$ *is discrete linearly compact with essential socle.*

2) *If condition* (*f*) *is fulfilled, then* K_A *is an injective cogenerator of* Mod-A *and* ${}_TK$ *is a finite direct sum of modules of the form* $t_{\mathcal{F}}(E(S))$ *where* S *in an* $\mathcal{F}$*-torsion simple* T*-module.*

PROOF. (*a*) $\Rightarrow$ (*b*). $A = \mathrm{End}\,({}_RU)$ and U_A is an injective cogenerator by Proposition 4.4. Thus, by Proposition 3.4 *b*), ${}_RU$ is quasi-injective. Let L be a submodule of ${}_RU$ and let us show that $\mathrm{Ann}_U\,\mathrm{Ann}_A\,(L) = L$, from which it will follow that ${}_RU$ is s.q.i., by Lemma 4.6.

First of all observe that we have a good duality

$$\mathfrak{D}({}_RU) \underset{\Delta_2}{\overset{\Delta_1}{\rightleftarrows}} \mathcal{C}(U_A)$$

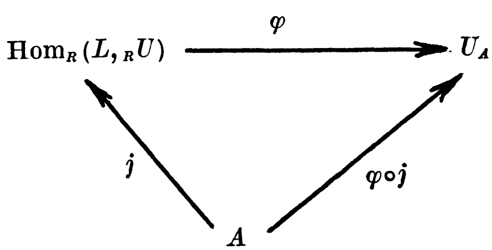
by Theorem 2.3 and since U_A is s.q.i.

Being L an ${}_RU$-discrete module, $\Delta_1(L)$ is $\mathrm{Hom}_A\,(L, {}_RU)$ endowed with the finite topology and $\Delta_2\Delta_1(L) = \mathrm{Chom}_A\,(\Delta_1(L), U_A)$. We claim that

$$\mathrm{Chom}_A\,(\Delta_1(L), U_A) = \mathrm{Hom}_A\,(\Delta_1(L), U_A)\,.$$

For every $a \in A$ denote by v_a the right multiplication by a in U. Since ${}_RU$ is quasi-injective every character of L is of the form $v_{a|L}$.

Let $\varphi \in \mathrm{Hom}_A\,(\Delta_1(L), U_A)$ and consider the diagram



where $j(a) = v_{a|L}$.

Obviously $\mathrm{Ann}_A(L) \leqslant \mathrm{Ker}(\varphi\circ j)$, thus

$$\bigcap_{y\in L} \mathrm{Ann}_A(y) \leqslant \mathrm{Ker}(\varphi\circ j) \leqslant A .$$

Now A is linearly compact discrete, $A/\mathrm{Ker}(\varphi\circ j)$ is a submodule of U_A and U_A is finitely embedded. Therefore it follows from Lemma 4.6 that there exists a finite subset F of L such that

$$\bigcap_{x\in F} \mathrm{Ann}_A(x) \leqslant \mathrm{Ker}(\varphi\circ j) .$$

Put $W(F) = \{\xi \in \mathrm{Hom}_R(L, {}_RU) \colon F\xi = 0\}$.

Note that $W(F) \leqslant \mathrm{Ker}\,\varphi$. Indeed if $F\xi = 0$ there exists $a \in A$ such that $\xi = v_{a|L}$ and $(\varphi\circ j)(a) = 0$. Therefore $\varphi(v_{a|L}) = 0$. Since $W(F)$ is an open submodule of $\Delta_1(L)$, φ is continuous.

Therefore

$$\Delta_2\Delta_1(L) = \mathrm{Hom}_A\big(\Delta_1(L), U_A\big) = \mathrm{Hom}_A\big(\mathrm{Hom}_R(L, {}_RU), U_A\big) .$$

Since ${}_RU$ is quasi-injective there exists the natural isomorphism

$$\psi\colon A/\mathrm{Ann}_A(L) \to \mathrm{Hom}_R(L, {}_RU)$$

given by $\psi(a + \mathrm{Ann}_A(L)) = v_{a|L}$.

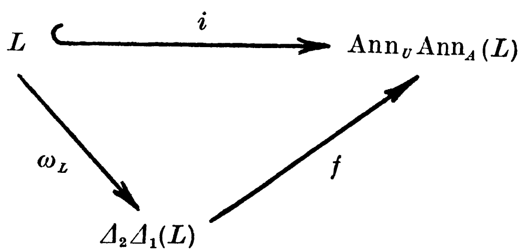
Using ψ we have the natural isomorphisms

$$\Delta_2\Delta_1(L) \xrightarrow{f_1} \mathrm{Hom}_A\left(\frac{A}{\mathrm{Ann}_A(L)}, U_A\right) \xrightarrow{f_2} \mathrm{Ann}_U \mathrm{Ann}_A(L) .$$

Putting $f = f_2\circ f_1$, f works as follows:

for every $\xi \in \Delta_2\Delta_1(L)$, $f(\xi) = (\xi\circ\psi\circ\pi)(1)$ where $\pi\colon A \to A/\mathrm{Ann}_A(L)$ is the canonical mapping.

Let us show that the diagram



is commutative, where i is the inclusion and ω_L is the natural morphism. Since Δ_U is a duality, ω_L is an isomorphism.

Let $x \in L$. It is

$$(f \circ \omega_L)(x) = (\omega_L(x) \circ \psi \circ \pi)(1) = \omega_L(x)[(\psi \circ \pi)(1)] =$$
$$= [(\psi \circ \pi)(1)](x) = [\psi(1 + \mathrm{Ann}_A(L))](x) = v_{1|L}(x) = i(x) .$$

(*b*) $\Rightarrow$ (*c*) is obvious.

(*c*) $\Rightarrow$ (*d*) is obvious.

(*d*) $\Rightarrow$ (*a*) follows from Lemma 4.2.

(*b*) $\Rightarrow$ (*f*) We know that $A = \mathrm{End}({}_B U)$ and that ${}_B U$ is s.q.i.

Since (*a*) $\Leftrightarrow$ (*b*), it follows from Proposition 4.4 that U_A is an injective cogenerator of Mod-A. Then by Proposition 4.3 ${}_B U$ is d.l.c. with essential socle.

(*f*) $\Rightarrow$ (*a*) By Proposition 4.3.

(*e*) $\Leftrightarrow$ (*a*) follows by Proposition 3.4 *b*).

1) Recall that A is semiperfect if $A/J(A)$ is semisimple artinian and the idempotents of $A/J(A)$ can be lifted in A.

If A_A is d.l.c. then $A/J(A)$ is semisimple artinian by Proposition 4.4. On the other hand, by (*f*), A is the endomorphism ring of a quasi-injective module, thus by a well known result the idempotents of $A/J(A)$ can be lifted in A.

2) Follows from Proposition 4.3.

Remark. The equivalence between conditions (*a*) and (*f*) has been found by Sandomierski ([5], Theorem 3.10 pg. 344). Moreover it is well known that a d.l.c. ring is semiperfect.

5. Further results.

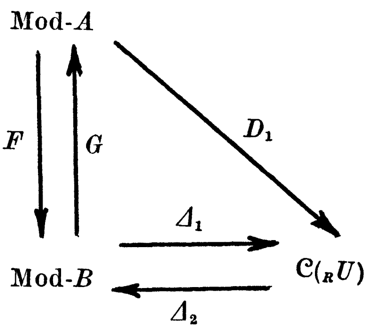
5.1. Proposition. *Let B and A be two Morita equivalent rings and suppose that B_B is discrete linearly compact. Then A_A is discrete linearly compact.*

Proof. This proposition may be obtained using some results of Sandomierski ([5], Corollary 1, pg. 336).

We give here a simple direct proof by means of good dualities.

Let Mod-$A \underset{G}{\overset{F}{\rightleftarrows}}$ Mod-B an equivalence and ${}_AP_B$ a faithfully balanced bimodule such that P_B and ${}_AP$ are both progenerators and $F = - \otimes_A P_B$, $G = \mathrm{Hom}_B(P_B, -)$ (see [1], Theorem 22.2).

Let U_B be the minimal cogenerator of Mod-B and $R = \mathrm{End}(U_B)$. Then by Theorem 4.7, Δ_U is a good duality. Consider the diagram



where $D_1 = \Delta_1 \circ F$.

Clearly D_1 is a duality and for every $M \in$ Mod-A

$$D_1(M) = \Delta_1\Big(M \otimes_A P_B\Big) = \mathrm{Hom}_B\Big(M \otimes_A P_B, U_B\Big) \cong$$

$$\cong \mathrm{Hom}_A(M, \mathrm{Hom}_B(P_B, U_B)) \cong \mathrm{Hom}_A(M, G(U_B)),$$

the isomorphisms being canonical and topological. Put $G(U_B) = K_A$. Since U_B is an injective cogenerator of Mod-B, K_A is an injective cogenerator of Mod-A (see [1], Proposition 21.6). Clearly $\mathrm{End}(K_A) = R$. Let us show that $\mathcal{C}({}_RK) = \mathcal{C}({}_RU)$ and that ${}_RK_A$ is faithfully balanced. It is $\Delta_1(P_B) = \mathrm{Hom}_B(P_B, U_B) \cong {}_RG(U_B) = {}_RK$ so that $\mathcal{C}({}_RK) \subseteq \mathcal{C}({}_RU)$. Moreover $A = \mathrm{End}({}_RK)$. In fact

$$\mathrm{End}({}_RK) = \mathrm{End}(\Delta_1(P_B)) \cong \mathrm{End}(P_B) = A, \quad \text{canonically}.$$

Thus ${}_RK_A$ is faithfully balanced.

Since P_B is projective and finitely generated, B is a direct summand of P_B^m where m is a positive integer. Then ${}_RU = \mathrm{Hom}_B(B, U_B)$ is a direct summand of $\mathrm{Hom}_B(P_B^m, U_B) = {}_RK^m$, therefore $\mathcal{C}({}_RU) \subseteq \mathcal{C}({}_RK)$.

Thus $\mathcal{C}({}_RK) = \mathcal{C}({}_RU)$.

We know that $D_1 = \mathrm{Hom}_A(-, K_A)$ endowed with the finite topology. On the other hand since K_A is a cogenerator, $\mathfrak{D}(K_A) = \text{Mod-}A$. Therefore, by 2.1, D_1 gives a good duality between Mod-A and $\mathcal{C}({}_RK)$. Thus, by Theorem 4.7, A_A is d.l.c.

5.2. Recall that a semiperfect ring B is a basic ring if $B/J(B)$ is a ring direct sum of division rings. It is well known (see [1], Proposition 27.14) that any semiperfect ring is Morita equivalent to a basic ring, which is unique up to isomorphisms. Our aim is to give a description of the basic ring of a right d.l.c. ring by means of a representation of it as endomorphism ring.

5.3. PROPOSITION. *Let A be a right d.l.c. ring, U_A the minimal cogenerator of* Mod-A, $R = \mathrm{End}(U_A)$, *$\mathcal{F}$ the filter of open left ideals in the ${}_RU$-topology of R. Then the basic ring B of A is isomorphic to the endomorphism ring of the minimal cogenerator of $\mathcal{C}_{\mathcal{F}}$, i.e.*

$$B \cong \mathrm{End}_R\left(\bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))\right)$$

where $(S_i)_{i=1,\dots,n}$ is a system of representatives of the non isomorphic simple $\mathcal{F}$-torsion left R-modules.

PROOF. Let $\{S_1, \dots, S_n\}$ be a system of representatives as above. Then by Theorem 4.7 and Proposition 4.3, A is the endomorphism ring of the left R-module

$${}_RU = \bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))^{m_i}$$

where m_i are suitable positive integers (in general >1, as may be showed by examples).

Put ${}_RK = \bigoplus_{i=1}^{n} t_{\mathcal{F}}(E(S_i))$, $B = \mathrm{End}({}_RK)$.

It is clear that ${}_RK$ is strongly quasi-injective, discrete linearly compact with essential socle.

Thus K_B is an injective cogenerator of Mod-B by Proposition 4.3. Note that ${}_RK_B$ is faithfully balanced since the ${}_RK$-topology of R coincides with the ${}_RU$-topology and using Proposition 2.5.

Moreover it is obvious that $\mathcal{C}({}_RK) = \mathcal{C}({}_RU)$. Since Δ_U is a good duality between Mod-A and $\mathcal{C}({}_RU) = \mathcal{C}({}_RK)$ and Δ_K is a good duality between Mod-B and $\mathcal{C}({}_RK)$, it follows that A and B are Morita equivalent so that B_B is d.l.c., hence semiperfect.

To conclude it is enough to show that $B/J(B)$ is a ring direct sum of division rings (see [1], Propositions 27.14 and 27.15).

For every $i = 1, \dots, n$ put $P_i = \mathrm{Ann}_B(S_i)$ and consider the exact sequence

$$0 \to S_i \to {}_RK \to {}_RK/S_i \to 0\,.$$

Since ${}_RK$ is quasi-injective and S_i is fully invariant in ${}_RK$, applying $\mathrm{Hom}_R(-, {}_RK)$ we get the exact sequence

$$0 \to P_i \to B \to \mathrm{End}_R(S_i) \to 0\,.$$

Then $D_i = B/P_i \cong \mathrm{End}_R(S_i)$ is a division ring and P_i is a maximal ideal of B.

We claim that $J(B) = \bigcap_{i=1}^{n} P_i$. It is clear that $J(B) \leqslant \bigcap_{i=1}^{n} P_i$. On the other hand let $b \in J(B)$. Since ${}_RK$ is quasi-injective, Ker (b) is essential in ${}_RK$, thus ker (b) contains $\bigoplus_{i=1}^{n} S_i$ which is the essential socle of ${}_RK$. Therefore $b \in \bigcap_{i=1}^{n} P_i$. Then $B/J(B)$ is the ring direct sum of the division rings D_i.

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