RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 64 (1981), p. 193-199
<http://www.numdam.org/item?id=RSMUP_1981__64__193_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 64 (1981)

Classification of Torsion Free Abelian Minimax Groups.

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SUMMARY - The Kurosh-Malcev-Derry method describing torsion free abelian groups of finite rank is used in modified form to get invariants of groups having only homomorphic images of finite rank, the sum of all p-ranks and the torsion free rank. Groups given by such invariants will be classified giving a necessary and sufficient condition to be isomorphic, and as application a method will be given to calculate numerically a quasi-isomorphism between algebraic groups or to exclude it.

1. Introduction.

Given any two torsion free abelian groups, briefly groups, by some description the classification problem arises to decide whether these groups are isomorphic or not. The following example shows that there are groups which are not known to be isomorphic. Let $\pi = 3{,}14\ldots$ be the circle number and $e = 2{,}71\ldots$ be the Euler number, let A and B be two groups of rank 1 given by the types $\boldsymbol{t}(A) := (3, 1, 4 \ldots)$, $\boldsymbol{t}(B) := (2, 7, 1 \ldots)$. Then A and B are isomorphic by [3; 85.1] if and only if $\pi - e = a/2^{\alpha}5^{\beta}$ rational and this is unknown. Obviously there are relations between the p-components of each of these types which cannot be handled. Therefore, whenever classification without considering such relations will be done, the restriction to minimax groups is indicated. A group of finite rank is called minimax group, if the quotient relative to a free abelian subgroup of maximal rank has only finitely many primary components not equal to 0.

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For algebraic torsion free abelian minimax groups a solution of the classification problem is given by theorem 3, firstly giving an arithmetical relation between two characteristics, i.e. descriptions of groups by lemma 1, of isomorphic groups, secondly presenting a method to calculate a quasi-isomorphism explicitely if it exists, see theorem 4.

2. Characteristics.

In order to describe groups the Kurosh-Malcev-Derry method [3; § 93] will be used in modified form. Let the localisation of the group A of finite rank n with maximal linearly independent elements $u_1, \dots, u_n$, called basis, be

$$u_1, \dots, u_n \in A \lneq \boldsymbol{Q}_p^* \otimes A = D_p \oplus R_p\,, \qquad \dim D_p = k_p\,,$$

where the natural embedding is taken. $\boldsymbol{Q}_p^*$ is the ring of p-adic integers ($\boldsymbol{K}_p^*$ the field of p-adic numbers) and the $\boldsymbol{Q}_p^*$-module $\boldsymbol{Q}_p^* \otimes A$ decomposes in a divisible part D_p and a (countably generated torsion free) reduced $\boldsymbol{Q}_p^*$-module R_p which is free by a theorem of Prüfer [3; 93.3]. A decomposition basis of the p-adic module can then be written as p-adic linear combination of the basis of the group.

LEMMA 1. *Let A be a torsion free abelian minimax group of rank n with $A/F \cong \bigoplus_{p \in P} \bigoplus_{k_p} Z(p^\infty)$ for a free abelian subgroup F (i.e. P is a finite set of primes and $1 \leq k_p \leq n$). Then there exist a basis $u_1, \dots, u_n$ of A and $\alpha_{pij} \in \boldsymbol{Q}_p^*$ where $1 \leq i \leq k_p < j \leq n$, $p \in P$, such that for all $p \in P$:*

$$\{u_i + \sum_{j > k_p}^{n} \alpha_{pij} u_j \,|\, 1 \leq i \leq k_p\}$$

is a basis of the divisible part of $\boldsymbol{Q}_p^* \otimes A$, uniquely determined by the basis $u_1, \dots, u_n$ of A.

Especially

$$A = \langle u_1, \dots, u_n, p^{-\infty}\Big(u_i + \sum_{j > k_p}^{n} \alpha_{pij} u_j\Big) \,|\, 1 \leq i \leq k_p,\ p \in P\rangle\,.$$

$\Big(A$ is generated by the given elements briefly noted using [3; § 88] i.e. $\Big\{p^{-m}\Big(u_i + \sum\limits_{j>k_p}^{n} \alpha_{pij}^{(m)} u_j\Big) | m \in \boldsymbol{N}\Big\}$ with the $(m-1)$-th partial sums $\alpha_{pij}^{(m)}$ of the standard expansion of the p-adic integers $\alpha_{pij}\Big)$.

Proof. There is a basis $u_1, \ldots, u_n$ of the minimax group A such that $A/\langle u_1, \ldots, u_n\rangle$ is a divisible torsion group like above.

Let $\boldsymbol{Q}_p^* \otimes A = \bigoplus\limits_{i=1}^{k_p} \boldsymbol{K}_p^* v_{pi} \oplus \bigoplus\limits_{j>k_p}^{n} \boldsymbol{Q}_p^* v_{pi}$ and obviously there are $\gamma_{pij} \in$ $\in \boldsymbol{Q}_p^*$ with $v_{pi} = \sum\limits_{j=1}^{n} \gamma_{pij} u_j$ and the sets $\{\gamma_{pij} | 1 \leqq j \leqq k_p\}$ can be supposed to contain a unit. Let be $S := \{p \in P | \gamma_{p11} \text{ unit}\}$, $q \in P \setminus S$, $w := \prod\limits_{p \in S} p$ and γ_{p1j} unit. Relative to the new basis $u_j' = u_j - wu_1$, $u_i' := u_i$ else, changing the γ_{pij} to the δ_{pij} where $\delta_{p11} = \gamma_{p11} + w\gamma_{p1j}$, δ_{p11} is a unit for $p \in S \cup \{q\}$. By induction can be assumed all γ_{p11} to be units. Further by row combinations $\gamma_{p11} = 1$, $\gamma_{pi1} = 0$ for all $p \in P$ and the sets $\{\gamma_{pij} | 1 < j \leqq k_p\}$ can be supposed to contain a unit where $1 < i \leqq n$.

Again by induction the given basis of the divisible parts of the modules $\boldsymbol{Q}_p^* \otimes A$ are obtained. Still easier for the (free) reduced parts R_p is gotten $R_p = \bigoplus\limits_{k>k_p} \boldsymbol{Q}_p^* u_k$.

By [3; 93.1 and 2] the group A can be proved to be generated by the given set of vectors. Finally the assumtion of a second basis of the divisible part of $\boldsymbol{Q}_p^* \otimes A$ given by numbers α_{pij}' leads at once to the contradiction $\bigoplus\limits_{k_p} Z(p^\infty) \neq A/\langle u_1, \ldots, u_n\rangle$.

Following lemma 1 $\mathrm{M} := \{(\alpha_{pij})_{1 \leqq i \leqq k_p < j \leqq n} | p \in P\}$ is called characteristic of the group A relative to the basis $u_1, \ldots, u_n$, briefly noted by $A = \langle u_1, \ldots, u_n | \mathrm{M}\rangle$ or $A \sim \mathrm{M}$.

3. Classification.

Lemma 2. Let $a_1, \ldots, a_n$ be rationals. There holds $a_1u_1 + \ldots + a_nu_n \in \langle u_1, \ldots, u_n | \mathrm{M}\rangle$ if and only if for all primes p and all j with $n \geqq j > k_p$ $(:= 0$ if $p \notin P)$ hold:

$$a_j - \sum_{i=1}^{k_p} a_i \alpha_{pij} \in \boldsymbol{Q}_p^* .$$

PROOF. Let $a = \sum a_p$ be a decomposition of a in partial fractions. Then $g = \sum_{j=1}^{n} a_j u_j \in \langle u_1, \dots, u_n | \mathrm{M} \rangle$ if and only if by lemma 1:

$$g = \sum_p \sum_{j=1}^{n} a_{pj} u_j = \sum_p \sum_{i=1}^{k_p} a_{pi} \Big(u_i + \sum_{j>k_p}^{n} \alpha_{pij}^{(l_{pj})} u_j \Big) + v$$

where $v \in \langle u_1, \dots, u_n \rangle$ and l_{pj} are non negative integers. Comparing the coefficients of $u_{k_p+1}, \dots, u_n$ the lemma is proved.

Two characteristics describe isomorphic groups if and only if they are characteristics of one and the same group relative to different basis. Therefore basis transformations have to be considered to classify groups.

THEOREM 3. *Let* $\mathrm{M} = \{(\alpha_{pij})_{1 \leq i \leq k_p < j \leq n} | p \in P\}$ *and* M' *be characteristics where* $n = n'$, $P = P'$ *and* $k_p = k_p'$ *for all primes.* M *and* M' *describe isomorphic groups if and only if there exists a regular rational* $n \times n$*-matrix* (a_{ij}) *such that with*

$$c_{pij} := a_{ij} + \sum_{k>k_p}^{n} \alpha_{pik} a_{kj} \qquad (1 \leq i \leq k_p,\ 1 \leq j \leq n,\ p \in P)$$

hold:

(i) *the rows of* (a_{ij}) *and of* $(a_{ij})^{-1}$ *satisfy the condition of lemma* 2 *relative to* M, M' *respectively.*

(ii) *For all primes* $p \in P$: $\det (c_{pij \leq k_p}) \neq 0$ *and*

$$(\alpha'_{pij}) = (c_{pij \leq k_p})^{-1} (c_{pij > k_p}) .$$

PROOF. The typical situation of a basis transformation is

$$A = \langle u_1, \dots, u_n, p^{-\infty}\Big(u_i + \sum_{j>k_p}^{n} \alpha_{pij} u_j\Big) | 1 \leq i \leq k_p,\ p \in P \rangle =$$
$$= \langle u_1', \dots, u_n', p^{-\infty}\Big(u_i' + \sum_{j>k_p}^{n} \alpha'_{pij} u_j'\Big) | 1 \leq i \leq k_p,\ p \in P \rangle ,$$

by lemma 1, where for $1 \leq i \leq n$

$$u_i' = \sum_{j=1}^{n} b_{ij} u_j , \quad u_i = \sum_{j=1}^{n} a_{ij} u_j' , \qquad (b_{ij}) = (a_{ij})^{-1} ,$$

and

$$A = \left\langle \sum_{j=1}^{n} a_{ij} u'_j ,\ p^{-\infty} \left[\sum_{j=1}^{n} \left(a_{lj} + \right.\right.\right.$$

$$\left.\left.\left. + \sum_{k>k_p}^{n} \alpha_{plk} a_{kj} \right) u'_j \right] \middle| 1 \leqq i \leqq n,\ 1 \leqq l \leqq k_p,\ p \in P \right\rangle .$$

Condition (i) follows by lemma 2, and the last form of A has to be transformed by the method proving lemma 1 but obviously without changing the basis $u_1, \ldots, u_n$, i.e. multiplication with the inverse of $(c_{pij \leqq k_p})$ from the left and the theorem is proved.

4. Isomorphism problem.

There is another aspect of the classification namely the calculation of an isomorphism between two minimax groups given by characteristics or the proof for these groups to be not isomorphic. In fact the basis of one of the groups has to be changed such that the resulting characteristic equals the second one. In view of theorem 3 (ii) where the algebraic dependence of the p-adic numbers α_{pij} and α'_{pij} is stated there can be expected a positive answer only for algebraic numbers (over the rationals), because the proof of the algebraic dependence of transcendent numbers is a number theoretical problem without general solution.

DEFINITION. A torsion free abelian minimax group is called algebraic if all the numbers α_{pij} occuring in the characteristic are algebraic (over the rationals).

That is an invariant of the group because basis transformations are algebraic transformations.

First the quasi-isomorphism problem will be settled i.e. whether two groups are quasi-isomorphic.

THEOREM 4. *Algebraic torsion free abelian minimax groups can be numerically classified up to quasi-isomorphism.*

PROOF. A method has to be given to calculate a matrix (a_{ij}) such that following theorem 3: $(c_{pij \leqq k_p})(\alpha'_{pij}) = (c_{pij > k_p})$ for all primes $p \in P$. Let γ be a primitive element of the algebraic number field $\boldsymbol{Q}(\alpha_{pij}|1 \leqq$

$\leqq i \leqq k_p < j \leqq n, p \in P) = \boldsymbol{Q}(\gamma)$. The number γ is a rational linear combination of the α_{pij}'s where only finitely many possibilities are excluded therefore γ can be calculated. By [5; lemmata 6, 7 and 8] the minimum polynomial of γ and the representations of the α_{pij}'s and α'_{pij}'s by rational linear combinations of powers of γ can be numerically obtained whenever the α_{pij}'s and α_{pij}'s are given by means of [5; propositions 4 and 5]. Substituting these expressions in the linearized form of theorem 3 (ii) noted above and using the minimum polynomial of γ to get only linearly independent powers of γ a homogeneous linear system of equations for the a_{ij} with rational coefficients is obtained having the coefficients of the powers of γ to be zero, and the general solution $a_{ij} = t \cdot a_{ij}(t_1, \ldots, t_r)$ with rational parameters t, $t_1, \ldots, t_r$ can be calculated i.e. the groups are quasi-isomorphic or the system can be proved to have no solution, i.e. the groups are not quasi-isomorphic.

The second part of the isomorphism problem is not solved here namely the proof whether two given quasi-isomorphic groups are isomorphic. I suppose the problem can be settled, especially for groups of rank 2, see also [1; 9.6], but not without a certain technical effort shown by some examples.

The following pairs of groups G, H are quasi-isomorphic by [2; 2.3]. Let be

$$G := \langle u, v, 5^{-\infty}(v + \sqrt{-1}\, u)\rangle\,, \qquad H := \left\langle G, \frac{v + 8u}{13}\right\rangle .$$

It can be explicitely calculated that

$$u' := \frac{2u - 3v}{13}\,, \qquad v' := \frac{3u + 2v}{13}$$

is a new basis of $H = \left\langle u', v', 5^{-\infty}(v + \sqrt{-1}\, u')\right\rangle \cong G$. Not isomorphic are groups G and H if

$$G := \langle u, v, 5^{-\infty}(v + \sqrt{-1}\, u)\,, \qquad H := \left\langle G, \frac{v}{13}\right\rangle$$

or

$$H := \left\langle G, \frac{v + xu}{7}\right\rangle$$

for all integers x, similarly

$$G := \langle u, v, 5^{-\infty}(v + \sqrt[3]{2}\,u)\rangle, \qquad H := \langle G, p^{-n}(v + xu)\rangle$$

with prime $p \neq 5$, n natural and x integer.

$\sqrt{-1}$ and $\sqrt[3]{2}$ are 5-adic integers by [5; Proposition 4]. See for the relevant basis transformations [4] or [6].

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Manoscritto pervenuto in redazione il 9 giugno 1980.