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Special Cases of Subproducts.

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1. Introduction.

In spite of the importance of subdirect products of modules we do not know much of their structure in general. An exception is a subdirect product $M = M_1 \mathbin{\underset{\sim}{\times}} M_2$ of two modules M_1, M_2. In that case there is a module F and epimorphisms $\alpha_i\colon M_i \to F (i = 1, 2)$, such that $M = \{(m_1, m_2) | \alpha_1 m_1 = \alpha_2 m_2\}$. For general subdirect products such a common factor module F does not exist. If however $M = \underset{\widetilde{i\in I}}{\times} M_i$ is a subdirect product of the M_i $(i \in I)$, and F a module with epimorphisms $\alpha_i\colon M_i \to F(i \in I)$, such that $M = \left\{(m_i)_{i\in I} \in \prod_{i\in I} M_i | \alpha_i m_i = \right.$ $\left. = \alpha_j m_j,\ \forall i, j \in I\right\}$, then M is called a *special subdirect product*, denoted by $M = \underset{\widetilde{i\in I}}{\times} M_i(\alpha_i, F)$.

If M is a submodule of the finite direct sum $M^* = \bigoplus_{i=1}^{k} M_i$, then M can be characterized in the following efficient way (¹): Define $F = M^*/M$, and $\alpha_i\colon M_i \to F$ by $\alpha_i(m_i) = m_i + M$; then an element $(m_1, m_2, \ldots, m_k) \in M^*$ belongs to M exactly if $\alpha_1 m_1 + \ldots + \alpha_k m_k = 0$. In other words: a submodule M of the finite direct sum M^* can be characterized by means of homomorphisms $\alpha_i\colon M_i \to F$ and equations of the form $\alpha_1 x_1 + \ldots + \alpha_k x_k = 0$.

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(¹) See: L. FUCHS - F. LOONSTRA, *On a class of submodules in direct products*, Rend. Accad. Naz. dei Lincei, Serie VIII, **60** (1976), pp. 743-748.

In the following we generalize this procedure for a set $\{M_i\}_{i\in I}$ of modules and epimorphisms $\alpha_i: M_i \to F$ onto an R-module F in case R is commutative.

Let R be a commutative ring $(1 \in R)$, $\{M_i\}_{i\in I}$ and F a nonempty set of non-zero unitary left R-modules, and $\alpha_i: M_i \to F (i \in I)$ a set of R-epimorphisms. Let $M^* = \prod_{i\in I} M_i$, and M the submodule of M^* defined as follows:

$$M = \left\{m^* = (m_i)_{i\in I} \in M^* \mid \sum_{i\in I} r_{ji}\alpha_i(m_i) = 0,\ j \in J\right\}, \tag{1}$$

where I and J are index sets. We suppose that for each $j \in J$ almost all r_{ji} are zero. The R-module M, define by (1) is called a *subproduct* of the M_i, denoted by

$$M = \left\{M_i;\ \alpha_i;\ F;\ \sum_{i\in I} r_{ji}\alpha_i(m_i) = 0;\ j \in J\right\}. \tag{2}$$

The relations

$$\sum_{i\in I} r_{ji}\,\alpha_i(m_i) = 0 \qquad (j \in J)$$

correspond with a homogeneous system of equations over F:

$$\sum_{i\in I} r_{ji}x_i = 0 \qquad (j \in J)\,. \tag{3}$$

We denote by

$$X = \langle \ldots, x_i, \ldots\rangle_{i\in I}\,, \qquad g_j = \sum_{i\in I} r_{ji}x_i \ (j \in J)\,, \qquad Y = \langle \ldots, g_j, \ldots\rangle_{j\in J}\,. \tag{4}$$

The solutions $(\ldots, f_i, \ldots)_{i\in I}$ of (3) form an R-module S and they correspond in a one to one way with the elements of the R-module

$$\operatorname{Hom}_R(X/Y, F)\,.$$

Indeed, if $(f_i)_{i\in I}$ satisfies (3), then there is a homomorphism $\varphi: X/Y \to F$, defined by

$$\varphi(x_i + Y) = \varphi(\bar{x}_i) = f_i \qquad (i \in I)\,.$$

The relations (3) assure that φ is well-defined. Conversely, if

$\psi \in \mathrm{Hom}_R(X/Y, F)$, then

$$\psi(\bar{x}_i) = f_i \qquad (i \in I)$$

determines a solution $(f_i)_{i\in I}$ of (3). We have

$$\text{(5)} \qquad \mathrm{Hom}_R(X/Y, F) \cong S\left\{(f_i)_{i\in I} \middle| \sum_{i\in I} r_{ji} f_i = 0;\ j \in J\right\}.$$

The elements $m = (m_i)_{i\in I} \in M$ are determined by the relations $\alpha_i(m_i) = f_i$ $(i \in I)$, if $(f_i)_{i\in I}$ is a solution of (3). The R-module S of all solutions $(f_i)_{i\in I}$ of (3) can be represented as

$$S = \underset{\widetilde{i\in I}}{\times} F_i,$$

where F_i is the submodule of F consisting of the i-components of all solutions $(f_i)_{i\in I}$ of (3). If $N_i = \alpha_i^{-1} F_i$, then

$$\text{(6)} \qquad M = \underset{\widetilde{i\in I}}{\times} N_i \left(\alpha_i;\ F_i;\ \sum_{i\in I} r_{ji} \alpha_i(n_i) = 0;\ j \in J\right).$$

The subproduct M, defined by (1) can be considered as the intersection of the one-relation subproducts $M^{(j)}$, where

$$\text{(7)} \qquad M^{(j)} = \left\{M_i;\ \alpha_i;\ F;\ \sum_{i\in I} r_{ji} \alpha_i(m_i) = 0,\ \text{fixed } j \in J\right\},$$

and

$$\text{(8)} \qquad M = \bigcap_{j\in J} M^{(j)}.$$

Let M be the subproduct (1) with corresponding system (3) of equations over F; using the same system $\{M_i;\ \alpha_i\colon M_i \to F\}_{i\in I}$ we may consider another system of relations $\sum_i r_{j'i} \alpha_i(m_i) = 0$ $(j' \in J')$ with the corresponding system of equations over F

$$\text{(9)} \qquad \sum_i r_{j'i} x_i = 0 \qquad (j' \in J').$$

Both systems of relations lead to the same subproduct M if any solution $(f_i)_{i\in I}$ of (3) is a solution of (9) and conversely. It is clear that a necessary and sufficient condition therefore is, that there exists an R-isomorphism

$$\nu\colon \mathrm{Hom}_R(X/Y, F) \cong \mathrm{Hom}_R(X/Y', F),$$

where $Y' = \langle \ldots, h_{j'}, \ldots \rangle$, $h_{j'} = \sum_i r_{j'i} x_i$, such that corresponding elements φ and $\nu(\varphi) = \varphi'$ have the property $\varphi(x_i + Y) = \varphi'(x_i + Y')$, $\forall i \in I$.

2. Relation between subproduct and subdirect product.

We formulate a relation between the modules $\mathrm{Hom}_R(X/Y, F)$ and $\mathrm{Hom}_R(X/Y_j, F)$, where $Y_j = \langle g_j \rangle$, $j \in J$. The elements $\varphi^{(j)} \in \in \mathrm{Hom}_R(X/Y_j, F)$ correspond in a one to one way with the solutions

$$(\ldots, f_i^{(j)}, \ldots)_{i \in I}$$

of the equation $g_j = 0$. If we take an element $\varphi^{(j)}$ of each module $\mathrm{Hom}_R(X/Y_j, F)$, where $\varphi^{(j)}$ corresponds with a solution of the equation $g_j = 0$,

$$\varphi^{(j)} \leftrightarrow (\ldots, f_i^{(j)}, \ldots)_{i \in I},$$

then the system

$$(\ldots, \varphi^{(j)}, \ldots, \varphi^{(k)}, \ldots), \qquad j, k \in J$$

defines an element of the subproduct M if and only if for any two indices $j, k \in J$ we have

$$\varphi^{(j)}(\ldots, \bar{x}_i, \ldots) = \varphi^{(k)}(\ldots, \bar{\bar{x}}_i, \ldots) = (\ldots, f_i, \ldots),$$

where

$$\bar{x}_i = x_i + Y_j, \qquad \bar{\bar{x}}_i = x_i + Y_k \qquad (\forall j, k \in J;\ \forall i \in I).$$

For in that case the corresponding system $(\ldots, f_i, \ldots)$ satisfies all equations (3). If we define the map

$$\beta^{(j)}\colon \mathrm{Hom}_R(X/Y_j, F) \to F \times F \times F \times \ldots \times F \times \ldots$$

by

$$\beta^{(j)}(\varphi^{(j)}) = (\ldots, f_i^{(j)}, \ldots)_{i\in I}, \qquad j \in J,$$

where $(f_i^{(j)})$ is the corresponding solution of $g_j = 0$, then $(\ldots, \varphi^{(j)}, \ldots)_{j\in J}$ defines a solution $(f_i)_{i\in I}$ of (3) if and only if

$$\beta^{(j)}\varphi^{(j)} = \beta^{(k)}\varphi^{(k)} \qquad (\forall j, k \in J).$$

Denoting by $H = \mathrm{Hom}_R(X/Y, F)$, $H_j = \mathrm{Hom}_R(X/Y_j, F)$, $j \in J$, then we find

2.1. $H = \mathrm{Hom}_R(X/Y, F)$ *is a special subdirect product* $H = \mathop{\widetilde{\times}}_{j\in J} H_j\left(\beta^{(j)}, \prod_{|I|} F\right)$ *of the* $H_j = \mathrm{Hom}_R(X/Y_j, F)$ *by means of the* $\beta^{(j)}$, $j \in J$.

To determine conditions therefore that M is a subdirect product $\mathop{\widetilde{\times}}_{i\in I} M_i$, we consider the equations (3), § 1; the solutions $(f_i)_{i\in I}$ have to form a subdirect product, i.e. that

$$S = \mathop{\widetilde{\times}}_{i\in I} F^{(i)}, \qquad \text{where} \qquad F^{(i)} = F \qquad \text{for } all\ i \in I.$$

Therefore, for any $f_i \in F$ there must be a homomorphism $\varphi \in \mathrm{Hom}_R(X/Y, F)$ such that $\varphi(x_i + H) = f_i$. If $\varphi: x_i + H \mapsto f_i$, then this map must induce a homomorphism $\varphi: \langle \bar{x}_i\rangle \to F$. We now formulate

2.2. *Let the R-module M be defined by* (2), § 1; *then M is a subdirect product* $M = \mathop{\widetilde{\times}}_{i\in I} M_i$ *if the following conditions are satisfied*:

$$\begin{array}{ccc} 0 \to \langle \bar{x}_i\rangle & \to & X/Y \\ \varphi\downarrow & & \swarrow \tilde{\varphi} \\ \langle f_i\rangle & & \\ \downarrow & & \\ F & & \end{array}$$

(*a*) $o(\bar{x}_i) \subseteq \mathrm{Ann}_R F \quad (\forall i \in I)$; (*b*) F *is injective*.

PR.: Mapping $\overline{x}_i = x_i + Y$ onto an element $f_i \in F$ there exists an R-homomorphism $\varphi\colon \langle \overline{x}_i \rangle \to F$ (by (a)), and φ has an extension $\tilde{\varphi}\colon X/Y \to F$ inducing φ. The two conditions (a) and (b) are therefore sufficient therefore that M is a subdirect product. If the conditions (a) and (b) are satisfied we see moreover that $\operatorname{Hom}_R(X/Y, F) \neq 0$.

2.3. *Necessary conditions therefore that any subproduct*

$$M = \Big\{M_i;\ \alpha_i;\ F;\ \sum_{i \in I} r_{ji}\alpha_i(m_i) = 0,\ j \in J\Big\}$$

defines a subdirect product $M = \underset{i \in I}{\widetilde{\times}} M_i$, *are*

$$(a')\quad o(\overline{x}_i) \subseteq \operatorname{Ann}_R F \quad (\forall i \in I); \qquad (b')\quad F \text{ is divisible}.$$

PR.: Since the equations (3), § 1 must have a solution with $x_i = f_i$, where f_i is any prescribed element of F, the map $\varphi\colon \overline{x}_i \mapsto f_i$ defines a homomorphism $\varphi\colon \langle \overline{x}_i \rangle \to F$, and that implies (a'). Choosing—in particular—a subproduct

$$M = \{M_1, M_2;\ \alpha_i\colon M_i \to F (i = 1, 2);\ r_1\alpha_1(m_1) + \alpha_2(m_2) = 0\},$$

the corresponding equation (over F) $r_1x_1 + x_2 = 0$ must be solvable for any $x_2 = f_2 \in F$, i.e. F must be divisible.

Summarizing the last two results we find

2.4. *Necessary and sufficient conditions therefore that any subproduct M, defined by means of a system* (2), § 1 *is a subdirect product, are*

$$(a)\quad o(\overline{x}_i) \subseteq \operatorname{Ann}_R F \quad (\forall i \in I), \qquad (b)\quad F \text{ is injective}.$$

We continue this § with the following question: suppose that the subproduct M, defined by (2), § 1, is a subdirect product $M = \underset{i \in I}{\widetilde{\times}} M_i$. Since $M = \bigcap_{j \in J} M^{(j)}$, where $M^{(j)}$ is a one-relation subproduct, defined by the j-th equation

$$\sum_i r_{ji}x_i = 0,$$

it is easy to prove that every $M^{(j)}$ is a subdirect product of the $\{M_i\}_{i \in I}$.

Using the notations of 2.1.: $H = \underset{j \in J}{\tilde{\times}} H_j \left(\beta^{(j)}, \prod_{|I|} F\right)$, where $\beta^{(j)}\varphi^{(j)} = (\ldots, f_i^{(j)}, \ldots)_{i \in I}$ is a solution of the j-th equation $\sum r_{ji} x_i = 0$. Now $\varphi \in H$ can be represented as

$$\varphi = (\ldots, \varphi^{(j)}, \ldots)_{j \in J},$$

with $\beta^{(j)}\varphi^{(j)} = \beta^{(k)}\varphi^{(k)}$ for all pairs $j, k \in J$. If $\varphi \leftrightarrow (\ldots, f_i, \ldots)_{i \in I}$, then by definition of $\beta^{(j)}$, all the components of $(\ldots, f_i^{(j)}, \ldots)$ must be—for each $i \in I$—the same as those of $(\ldots, f_i, \ldots)_{i \in I}$. But then all the $M^{(j)}$ are subdirect products. We have therefore

2.5. *If the subproduct* $M = \bigcap_j M^{(j)}$ *is a subdirect product, then all the one-relation subproducts* $M^{(j)}$ *are subdirect products of the* M_i $(i \in I)$.

We prove the converse: suppose $M = \bigcap_{j \in J} M^{(j)}$ defines a subproduct, and that each $M^{(j)}$ is a subdirect product

$$M^{(j)} = \left(\underset{i \in I}{\tilde{\times}} M_i\right)^{(j)}, \quad j \in J.$$

We prove that M is also a subdirect product of the M_i $(i \in I)$. M is completely determined by the R-module $\mathrm{Hom}_R(X/Y, F)$, since $\varphi \in \mathrm{Hom}_R(X/Y, F)$ determines a solution of the equations (3), § 1: $(\ldots, f_i, \ldots)_{i \in I}$ by means of $\varphi(\bar{x}_i) = \varphi(x_i + Y) = f_i$ $(i \in I)$. The modules $\mathrm{Hom}_R(X/Y_j, F)$ correspond (for each $j \in J$) with a subdirect product

$$\left(\underset{i \in I}{\tilde{\times}} F^{(i)}\right)^{(j)}, \quad F^{(i)} = F \qquad \text{for all } j \in J.$$

Any $\varphi \in H$ corresponds in a one to one way with

$$\varphi \leftrightarrow (\ldots, \varphi^{(j)}, \ldots, \varphi^{(k)}, \ldots)_{j,k \in J}, \tag{10}$$

where $\beta^{(j)}\varphi^{(j)} = \beta^{(k)}\varphi^{(k)}$ $(\forall j, k \in J)$.

Since every $\varphi^{(j)} \in H_j$ can occur as j-th component of an element $\varphi \in H$, and since H_j corresponds with a subdirect product $M^{(j)}$, any prescribed $f_i = f_i^{(j)}$ of F can occur as j-th component (corresponding to $\varphi^{(j)}$). But then the same f_i corresponds to every $\varphi^{(k)}$ in (10). That implies that M is a subdirect product of the M_i $(i \in I)$. The result is:

2.6. *The subproduct* $M = \bigcap_j M^{(j)}$ *is a subdirect product* $M = \underset{i \in I}{\widetilde{\times}} M_i$ *if and only if all the one-relation subproducts* $M^{(j)}$ $(j \in J)$ *are subdirect products.*

EXAMPLE. An interesting example of a subdirect product is the following one-relation subproduct of the two R-modules M_1 and M_2:

$$M = \{M_1, M_2;\ \alpha_i\colon M_i \to F\ (i = 1, 2);\ r_1\alpha_1(m_1) + r_2\alpha_2(m_2) = 0\}$$

$$\text{with } r_1F_1 = r_2F_2\,,$$

where α_1 and α_2 are epimorphisms. The last condition implies that M is a subdirect product of M_1 and M_2. Define

$$N_1 = \{m_1 \in M_1 | (m_1, 0) \in M\}\,, \qquad N_2 = \{m_2 \in M_2 | (0, m_2) \in M\}\,,$$

then $N_1 = \{m_1 \in M_1 | r_1\alpha_1(m_1) = 0$, etc. Define

$$F_1 = \{f_1 \in F | r_1 f_1 = 0\}\,, \qquad F_2 = \{f_2 \in F | r_2 f_2 = 0\}\,,$$

then $\alpha_1 N_1 = F_1$, $\alpha_2 N_2 = F_2$. Now it is easy to prove the isomorphism $\phi\colon M_1/N_1 \cong M_2/N_2$, defined by $\phi(m_1 + N_1) = m_2 + N_2$ if and only if $(m_1, m_2) \in M$. Moreover $F/F_1 \cong F/F_2$.

3. Essential subproducts.

We suppose that M is a subproduct (§ 1, (2)) of a *finite* number of R-modules $\{M_i\}$, $i = 1, 2, \ldots, k$, while the epimorphisms $\alpha_i\colon M_i \to F$ have kernels $\operatorname{Ker}(\alpha_i)$ which are closed in M_i,

$$\operatorname{Ker}(\alpha_i) \subseteq_{\mathrm{cl}} M_i \qquad (i = 1, \ldots, k)\,, \tag{11}$$

i.e. $\operatorname{Ker}(\alpha_i)$ has no proper essential extension in M_i. We want to study the conditions for M to be an essential subproduct of the M_i $(i = 1, \ldots, k)$. We know that

$$M \subseteq_{\mathrm{e}} M^* \leftrightarrow M \cap M_i \subseteq_{\mathrm{e}} M_i \qquad (i = 1, \ldots, k)\ \text{[2]}\,.$$

[2] F. LOONSTRA, *Essential submodules and essential subdirect products*, Symposia Math., **23** (1979), pp. 85-105.

$M \cap M_i$ is characterized by the fact that $m_l = 0$ $(l \neq i)$ and

$$r_{ji}\alpha_i(m_i) = 0 \qquad (\forall j \in J)\,;$$

this last condition is equivalent with

$$r_{ji}f_i = 0\,, \quad f_i = \alpha_i(m_i)\,, \quad (\forall j \in J)\,.$$

Defining for each $i = 1, \ldots, k$ the ideal L_i of R by

$$L_i = \langle r_{1i}, r_{2i}, \ldots, r_{ji}, \ldots \rangle_{j \in J}\,, \quad i = 1, \ldots, k\,,$$

and the submodule $F_i \subseteq F$ by

$$F_i = \{f \in F | L_i f = 0\}\,, \quad i = 1, \ldots, k\,,$$

we have

$$m_i \in M \cap M_i \Leftrightarrow \alpha_i(m_i) \in F_i\,, \quad i = 1, \ldots, k\,;$$

i.e. $M \cap M_i$ is characterized by

$$\alpha_i(M \cap M_i) = F_i\,, \quad i = 1, \ldots, k\,.$$

Since $\mathrm{Ker}\,(\alpha_i) \subseteq_{\mathrm{cl}} M_i$ $(i = 1, \ldots, k)$, it follows that

$$F_i \subseteq_{\mathrm{e}} F$$

since $M \cap M_i \subseteq_{\mathrm{e}} M_i$ $(i = 1, \ldots, k)$.

If therefore the subproduct M is defined by epimorphisms α_i with closed kernels, then

(12) $$M \cap M_i \subseteq_{\mathrm{e}} M_i \rightarrow F_i \subseteq_{\mathrm{e}} F \quad (i = 1, \ldots, k)\,.$$

Since the converse of (12) is always true, we find

(13) $$M \cap M_i \subseteq_{\mathrm{e}} M_i \leftrightarrow F_i \subseteq_{\mathrm{e}} F \quad (i = 1, \ldots, k)\,.$$

Summarizing we have the following result:

3.1. *Let* $M = \left\{M_i\ (i = 1, \ldots, k);\ \alpha_i;\ F \mid \sum_{i=1}^{k} r_{ji}\alpha_i(m_i) = 0;\ j \in J\right\}$ *be a subproduct of the* $M_1, \ldots, M_k$ *with epimorphisms* $\alpha_i\colon M_i \to F$, *such that the kernels* $\mathrm{Ker}(\alpha_i)$ *are closed in* M_i $(\forall i)$, L_i *the idal of* R *generated by the* $r_{1i}, r_{2i}, \ldots, r_{ji}, \ldots$ $(i = 1, \ldots, k)$, *and* F_i *the submodule of* F, *defined by* $F_i = \{f \in F \mid L_i f = 0\}$, $i = 1, \ldots, k$; *then* M *is an essential subproduct of the* $M_1, \ldots, M_k$, *if and only if* $F_i \subseteq_e F$ $(i = 1, \ldots, k)$.

Remark 1. Since $\mathrm{Ker}(\alpha_i) \subseteq_{cl} M_i$ and $M \cap M_i \subseteq_e M_i$, we see that F_i cannot be the zero submodule of F.

Suppose that $\{M_i \mid i = 1, \ldots, k;\ \alpha_i\colon M_i \to F\}$ is a finite system of R-modules and $\{\alpha_i\ (i = 1, \ldots, k)\}$ a system of epimorphisms, and les $\{F_i \mid i = 1, \ldots, k\}$ be a system of k essential submodules of F. Thit system $\{M_i; \alpha_i, F, F_i\}$ determines uniquely an essential subproduct M (of the M_i) such that M corresponds in the above sense with the prescribed submodules $F_i \subseteq_e F$ $(i = 1, \ldots, k)$. Indeed, for each of the submodules $F_i \subseteq_e F$ we define the ideal $L_i \subseteq R$ by

$$L_i = \{r_{ji} \in R \mid r_{ji} F_i = 0\}_{j \in J}.$$

That implies that—for each $j \in J$—we have a finite system of elements of R

$$\{r_{j1}, r_{j2}, \ldots, r_{ji}, \ldots, r_{jk}\}, \qquad j \in J.$$

We define a subproduct M as follows

$$M = \left\{m^* = (m_1, m_2, \ldots, m_k) \in M^* \mid \sum_{i=1}^{k} r_{ji}\alpha_i(m_i) = 0;\ j \in J\right\}.$$

Since $F_i \subseteq_e F$ $(i = 1, \ldots, k)$ it is now easy to see that the constructed subproduct M is an essential subproduct, for $F_i \subseteq_e F$ and $\alpha_i(M \cap M_i) = F_i$ implies $M \cap M_i \subseteq_e M_i$ $(i = 1, \ldots, k)$.

The corresponding one-relation subproducts $M^{(j)}$ are determined by the j-equation

$$\sum_{i=1}^{k} r_{ji} x_i = 0.$$

Remark 2. If R is a principal ideal ring, then $L_i = \langle r_i \rangle$, and that means that M can be described by means of *one* equation

$$r_1x_1 + \dots r_kx_k = 0\,.$$

3.2. *Let* $M = \left\{M_i\ (i = 1, \dots, k), \alpha_i; F \,\middle|\, \sum_{i=1}^{k} r_{ji}x_i = 0,\ j \in J\right\}$ *be a subproduct of the* $M_1, \dots, M_k$; *then*

(i) *The one-relation subproducts* $M^{(j)}$, $j \in J$ *are essential subproducts if* M *is an essential subproduct.*

(ii) *If the* $M^{(j)}$, $j \in J$, *are essential subproducts, and* J *is a finite set, then* M *is an essential subproduct.*

Proof: (i) This follows from the fact that $M \cap M_i \subseteq_e M_i$ $(i = 1, \dots, k)$ and the fact that $M \cap M_i \subseteq M^{(j)} \cap M_i \subseteq M_i$ $(i = 1, \dots, k)$ for all $j \in J$. Then $M^{(j)} \cap M_i \subseteq_e M_i$ $(i = 1, \dots, k)$ for all $j \in J$.

(ii) If $M^{(j)} \cap M_i \subseteq_e M_i$ $(i = 1, \dots, k)$ for all $j \in J$ (where J is finite!), then we have for the intersection

$$\bigcap_j (M^{(j)} \cap M_i) \subseteq_e M_i\,,$$

or $M \cap M_i \subseteq_e M_i$ $(i = 1, \dots, k)$.

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