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# The Law of Exponential Decay for Expanding Mappings.

A. Lasota (*) - James A. Yorke (**)

## Introduction.

The law of exponential decay has a long history. From its very beginning it was related with the theory of probability [1]. It was recently discovered that the phenomenon of the exponential decay may also be observed in deterministic dynamical systems working without any random perturbations. In the simplest case such a system may be described by a transformation $T: A \to X$ where $A$ is a subset of a measure space $(X, \mu)$. Suppose that our system « usually » works on $A$ which simply means that the measure of the set $T(A)\setminus A$ (or better, $A\setminus T^{-1}(A)$) is small. Then, of course, « on the average » starting with $x \in A$ we must wait a long time (large $n$) to observe $T^n(x) \notin A$. Now we may ask what is the probability of that event, in other words, what is the distribution of the random variable of ejection times,

$$\xi(x) = \inf\{n: T^{n+1}(x) \notin A\} .$$

This distribution can be found when a distribution of the initial points

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is given. We shall show, however, for any smooth initial distribution that if $\mu(A \setminus T^{-1}(A))$ is small (goes to zero), then the distribution of $\xi$ is always nearly exponential. The class of transformations $T$ for which we shall prove this theorem is restricted to so-called expanding mappings [2-5] and related examples but the results of Bowen-Ruelle [6, 7] and Franco-Sanchez [8] indicate that the same should be true for a fairly broad class of dynamical systems.

By proving the law of exponential decay for deterministic systems it is possible to explain many interesting phenomena. To demonstrate the potential scope of the problem, we shall mention one physical and one biological esample, both of which appear to satisfy the expotential decay law. These examples do not however satisfy our conditions on derivatives so their final resolution is indeterminate. The well known system of Lorenz equations [9] depends upon a parameter $R$ having the role of a Rayleigh number. For small values of $R$ the system has a stable attracting equilibrium but for $R$ sufficiently large admits chaotic trajectories. The transition value between chaotic and stable regions is $R_1 \simeq 24.06$ (see [10-12]). For values of $R$ slightly below $R_1$ the trajectories can appear chaotic for a while but then are seen to change behavior suddenly; they then damp down to one of the fixed points. The length of the time interval in which the trajectory is chaotic depends upon the initial conditions. When sufficiently many initial conditions are chosen, we can find the corresponding distribution of the lifetimes of chaos. Except for the part of the distribution corresponding to very short life times, it is always exponential. See [12], Figure 3.1.

Our second example is a highly idealized discrete model of blood cell renewal given by

$$u_{n+1} = (1-\theta)u_n + (0.4 + u_n)^8 e^{-u_n} \tag{0.1}$$

in which $u_n$ denotes the total mass of certain blood cells on day $n$ and $\theta$ is the coefficient of destruction [13]. In health $\theta$ is small (less than 0.1); in disease it is large. The severe disease ($\theta$ around 0.8) is characterized by chaotic behavior of trajectories. For $\theta \gg \theta_1 \sim 0.869$ the system dies (trajectories go quickly to zero) but for values $\theta$ slightly greater than $\theta_1$, the trajectories are chaotic for a while and then suddenly go to zero. Once more it can be found by numerical computation that the distribution of life times of chaos is exponential. This fact has an important consequence from the therapeutic point of view. It shows that even if the blood renewal system may be considered as deterministic, in some fatal diseases the life time of a patient is a random value. The patient appears moderately well for some time with cell count low because of rapid destruction. Cell production is high but oscillates irregularly. Then for no special reason, blood production suddenly fails and the patient dies. This scenerio may be of heuristic value in interpreting many medical crises.

The paper is divided into six parts. First we formulate our theorem in a special case for expanding mappings on the unit interval. This allows us to explain the main idea without going into technical details. In the next section we formulate assumptions concerning families of expanding mappings acting on an open subset $A \subset R^d$. Sections 4-6 are devoted to the proof. Specifically in Section 4 we recall some results from [4] concerning conditionally invariant measures. In Section 5 using a generalization of the Dini Theorem we prove uniform convergence of the sequence of iterates of the Frobenius-Perron operator and, finally, in Section 6 we complete the proof.

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## 1. Law of exponential decay for Rényi transformations.

A mapping $T$ of the closed interval $[0, 1]$ into itself will be called a *Rényi transformation*, if there exists a partition $0 = a_0 < \ldots < a_p = 1$ of the unit interval such that for each integer $i$ $(i = 1, \ldots, p)$ the restriction $T_i$ of $T$ to the open interval $(a_{i-1}, a_i)$ can be extended as a $C^2$ function to the closed interval $[a_{i-1}, a_i]$ and

$$(1.1) \qquad T_i([a_{i-1}, a_i]) = [0, 1], \text{ and } \inf_{(a_{i-1}, a_i)} |T'(x)| > 1 \qquad i = 1, \ldots, p .$$

A function $f\colon [0, 1] \to R$ (reals) will be called a *smooth density* if it is Lipschitzean and satisfies

$$\min f > 0 , \qquad \int_0^1 f \, dm = 1$$

where $m$ denotes the Lebesgue measure on $[0, 1]$.

THEOREM 1. Let $T$: $[0,1] \to [0,1]$ be a Rényi transformation and let $\varepsilon = 0$. Let $T_\varepsilon = (1+\varepsilon)T$ and

$$\xi_\varepsilon(x) = \inf\{n: T_\varepsilon^{n+1}(x) > 1,\ n = 0, 1, \ldots c\,.$$

Then there exists a constant $\sigma > 0$ such that for each smooth density $f$ the cumulative distribution

$$F_\varepsilon(z) = \mu_f\{x: \xi_\varepsilon(x) < z\}, \qquad d\mu_f = f\,dm$$

satisfies

$$\lim_{\varepsilon\to 0} F_\varepsilon(z/\varepsilon) = 1 - e^{-\sigma z}, \qquad z \geqslant 0\,. \tag{1.2}$$

This theorem is a consequence of a more general result which will be stated in Section 3. It is worthwhile to notice that the statement of Theorem 1 is quite elementary and does not require any notions of ergodic theory such as invariant measure, exactness and so on. The proof is, however, far from simple and is based on some delicate properties of (conditionally) invariant measures.

REMARK 1. Certain piecewise monotonic transformations with $|T'| > 1$ were studied by A. Rényi [14]. He proved these maps have an absolutely continuous invariant measure and it is unique. More general results in this area are given in [15-19].

AN UNSOLVED PROBLEM. Let $T_\varepsilon = (1+\varepsilon)4x(1-x)$ and define $\xi_\varepsilon$ and $F_\varepsilon$ as in Theorem 1. We conjecture that

$$\lim F_\varepsilon(z/\varepsilon^{\frac{1}{2}}) = 1 - e^{-\sigma z}$$

for some constant $\sigma > 0$.

## 2. Expanding mappings in $R^d$.

In what follows we shall consider mappings from an open set $A \subset R^d$ into $R^d$. We shall assume that

$$A = \bigcup_{i=1}^{p} A_i$$

where $A_i$ are disjoint open connected sets. Each $A_i$ is assumed to be « arc-wise bounded » which means that there is a number $\delta_i$ such that any two points in $A_i$ can be joined by a polygonal arc of length at most $\delta_i$.

We say a matrix $M$ is $\lambda$ *expanding* if $|M^{-1}| < \lambda^{-1}$ where $|M| = \sup\{|Mx| : |x| = 1\}$ and $|\cdot|$ stands for the norm in $R^d$. A mapping $T: A \to R^d$ will be called *expanding* if it is twice continuously differentiable on $A$, there is a uniform bound (say $\beta$) on all first and second derivatives, and if the following conditions are satisfied

$E1$: $A \subset T(A)$.

$E2$: If $x_i \to x_0 \in \partial A$ (where $\partial A$ denotes the boundary of $A$) and if $T(x_i) \to y$ for some $y$, then $y \notin A$.

$E3$: There exists $\lambda > 1$ such that for $x \in A$ the Jacobian matrix $DT(x)$ is $\beta$ expanding.

We say that an expanding mapping $T: A \to R^d$ is *exact* if there exists an integer $N$ such that $T^N(A_i) \supset A$ for all $i$.

A simple example of an expanding mapping is $T_\varepsilon = (1+\varepsilon)T$ where $T$ is a Rényi transformation and $\varepsilon \geqslant 0$. We need only to replace $[0, 1]$ by the union of open intervals $(a_{i-1}, a_i)$ in which $T$ is differentiable. The mappings $T_\varepsilon$ are also exact with $N = 1$.

REMARK 2. The expanding mappings described here were studied in [4] with a weaker assumption that $T$ is twice differentiable only on the set $A \cap T^{-1}(A)$. Since we shall consider families of transformations this assumption is somewhat inconvenient.

REMARK 3. The term « exact » was used by Rochlin [20] to describe an important and deep property of some measure preserving transformations (exact endomorphisms). Our definition is much simpler and from the formal point of view is more related to topology than to measure. These notions are, however, strongly related. For example a Rényi transformation becomes an exact endomorphism in the sense of Rochlin when an invariant measure is properly chosen (e.g. when absolutely continuous). The same is true for expanding mappings on manifolds [3].

A family of mapping $T_\varepsilon: A \to R^d$ $(\varepsilon \in [0, \varepsilon_0])$ will be called *uniformly expanding* if the following conditions are satisfied.

$F1$: For each $\varepsilon \in [0, \varepsilon_0]$ the mapping $T_\varepsilon$ is expanding and the constant $\beta$ in assumption $E3$ does not depend upon $\varepsilon$.

$F2$: The functions $T_\varepsilon(x)$ as well as the first derivatives with respect to $x$ depend continuously upon $x$. More precisely we assume that the functions $(\varepsilon, x) \to T_\varepsilon(x)$ and $(\varepsilon, x) \to DT_\varepsilon(x)$ are uniformly continuous (and consequently uniformly bounded) and $(\varepsilon, x) \to D^2T_\varepsilon(x)$ is uniformly bounded. Again we shall denote the upper bound of all first and second derivatives by $\beta$.

It is in fact easily shown that exactness for just one $\varepsilon$ in $[0, \varepsilon_0]$ implies we must have exactness for all $\varepsilon$ with the same $N$ because of property $E2$.

## 3. Law of exponential decay for expanding mappings.

As in Section 1, a function $f\colon A \to R$ will be called a *smooth density* if it is Lipschitzean (and so $\sup_A f < \infty$) and

$$\inf_A f > 0 \qquad \text{and} \qquad \int_A f\, dm = 1 . \tag{3.1}$$

If $f$ is a smooth density then the measure $\mu$ defined by $d\mu_f = f\, dm$ ($m$ = Lebesgue measure) will be called a *smooth measure.* (The restriction « $\inf_A f > 0$ » is necessary to get uniqueness.)

It was proved in [4] that for each expanding exact map $T\colon A \to R^d$ there exists unique smooth density $f$ such that $d\mu_f = f\, dm$ is « conditionally invariant ». This means that there is a constant $\alpha$ such that

$$\mu_f\big(T^{-1}(E)\big) = \alpha \mu_f(E) \tag{3.2}$$

for all Borel sets $E$. Notice $\alpha$ is the constant $\mu_f\big(T^{-1}(A)\big) \leqslant 1$. A conditionally invariant measure is invariant when $\alpha = 1$.

Now we are in a position to state our main result.

THEOREM 2. Let $T_\varepsilon\colon A \to R^d$ $\big(\varepsilon \in [0, \varepsilon_0]\big)$ be a family of uniformly expanding, exact mappings and let

$$\xi_\varepsilon(x) = \inf\{n\colon T_\varepsilon^{n+1}(x) \notin A,\ n = 0, 1, 2, \ldots\} \qquad \text{for } x \in A . \tag{3.3}$$

Assume, moreover, that there exists a limit

$$\sigma = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{B_\varepsilon} f_0\, dm > 0 \tag{3.4}$$

where $B_\varepsilon = A \setminus T_\varepsilon^{-1}(A)$ and $f_0\,dm$ is the unique smooth measure, (conditionally) invariant with respect to $T_0$. Then for each smooth density $f$ the cumulative distribution

$$F_\varepsilon(z) = \mu_f\{x\colon \xi_\varepsilon(x) < z\}\,, \qquad d\mu_f = f\,dm \tag{3.5}$$

(for $z \geqslant 0$) satisfies

$$\lim_{\varepsilon\to 0} F_\varepsilon(z/\varepsilon) = 1 - e^{-\sigma z} \qquad \text{for } z \geqslant 0\,. \tag{3.6}$$

The proof of Theorem 2 will be given in Section 6. Sections 4 and 5 are devoted to some auxiliary results. Now we shall make only two simple remarks. First observe that (3.4) implies

$$\lim_{\varepsilon\to 0} \int_{B_\varepsilon} f_0\,dm = 0$$

and, consequently, since $DT_\varepsilon(x)$ is uniformly continuous function of $\varepsilon$,

$$\int_{B_0} f_0\,dm = 0 \qquad \text{with } B_0 = A \setminus T_0^{-1}(A)\,.$$

From this it follows $\mu_{f_0}\big(T^{-1}(A)\big) = \mu_{f_0}(A)$ which means that $d\mu_{f_0} = f_0\,dm$ is actually invariant.

Comparing the statements of Theorems 1 and 2 we may notice that in the first case there is no assumption which is analogous to (3.4). This is simply because in the case of Theorem 1 (3.4) is automatically satisfied. In fact, the set $B_\varepsilon = [0,1] \setminus T_\varepsilon^{-1}\big([0,1]\big)$ consists of a finite number of intervals $J_{1\varepsilon}, \ldots, J_{p\varepsilon}$ of the form

$$J_{i\varepsilon} = \begin{cases} [\varphi_{i\varepsilon}(1), a_i] & \text{if } T_i' > 1\,, \\ [a_{i-1}, \varphi_{i\varepsilon}(1)] & \text{if } T_i' < -1\,, \end{cases}$$

where $\varphi_{i\varepsilon}$ is the inverse function to $T_\varepsilon$ restricted to $[a_{i-1}, a_i]$. Each of these intervals has the length

$$|J_{i\varepsilon}| = \varepsilon|c_i| + o(\varepsilon) \qquad \Big(\lim_{\varepsilon\to 0} o(\varepsilon)/\varepsilon = 0\Big)$$

where $c_i = -1/T_i'(a_{i'})$ and

$$a_{i'} = \begin{cases} a_{i'} & \text{if } T_i' > 1\,, \\ a_{i-1} & \text{if } T_i' < -1\,. \end{cases}$$

Therefore

$$\int_{B_\varepsilon} f_0\, dm = \sum_{i=1}^{p} \left| \int_{a_{i'}}^{a_{i'}+\varepsilon c_i} f_0\, dm \right| + o(\varepsilon) = \varepsilon \sum_{i=1}^{p} |c_i| f_0(a_{i'}) + o(\varepsilon)\,.$$

Thus (3.4) is satisfied with

$$\sigma = \sum_{i=1}^{p} |c_i| f_0(a_{i'})\,.$$

## 4. Invariant measures for expanding mappings.

In the proof of Theorem 2 we shall use some results from [4] concerning expanding maps in $R^d$. To make our proof readable we shall not « adapt » those results to our case but we shall recall them in the original version.

Let $T\colon A \to R^d$ be an expanding mapping. We shall denote by $P$ the conditional Frobenius-Perron operator corresponding to $T$. It will be defined in two steps. First for each integrable $f\colon A \to R$ we set

$$\bar{P}f = d(\mu_f \circ T^{-1})/dm \qquad (d\mu_f = f\, dm)\,. \tag{4.1}$$

Now denote by $\|\cdot\|_L$ the norm in the space $L^1(A)$ of integrable functions. If $\|\bar{P}f\|_L > 0$, we may pass to the second step setting

$$Pf = \frac{\bar{P}f}{\|\bar{P}f\|_L}\,. \tag{4.2}$$

Notice $\int_A Pf\, dm = 1$. From these definitions follows immediately the most important property of Frobenius-Perron operator $P$, namely

*FP*1: $Pf = f$ if and only if $f\, dm$ is a conditionally invariant measure.

Since $DT$ is nonsingular, for each point $x \in A$ there is a neighborhoof $U$ of $x$ such that $T$ admits a finite number of local inverse functions $\varphi_i\colon U \to A$ and $\bigcup_i \varphi_i(U) = T^{-1}(U)$ where $\varphi_i(U)$ and $\varphi_j(U)$ are disjoint for $i \neq j$. Thus on $U$ the operator $P$ has an explicit formula

$$Pf(x) = \sum_i |\det D\varphi_i(x)| (f \circ \varphi_i(x)) / \int_{T^{-1}(A)} f \, dm \,. \tag{4.3}$$

From this and condition $E2$ it follows that $Pf$ is continuous if $f$ is continuous. For each $f > 0$, we have also $Pf > 0$ (as well as $\overline{P}f > 0$), because $\det D\varphi_i = (\det DT)^{-1} \neq 0$. When $f$ is a smooth density much more can be shown. Define the «regularity of $f$» to be

$$\operatorname{Reg} f = \sup \left\{ \frac{|f'(x)|}{f(x)} : x \in A \right\} \qquad (f' = \text{gradient } f) \,.$$

Using (4.3) it can be shown (see [4]) that

$$\operatorname{Reg}(Pf) \leqslant \frac{1}{\lambda} \operatorname{Reg} f + \max_{i,x} \frac{|J_i'|}{|J_i|} \tag{4.4}$$

where $J_i = \det D\varphi_i$. From the implicit function theorem it follows

$$\frac{|J_i'|}{|J_i|} \leqslant \frac{|(\det DT)'|}{|\det DT|^2} \leqslant \frac{|(\det DT)'|}{\lambda^2} \,.$$

So $|J_i'|/|J_i|$ is uniformly bounded by a constant $M_1$ which depends only upon $\lambda$ and the upper bound $\beta$ of the first and second derivatives of $T$. We may rewrite ineq. (4.4) in the form

$$\operatorname{Reg}(Pf) \leqslant \frac{1}{\lambda} \operatorname{Reg} f + M_1 \,, \qquad M_1 = M_1(\lambda, \beta) \,. \tag{4.5}$$

Iteration of this inequality yields

$$\operatorname{Reg}(P^n f) \leqslant \frac{\lambda M_1}{\lambda - 1} + \frac{1}{\lambda^n} \operatorname{Reg} f \leqslant \frac{\lambda M_1}{\lambda - 1} + \operatorname{Reg} f \,. \tag{4.6}$$

Thus the regularity of $P^n f$ may be bounded by a constant which depends only upon $\lambda, \beta$ and regularity of $f$ but is independent of $n$ and the particular choice of the function $T$. We shall rewrite this as the second important property of the Frobenius-Perron operator

$$FP2: \ \operatorname{Reg}(P^n f) \leqslant M_2 , \qquad M_2 = M_2(\lambda, \beta, \operatorname{Reg} f) .$$

The convenience in the use of regularity lies in the fact that it gives an immediate estimate for $f$ and its first derivatives, namely if $\operatorname{Reg} f \leqslant \varrho$, then

$$f(x) \leqslant e^{\varrho\delta_i} f(y) \qquad \text{for } x, y \in A_i .$$

and consequently, since $\int_A f\, dm = 1$

$$f(x) \leqslant \alpha e^{\gamma\varrho} , \qquad |f'(x)| \leqslant \alpha e^{\gamma\varrho} \tag{4.7}$$

where $\gamma = \max \delta_i$ and $\alpha = \max \left(1/m(A_i)\right)$. In particular from $FP2$ it follows that the sequence $P^n f$ is uniformly bounded with the first derivatives bounded by a constant which depends upon $\lambda, \beta$ and $\operatorname{Reg} f$ only.

Denote by $(C(A), \|\cdot\|)$ the space of bounded continuous functions $f: A \to R$ with the supremum norm. Using (4.2) it can be easily shown (see the proof of Prop. 1 in [4]) that for any two smooth densities $h$ and $g$

$$\|P^n h - P^n g\| \leqslant \|h - g\| \frac{\|\bar{P}^n 1\|_L}{\|\bar{P}^n h\|_L} \left( \|P^n 1\| + \|P^n g\| \right) .$$

Now using $FP2$ on $P^n 1$ and the obvious inequalities

$$\frac{\|\bar{P}^n 1\|_L}{\|\bar{P}^n 1\|_L} \leqslant \frac{1}{\inf_A h} , \qquad \|P^n g\| \leqslant \|P^n 1\| \sup_A g$$

we obtain the third important property of $P$, namely

$$FP3: \ \|P^n h - P^n g\| \leqslant M_4 \|h - g\| , \qquad M_4 = M_4(\lambda, \beta, g, h)$$

where the constant $M_4$ depends only upon $\lambda, \beta$, $\inf_A h$ and $\sup_A g$.

We shall close this section recalling the main result from [4] which will be stated here as the following property.

*FP*4: If $T: A \to R^d$ is expanding and exact, then for any smooth density $f$ the sequence $P^n f$ converges uniformly to the unique (in the set of smooth densities) fixed point $f^*$ of the operator $P$. Moreover

$$\operatorname{Reg} f^* \leqslant \varrho_0 \overset{\text{def}}{=} \lambda M_1/(\lambda - 1) \quad \text{and} \quad \inf_A f^* \geqslant d \overset{\text{def}}{=} e^{-\gamma\varrho_0}/\sup_A |\det(DT^N)| .$$

## 5. Generalized Dini theorem and its applications.

Let $T_\varepsilon: A \to R^d$ $(\varepsilon \in [0, \varepsilon_0])$ be a family of exact and uniformly expanding mappings depending continuously on $\varepsilon$. Let $P_\varepsilon = \overline{P}_\varepsilon / \|\overline{P}_\varepsilon\|_L$ be the conditional Frobenius-Perron operator for $T_\varepsilon$. Since the values $\lambda$ and $\beta$ are the same for the whole family we may immediately apply properties *FP*2, *FP*3 and *FP*4.

Given a smooth density $f$ consider the sequence $f_{n\varepsilon} = P_\varepsilon^n f$. According to *FP*2 we have $\operatorname{Reg} f_{n\varepsilon} < M_2$ and consequently by (4.7)

(i) $\sup\limits_{n,\varepsilon} \|f_{n\varepsilon}\| < \infty$ and $\sup\limits_{n,\varepsilon} \|f'_{n\varepsilon}\| < \infty$.

Moreover, according to *FP*4

(ii) For each $\varepsilon \in [0, \varepsilon_0]$ there exists $\lim\limits_{n\to\infty} f_{n\varepsilon} = f_\varepsilon$ (the unique smooth density, fixed point of $P_\varepsilon$); for fixed $\varepsilon$ the convergence $f_{n\varepsilon}(x) \to f(x)$ is uniform in $x$.

From *FP*4 and (4.7) it follows also

(iii) $\sup_\varepsilon \|f_\varepsilon\| < \infty$, $\quad \sup_\varepsilon \|f_\varepsilon\| < \infty \quad$ and $\quad \inf_A f_\varepsilon \geqslant d$.

Now we shall show that

(iv) For each fixed $n$ the mapping $\varepsilon \to f_{n\varepsilon} \in C(A)$ is continuous.

In fact according to the definition of $f_{n\varepsilon}$ we have

$$f_{n+1,\varepsilon}(x) = \frac{\overline{P} f_{n\varepsilon}(x)}{\|\overline{P} f_{n\varepsilon}\|_L} . \tag{5.1}$$

For the operator $\overline{P}$ we have the explicit formula

$$\overline{P}_\varepsilon f(x) = \sum |\det D\varphi_{i\varepsilon}(x)| \big(f(\varphi_{i\varepsilon}(x))\big) \tag{5.2}$$

where $\varphi_{i\varepsilon}$ are locally inverse functions to $T_\varepsilon$. The mappings $(\varepsilon, x) \to T_\varepsilon(x)$ and $(\varepsilon, x) \to DT_\varepsilon(x)$ are uniformly continuous and $|\det DT_\varepsilon(x)| \geqslant \lambda$; thus from the implicit function theorem it follows that for each $\bar{x} \in A$ and $\bar{\varepsilon} \in [0, \varepsilon_0]$ the functions $(\varepsilon, x) \to \varphi_{i\varepsilon}(x)$ and $(\varepsilon, x) \to D\varphi_{i\varepsilon}(x)$ are defined and continuous in a sufficiently small neighborhood of $(\bar{\varepsilon}, \bar{x})$. Since the denominators in (5.1) are always positive, this implies that for each fixed $x$ and $n$ the mapping $\varepsilon \to f_{n\varepsilon}(x) \in R$ is continuous. By (i) all the function $x \to f_{n\varepsilon}(x)$ are bounded with the first derivatives and form a precompact set in $C(A)$. Thus the mapping $\varepsilon \to f_{n\varepsilon} \in C(A)$ is also continuous. □

Our next step is to show that

(v) The mapping $\varepsilon \to f_\varepsilon \in C(A)$ is continuous.

Suppose not. From (iii) it follows that $f_\varepsilon$ are equibounded and equicontinuous. Thus there exists a value $\varepsilon$ and a sequence $\varepsilon_n \to \varepsilon$ such that the corresponding sequence $f_{\varepsilon_n}$ converges uniformly to a function $g \neq f_\varepsilon$. Since $\inf\limits_{\varepsilon, x} f_\varepsilon(x) \geqslant d$, we have $g \geqslant d$ and since $f_{\varepsilon_n}$ are uniformly Lipschitzean, $g$ is also Lipschitzean. Thus $g$ is a smooth density. Now, according to the definition of $f_\varepsilon$ we have

$$\frac{\bar{P}_{\varepsilon_n} f_{\varepsilon_n}}{\|\bar{P}_{\varepsilon_n} f_{\varepsilon_n}\|_L} = f_{\varepsilon_n} . \tag{5.3}$$

The operators $\bar{P}_\varepsilon$ are contractive in $L^1$, therefore

$$\|\bar{P}_{\varepsilon_n} f_{\varepsilon_n} - \bar{P}_\varepsilon g\|_L \leqslant \|f_{\varepsilon_n} - g\|_L + \|(\bar{P}_{\varepsilon_n} - \bar{P}_\varepsilon) g\|_L . \tag{5.4}$$

Since $f_{\varepsilon_n}$ converges uniformly to $g$ the first term on the right hand side of (5.4) converges to zero. To evaluate the second term observe that, according to formula (5.2) $\bar{P}_{\varepsilon_n} g(x)$ converges, at least pointwise, to $\bar{P}_\varepsilon g(x)$. Moreover, according to (i)

$$\bar{P}_{\varepsilon_n} g \leqslant P_{\varepsilon_n} g \leqslant \text{const} .$$

Thus by Lebesgue dominated convergence, $\bar{P}_{\varepsilon_n} g$ converges to $\bar{P}_\varepsilon g$ in $L^1$. Finally, by (5.4) $\bar{P}_{\varepsilon_n} f_{\varepsilon_n}$ converges to $\bar{P}_{\varepsilon_0} f$ in $L^1$. Passing to the limit in (5.3) we obtain

$$\frac{\bar{P}_{\varepsilon_0} g}{\|\bar{P}_{\varepsilon_0} f\|_L} = g .$$

(The denominator is positive, since $\overline{P}_\varepsilon g \geqslant d\overline{P}_\varepsilon 1 > 0$.) Thus $g$ is a smooth fixed point of $P_\varepsilon$ different from $f_\varepsilon$ which contradicts $FP4$. □

Out next step is to prove that the convergence in (ii) is uniform with respect to $\varepsilon$. We start with a lemma which is a simple modification of the Dini theorem.

LEMMA 1. Let $G$ and $G_n$ $(n = 1, 2, \ldots)$ be continuous functions defined in a compact interval $S$ with the range in a Banach space $(B, \|\cdot\|)$. Assume that

$$\lim_{n\to\infty} \|G_n(t) - G(t)\| = 0 \qquad \text{pointwise for } t \in J \tag{5.5}$$

and that for each $t \in J$

$$\sup_{k\geqslant n} \|G_k(t) - G(t)\| \leqslant M\|G_n(t) - G(t)\| \qquad n = 1, 2, \ldots \tag{5.6}$$

where $M$ is a constant independent of $n$ and $t$. Then

$$\lim_{n\to\infty} \sup_t \|G_n(t) - G(t)\| = 0 \tag{5.7}$$

and consequently the functions $G_n(t)$ and $G(t)$ are equicontinuous.

PROOF. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/M$ and write

$$J_n = \{t \in J: \|G_n(t) - G(t)\| < \delta\}, \qquad K_n = J \setminus \bigcup_{j=1}^{n} J_j .$$

The sets $K_n$ are compact and $K_{n+1} \subset K_n$ $(n = 1, 2, \ldots)$. If for some $n$ the set $K_n$ is empty, then for each $t \in J$

$$\inf_{j\leqslant n} \|G_j(t) - G(t)\| < \delta$$

and consequently by (5.6)

$$\sup_{k\geqslant n} \|G_n(t) - G(t)\| \leqslant \varepsilon \qquad \text{for } t \in J .$$

which finishes the proof. If all $K_n$ are nonempty, then also

$$K = \bigcap_{n=1}^{\infty} K_n$$

is a nonempty set. We have

$$\|G_n(t) - G(t)\| \geqslant \delta$$

for all $n$ and $t \in K$. This contradicts (5.5). □

REMARK 4. Lemma 1 reduces to the classical Dini Theorem for $M = 1$.

Now we are in a position to prove that

(vi) $\lim\limits_{n\to\infty} \sup\limits_{\varepsilon} \|f_{n\varepsilon} - f_\varepsilon\| = 0.$

We know that the mappings $\varepsilon \to f_{n\varepsilon} \in C(A)$ and $\varepsilon \to f_\varepsilon \in C(A)$ are continuous (properties (iv) and (v)) and that for each $\varepsilon \in [0, \varepsilon_0]$ the sequence $f_{n\varepsilon}$ converges in $C(A)$ to $f_\varepsilon$ (property (ii)). Setting $B = C(A)$, $G_n(\varepsilon) = f_{n\varepsilon}$, $G(\varepsilon) = f_\varepsilon$ and $S = [0, \varepsilon_0]$ we may apply Lemma 1. It is necessary only to verify inequality (5.6). Since $f_\varepsilon$ is a fixed point of $P_\varepsilon$, we have

$$f_{n\varepsilon} - f_\varepsilon = P_\varepsilon^n f - P_\varepsilon^n f_\varepsilon$$

setting $g = P_\varepsilon^n f$, $h = P_\varepsilon^n f_\varepsilon = f_\varepsilon$ and using $FP3$ we obtain

$$\|P_\varepsilon^{n+k} f - P_\varepsilon^{n+k} f_\varepsilon\| \leqslant M_4 \|P_\varepsilon^n f - P_\varepsilon^n f_\varepsilon\|$$

or

$$\|f_{n+k,\varepsilon} - f_\varepsilon\| \leqslant M_4 \|f_{n\varepsilon} - f_\varepsilon\| , \qquad n, k = 1, 2, \ldots$$

which is equivalent to (5.6). Observe that in this case $M_4$ depends only upon $\lambda$, $\beta$, $\inf f_\varepsilon \geqslant d$ and $\sup\limits_{n,\varepsilon} f_{n\varepsilon} < \infty$ but not upon $n$ and $\varepsilon$.

We may summarize our results in the following.

PROPOSITION 1. *If the family $T_\varepsilon \colon A \to R^d$ $(\varepsilon \in [0, \varepsilon_0])$ is uniformly expanding and exact, then for each smooth density $f$ and each $\varepsilon \in$*

$\in [0, \varepsilon_0]$ the sequence $f_{n\varepsilon} = P_\varepsilon^n f \in C(A)$ is convergent to the unique (smooth density) fixed point $f_\varepsilon$ of the Frobenius-Perron operator $P_\varepsilon$ (corresponding to $T_\varepsilon$). This convergence is uniform in $\varepsilon$, that is (vi) holds and the function $\varepsilon \to f_\varepsilon \in C(A)$ is continuous.

## 6. Proof of the Theorem 2.

From the definition of $F_\varepsilon(z)$ (see (3.3), (3.5)) it follows immediately that

$$1 - F_\varepsilon(z/\varepsilon) = \mu_f\{x: \xi_\varepsilon(x) \geqslant n_\varepsilon\} = \mu_f(T_\varepsilon^{-n_\varepsilon}(A)) = \prod_{k=1}^{n_\varepsilon} \frac{\mu_f(T_\varepsilon^{-k}(A))}{\mu_f(T_\varepsilon^{-k+1}(A))}$$

where $n_\varepsilon$ is the smallest integer $\geqslant z/\varepsilon$. According to the definition of $\bar{P}_\varepsilon$ and $\mu_f$ we have (writing $\bar{f}_{k\varepsilon}$ for $\bar{P}_\varepsilon^k f$)

$$\mu_f(T_\varepsilon^{-k}(A)) = \int_{T_\varepsilon^{-k}(A)} f\, dm = \int_A \bar{f}_{k\varepsilon}\, dm = \|\bar{f}_{k\varepsilon}\|_L$$

and consequently

$$1 - F_\varepsilon(z/\varepsilon) = \prod_{k=1}^{n_\varepsilon} \frac{\|\bar{f}_{k\varepsilon}\|_L}{\|\bar{f}_{k-1,\varepsilon}\|_L} = \prod_{k=0}^{n_\varepsilon-1} \int_A \bar{P}_\varepsilon \frac{\bar{f}_{k\varepsilon}}{\|\bar{f}_{k\varepsilon}\|_L}\, dm\,.$$

where $f_{n\varepsilon} = P_\varepsilon^n f$. We obtain finally

$$1 - F_\varepsilon(z/\varepsilon) = \prod_{k=0}^{n_\varepsilon-1} \int_A \bar{P}_\varepsilon f_{k\varepsilon}\, dm = \prod_{k=0}^{n_\varepsilon-1} \int_{T_\varepsilon^{-1}(A)} f_{k\varepsilon}\, dm = \prod_{k=0}^{n_\varepsilon-1} \left(1 - \int_{B_\varepsilon} f_{k\varepsilon}\, dm\right) \tag{6.1}$$

where $B_\varepsilon = A \setminus T_\varepsilon^{-1}(A)$. Since $f_0 \geqslant d$, from (3.4) it follows

$$\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} m(B_\varepsilon) \leqslant \sigma/d\,. \tag{6.2}$$

Now defining

$$u_\varepsilon = \frac{1}{\varepsilon}\int_{B_\varepsilon} (f_\varepsilon - f_0)\, dm\,, \qquad v_{k\varepsilon} = \frac{1}{\varepsilon}\int_{B_\varepsilon} (f_{k\varepsilon} - f_\varepsilon)\, dm\,, \qquad w_\varepsilon = \frac{1}{\varepsilon}\int_{B_\varepsilon} f_0\, dm - \sigma$$

we have $\int\limits_{B_\varepsilon} f_{k\varepsilon}\, dm = \sigma\varepsilon + (u_\varepsilon + v_{k\varepsilon} + w_\varepsilon)\,\varepsilon$ and

$$\sum_{k=0}^{n_\varepsilon-1} \int\limits_{B_\varepsilon} f_{k\varepsilon}\, dm = \sigma z - \sigma(z - \varepsilon n_\varepsilon) + (u_\varepsilon + w_\varepsilon)\,\varepsilon n_\varepsilon + \varepsilon \sum_{k=0}^{n_\varepsilon-1} v_{k\varepsilon} \,. \tag{6.3}$$

According to the definition of $n_\varepsilon$, $\lim\limits_{\varepsilon\to 0} (z - \varepsilon n_\varepsilon) = 0$.

From (6.2) and Proposition 1 it follows that

$$\lim_{\varepsilon\to 0} u_\varepsilon = 0\,, \qquad \lim_{k\to\infty} \sup_{\varepsilon} |v_{k\varepsilon}| = 0\,, \qquad \lim_{\varepsilon\to 0} w_\varepsilon = 0\,.$$

Thus all the terms on the right hand side of (6.3) converge to zero except $\sigma z$, and consequently

$$\lim_{\varepsilon\to 0} \sum_{k=0}^{n_\varepsilon-1} \int\limits_{B_\varepsilon} f_{k\varepsilon}\, dm = \sigma z\,.$$

From this and (6.1) (taking in account that $n_\varepsilon \to \infty$ as $\varepsilon \to 0$) we have

$$\lim_{\varepsilon\to 0} \big(1 - F_\varepsilon(z/\varepsilon)\big) = e^{-\sigma z}\,. \qquad \square$$

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