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A. MONTANARO
A. BRESSAN

Contributions to foundations of probability calculus on the basis of the modal logical calculus MC^{ν} or MC_{*}^{ν}

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Contributions to Foundations of Probability Calculus on the Basis of the Modal Logical Calculus MC^{ν} or MC^{ν}_{*}.

A. Montanaro - A. Bressan (*)

Summary - In [3] an axiom system Σ for probability calculus, based on the modal logical calculus MC^{ν}—cf. [2]—was proposed; and the same was done with a more natural (equivalent) version Σ_{*} of it, based on the extension MC^{ν}_{*} of MC^{ν}, which includes propositional variables. Furthermore a precise axiom, [3, A12.8] was proposed in [3] as a substitutum for the existence rule, used e.g. by Reichenbach in [15] and some probabilsts —cf. [7]. These axioms have no set-theoretic character and in this respect comply with Freudenthal and De Finetti's views—cf. [8], [6]. In [3] both Σ and [3, A12.8] were said to need checking. In the present work, devided in three parts, first Σ (as well as Σ^{*}) and [3, A12.8] are positively checked in Parts 1 and 2 respectively: the main theorems expected to be provable on their basis have been effectively proved. In Part 3 the two main notions of random variables—so to say the physical one and the mathematical notion—are analysed (and defined) by means of modal concepts: extensional and absolute relations—cf. [2]. In addition probability spaces are defined and some existence theorems for them are proved.

(*) Indirizzo degli A.A.: Seminario Matematico, Università - Via Belzoni 7 - 35100 Padova.

PART 1

Basic Theorems of a Recent Modal Version of the Probability Calculus, Based on MC^{ν} or MC_{*}^{ν}.

1. Introduction (**).

Most actual approaches to probability calculus introduce a probability space at the outset, which is a particular measured space. Then the axioms are simply the purely mathematical conditions defining these spaces. These approaches, started by Kolmogorov, and the corresponding wide application of measure theory to probability calculus, were substantially (practically) very useful; and progress in this direction is still being made and divulged—cf. recent treatises such as [9] and [10], that deal with measures also on (infinite dimensional) functional spaces.

The afore-mentioned approaches are not much interested in the theorical systematization of the conditions under which in a given real situation a given probability space can be used or how it can be used. Likewise random variables are usually introduced in a purely

(**) This work is based on the dissertation (thesis) of A. Montanaro, made for his degree in mathematics in the research group of logic, under the direction of A. Bressan. Bressan's contribution to this work belongs to the sphere of activity of the CNR (Consiglio Nazionale delle Ricerche) in the academic years 1977-78 and 1978-79.

The new results presented in Part 1—i.e. mainly the modal theorems of the probability calculus PC^* started and proved in NN. 6, 7—are substantially due to A. Montanaro. Bressan re-exposed the whole matter in a more concise way; in particular in some cases he selected theorems, slightly changed some of them, and especially made some of their proofs shorter. The same can be said of the theorems connected with the existence axiom presented in Part. 2. However Bressan showed how to reduce his existence axiom (scheme), already proposed in [3] as A12.8, to a single axiom. Both axioms were simplified later by Montanaro into A9.1 and A9.1′ respectively.

In Part. 3 the analysis of the notion of (casual or) random variables is due to Bressan, whereas the definition of the probability spaces relative to a proposition and the theorems connected with it are due to Montanaro.

mathematical way as functions on the above spaces. Some authors such as Freudenthal and De Finetti criticize this a priori set-theoretic approach, admittedly from the didactical point of wiew—cf. the introduction of Part 3—and prefer an approach that takes a probability function $\mathcal{T}$ of propositions as a primitive—cf. [8], [6].

The present work happens to be in agreement with the above views in that, not by didactical purposes but by theorical reasons of completeness, we prefer approaches to probability calculus that (i) take as a primitive the probability $\mathcal{T}_{\alpha,\beta}$ of the proposition β relative to the proposition α, or something similar, and (ii) state existence axioms (1). Such a theory is proposed by Reichenbach in [15] on the basis of von Mises's theory [14], and is accepted also by some probabilists—cf. e.g. [7]. Reichenbach assumes as *a* primitive $(\forall i)(x_i \in A \ni_p y_i \in B)$—abbreviated by $A \ni_p B$—where A is a class of trials, B a class of results, x_i a trial, and y_i the corresponding result $(i = 1, 2, \ldots)$. Let $\mathcal{N}_{A,n}$ be the number of those among x_1 to x_n that are in A and let $\mathcal{N}_{A,B,n}$ be the number of the x_i's such that $x_i \in A$, $y_i \in B$. Then $A \ni_p B$ means that p is the limit of $\mathcal{N}_{A,B,n}/\mathcal{N}_{A,n}$ for $n \to +\infty$.

Reichenbach's frequentistic theory is based on extensional logic so that it is natural to assume the above limit to exist when the sequence x_i includes all trials in A and is ordered in a natural way (respecting time order).

The reference to extensional logic, and hence the restriction of the probability calculus to the real world, causes severe limitations on this theory, which are usually not spoken of (2). They are similar

(1) Our existence axioms for probabilities are conditional assertions stating the existence of certain probabilities in case certain other probabilities exist. These axioms do not at all compel us to believe that absolute objective probabilities exist. In certain situations (subjective or even objective) probabilities can be regarded to exist in that this is satisfactory relative to certain purposes. Likewise, for certain mechanical purposes it is satisfactory to admit the existence of Eucledian inertial spaces, according to classical physics, whereas for other purposes, related with general relativity, this admission is not possible.

(2) *Reichenbach's frequentistic version of the probability theory, based on extensional logic is incompatible with the existence for every* $\delta \in 1 \vdash\!\dashv 2$, *of the probability* p_δ *that a given ball* S *launched in a given way in a region where the air has the density* $\delta(A_\delta)$, *on a plane* Π, *may fall within the circle* Γ *in* $\Pi(B_\delta)$.

Indeed by the use of extensional logic the existence of p_δ implies the existence of a sequence $x_i(\in A_\delta)$ of the above launches of S in air of density δ. Every experiment $x_i(\in A_\delta)$ is localized in a 3-dimensional region $R_{i,\delta}$; and to the unacceptable defects of Hermes's extensional axiomatizations of particle mechanics according to Mach and Painlevé—cf. [11] (and [12]) criticized by Rosser—cf. [16]. These defects disappear when modalities are used—cf. [12], [1], and [2].

By the reasons above it was natural to use the recent constructions of general modal calculi, in particular MC^ν, to formulate a modal probability theory, the more so as probability is often asserted intuitively to be a degree (quantity) of possibility; and a set of axioms for such a theory was proposed in [3]; more, for the sake of simplicity MC^ν was extented to a calculus MC^ν_* having propositional variables and variables for relations and functions among propositions and other entities, and so on, so that a probability implication $\alpha \ni_p \beta$ similar to Reichenbach's can be introduced naturally. In [3] also a precise existence axiom to be substituted for the so called existence rule used by Reinchenbach and some probabilists such as Dore—cf. [15], p. 53, [7]—,was proposed. Furthermore, there it is said that the whole theory, which contains essentially modal axioms (having no extensional analogues), ought to be tested, especially in connection with the existence axiom. This demand is met in the Part 1 and 2 of the present work, in connection with the main consequences of the axioms different from the existence axioms, and those of all axioms respectively. In Part 3 the concept of random (or casual) variables is analyzed and a notion of probability spaces relative to a proposition α is defined. Especially Part 3 happens to agree to a certain extent with the afore-mentioned views of Freudenthal and De Finetti.

The present work differs considerally from the paper [17], which is based on an extensional infinitary language.

Now we describe the content of Part 1 in more detail. The analogues for Parts 2 and 3 will be done in the introductions of these parts. In NN. 2-4, the modal language ML^ν_* and the logical calculus MC^ν_* which is based on it, and extends the calculus MC^ν presented in [2], is briefly remembered from [3].

In N. 4 the class El of certain analogues defined in MC^ν for the elementary possible cases, and some theorems holding for it (also in MC^ν_*), are remembered from [2].

$R_{i,\delta} \cap R_{j,\delta'} = \emptyset$ for $i \neq j$ or $\delta \neq \delta'$. Then if p_δ existed for all $\delta \in 1 \mapsto 2$, the 3-dimensional space could be divided into a class of nonvanishing 3-dimensional regions, that has the continuum power, which is absurd.

In N. 5 axioms A5.1 to A5.7 for the probability calculus PC^*, $[PC]$ are substantially taken from [3]. Axiom A5.8 and the considerations on it are added. In N. 6 basic modal theorems of probability calculus on $\ni_p$ are stated and proved. The same is done in N. 7 with the probability function $\mathcal{P}_{\alpha,\beta}$.

2. Formation rules for ML_*^ν.

We want to consider the modal calculus MC_*^ν introduced in [3] and based on the modal language ML_*^ν, an extension of ML^ν—cf. [2]. To this purpose we denote the n-tuple formed whith a_1 to a_n by $\langle a_1, \ldots, a_n \rangle$ and we define recursively τ_*^ν, the set of types for ML_*^ν, by the conditions (α) and (β) below, where n runs over $Z^+ = \{1, 2, \ldots\}$:

(α) $\{0, 1, \ldots, \nu\} \subset \tau_*^\nu$, and

(β) *if* $t_0, t_1, \ldots, t_n \in \tau_*^\nu$, *then* $\langle t_1, \ldots, t_n, t_0 \rangle \in \tau_*^\nu$.

We call 0 the *sentence* type and 1 to ν the *individual* types. For $t_0, t_1, \ldots, t_n \in \tau_*^\nu$ and $t_0 = 0$ $[t_0 \neq 0]$ we say that $\langle t_1, \ldots, t_n, t_0 \rangle$ is a *relation* [*function*] type. Of course these names suggest how the various types will be used. Following [5] we use, for $t_0, \ldots, t_n \in \tau_*^\nu$, the notations

$$(2.1) \qquad \begin{cases} (t_1, \ldots, t_n) =_D \langle t_1, \ldots, t_n, 0 \rangle , \\ (t_1, \ldots, t_n : t_0) =_D \langle t_1, \ldots, t_n, t_0 \rangle \quad \text{for } t_0 \neq 0 \ (^3) . \end{cases}$$

We also define the set τ^ν recursively by the conditions (a) and (b) below, where n runs over Z^+:

(a) $\{1, \ldots, \nu\} \subset \tau^\nu$,

(b) *if* $t_0, \ldots, t_n \in \tau^\nu$ *and* $t_0 \in \tau^\nu \cup \{0\}$, *then* $\langle t_1, \ldots, t_n, t_0 \rangle \in \tau^\nu$.

Of course

$$(2.2) \qquad \tau^\nu \subset \bar{\tau}^\nu \subset \tau_*^\nu \quad \text{where} \quad \bar{\tau}^\nu = \tau^\nu \cup \{0\} .$$

(3) Hence in harmony with [5] $\langle 0, 0 \rangle$ is (0)—the assertion $\langle 0, 0 \rangle = (0:0)$ is meaningless. Thus e.g. negation is a property of propositions (and not a function among propositions).

The symbols of $ML^\nu_*[ML^\nu]$ are "("," ")", "$\sim$", "$\wedge$", "$\square$" (necessity), "$=$" (identity), the reversed iota "$\imath$" (for descriptions), the variables v_{tn} and constants c_{tn} where $n \in Z^+$ and $t \in \tau^\nu_*[\tau^\nu]$. The class $\mathcal{E}_t$ $(= \mathcal{E}^{*\nu}_t)$ of the *designators* or wfes (*well-formed expressions*) of type $t\,(\in \tau^\nu_*)$ for ML^ν_* can be defined by conditions (f_1) to (f_8) below (formation rules) where n runs over Z^+ and $t_0, t_1, \ldots, t_n$ run over τ^ν_*.

(f_1) v_{tn}, $c_{tn} \in \mathcal{E}_t$.

(f_2) *If* $\Delta, \Delta' \in \mathcal{E}_t$, *then* $(\Delta = \Delta') \in \mathcal{E}_0$.

(f_3) *If* $\Delta_i \in \mathcal{E}_{t_i}$ $(i = 0, \ldots, n)$ *and* $\Delta \in \mathcal{E}_{\langle t_1, \ldots, t_n, t_0 \rangle}$, *then* $\Delta(\Delta_1, \ldots, \Delta_n) \in \mathcal{E}_{t_0}$.

(f_4) *If* $\Delta \in \mathcal{E}_0$, *then* $(\imath v_{tn})\Delta \in \mathcal{E}_t$.

$(f_{5\text{-}8})$ *If* $\Delta, \Delta' \in \mathcal{E}_0$, *then* $\sim \Delta$, $(\Delta \wedge \Delta')$, $(v_{tn})\Delta$, $\square \Delta \in \mathcal{E}_0$.

In case $\Delta \in \mathcal{E}_t$, we say that Δ is a *matrix* or *wff* [*term*] for $t = 0$ $[t \neq 0]$; and if $t \neq 0$, Δ is said to be an *individual expression*, *attribute*, or *functor* according as t is a type for individuals, relations, or functions respectively.

The expressions of type t $(\in \bar{\tau}^\nu)$ for ML^ν are those for ML^*_ν where only symbols of ML^ν occur, and where "$=$" occurs only between terms as its arguments—cf. e.g. [2], p. 12.

The connectives $\wedge$, $\supset$, and $\equiv$, and the existence [possibility] sign $\exists$ [$\diamond$] are understood to be introduced in the usual way. The cohesive powers of (v_{tn}) (universal quantifier), $(\imath v_{tn})$ (descriptor operator), $\square$, $\sim$, $\wedge$, $\vee$, $\supset$, and $\equiv$ are considered to decrease in the written order; e.g.:

$$(2.3)\qquad (x)\alpha \wedge \beta \vee \gamma \quad \text{stands for} \quad \big([(x)\alpha] \wedge \beta\big) \vee \gamma \qquad (\alpha, \beta, \gamma \in \mathcal{E}_0)\,.$$

In addition we write $\bigwedge_{i=1}^{n} \alpha_i$, $\bigvee_{i=1}^{n} \alpha_i$, and $(\forall x_1, \ldots, x_n)\alpha$ for $\alpha_1 \wedge \ldots \wedge \alpha_n$, $\alpha_1 \vee \ldots \vee \alpha_n$, and $(x_1) \ldots (x_n)\alpha$ respectively; sometimes we drop $\wedge$; furthermore a dot may replace a left [right] handed parenthesis, the mate of which is to be restored at the farthest possible place within all pairs of parentheses that include the dot itself:

$$(2.4)\qquad [(x).\alpha \wedge .\beta \vee \gamma] \supset \beta \quad \text{stands for} \quad \{(x)[\alpha \wedge (\beta \vee \gamma)]\} \supset \beta\,.$$

We shall use $\in$, $\in^\cap$, $\in^\cup$, $=^\cap$, $\wedge^\cap$ to $\equiv^\cap$, and $\wedge^\cup$ to $\equiv^\cup$ in accordance with the definitions

$$(2.5)\qquad \Delta \in \Delta' \equiv_D \Delta'(\Delta) \quad \textit{if} \quad \Delta \in \mathcal{E}_t \quad \textit{and} \quad \Delta' \in \mathcal{E}_{(t)} \quad \textit{for} \quad t \in \tau^{\nu}_{*},$$

$$(2.6)\qquad \begin{cases} \Delta_1 =^{\cap} \Delta_2 \equiv_D \Box\, \Delta_1 = \Delta_2\,, & \Delta \in^{\cap} \Delta' \equiv_D \Box\, \Delta \in \Delta'\,, \\ \alpha \supset^{\cap} \beta \equiv_D \Box\, (\alpha \supset \beta)\,, & \end{cases}$$

$$(2.7)\qquad \begin{cases} \Delta_1 =^{\cup} \Delta_2 \equiv_D \Diamond \Delta_1 = \Delta_2\,, & \Delta \in^{\cup} \Delta' \equiv_D \Diamond \Delta \in \Delta'\,, \\ \alpha \wedge^{\cup} \beta \equiv_D \Diamond\, (\alpha \wedge \beta)\,. & \end{cases}$$

Bound variables are understood to be defined in the usual way in connection with both (v_{tn}) and $(\imath v_{tn})$ $(t \in \tau^{\nu}_{*},\ n \in Z^{+})$; and the same holds for free variables and for terms that are free for a variable in a wff—cf. e.g. [13], pagg. 47-48, closed matrices are called sentences.

CONVENTION 2.1. *If we denote a matrix by « $\Phi(x, y)$ » where « x » and « y » express variables, and then we use « $\Phi(\Delta, \Delta')$ », it is understood that $\Delta[\Delta']$ is a term free for $x[y]$ in $\Phi(x, y)$ and that $\Phi(\Delta, \Delta')$ is obtained from $\Phi(x, y)$ by substituting Δ for x and Δ' for y.*

DEF. 2.1. *A matrix α is said to be modally closed if it is constructed starting out from matrices of the form $\Box\, \beta$ by means of $\sim$, $\wedge$, $\Box$, and (v_{tn}).*

We define $\exists^{(1)}\, [\exists^{(1)\cap}]$ (*there is* [*is strictly*] *at most one*) and $\exists_1\, [\exists_1^{\cap}]$ (*there is exactly* [*strictly*] *one*) by

$$(2.8)\qquad \left\{\begin{matrix} (\exists^{(1)}x) \\ (\exists^{(1)\cap}\, x) \end{matrix}\right. \Phi(x) \equiv_D (\forall x, y)[\Phi\,(x)\Phi(y)] \supset \left\{\begin{matrix} x = \ y \\ x =^{\cap} y \end{matrix}\right. ,$$

$$\left\{\begin{matrix} (\exists_1 x) \\ (\exists_1^{\cap} x) \end{matrix}\right. \alpha \equiv_D (\exists x)\alpha_{\wedge} \left\{\begin{matrix} (\exists^{(1)}x)\alpha \\ (\exists^{(1)\cap}x)\alpha \end{matrix}\right. ,$$

the non-existing object by

$$(2.9)\qquad a^* =_D a^*_t =_D (\imath v_{t1})(v_{t1} \neq v_{t1})\,,$$

and the relational and functional lambda expressions by

$$(2.10)\qquad (\lambda x_1, \ldots, x_n)\, \alpha \equiv_D (\imath R)(\forall x_1, \ldots, x_n).R(x_1, \ldots, x_n) \equiv \alpha\,,$$

$$(2.11)\qquad (\lambda x_1, \ldots, x_n)\Delta =_D (\imath f)\ (\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) = \Delta\,,$$

where $\alpha \in \mathcal{E}_0$, $\Delta \in \mathcal{E}_{t_0}$ *with* $t_0 \neq 0$, x_1 *to* x_n *are distinct variables of the respective types* t_1 *to* t_n, *and* $R[f]$ *is the first variable of type* $(t_1, \ldots, t_n)$ $[(t_1, \ldots, t_n : t_0)]$ *that does not occur in* $\alpha[\Delta]$.

We can now define the *modal product* [*sum*] $R^{\cap}[R^{\cup}]$ of the relation R of type $(t_1, \ldots, t_n)$:

$$\begin{cases} R^{\cap} =_D (\lambda x_1, \ldots, x_n) \Box R(x_1, \ldots, x_n) \,, \\ R^{\cup} =_D (\lambda x_1, \ldots, x_n) \Diamond R(x_1, \ldots, x_n) \,. \end{cases} \tag{2.12}$$

3. An axiom system for the logical calculus MC^{ν}_{*} [MC^{ν}] based on ML^{ν}_{*} [ML^{ν}].

Modus ponens is the only inference rule for MC^{ν} and MC^{ν}_{*}. We now write a set of axioms for MC^{ν} which is taken substantially from [2, N. 12] and holds also for MC^{ν}_{*}. Then we add one axiom for MC^{ν}_{*}. Below α, β, and γ are arbitrary matrices and x, y, and z are distinct variables; furthermore types are always understood to be such that the written expressions are well formed.

An arbitrary strings of universal quantifiers and $\Box$ *s* is denoted by $(\Box)$, or by ( ) if the $\Box$*'s* are lacking. In any case it has the least cohesive power.

A3.1-3 $(\Box)\ \alpha \supset \alpha\alpha$, $(\Box)\ \alpha\beta \supset \alpha$, $(\Box)\ (\alpha \supset \beta) \supset \sim (\beta\gamma) \supset \sim (\gamma\alpha)$,

A3.4-5 $(\Box)\ (x)(\alpha \supset \beta) \supset .(x)\alpha \supset (x)\beta$, $(\Box)\ (\alpha \supset {}^{\cap}\beta) \supset (\Box\alpha \supset \Box\beta)$,

A3.6 $(\Box)\ \alpha \supset (x)\beta$ *if* x *is not free in* α,

A3.7 $(\Box)\ \alpha \supset \Box\alpha$ *if* α *is modally closed* [Def. 2.1],

A3.8 $(\Box)\ (x)\Phi(x) \supset \Phi(\Delta)$—*see convention* 2.1,

A3.9 $(\Box)\ \Box\alpha \supset \alpha$.

The next three axioms concern identity and descriptions.

A3.10-11 $(\Box)\ x = x$, $(\Box)\ x = y = z \supset z = x$,

A3.12 $(\Box)\ x = {}^{\cap} y \wedge \Phi(x) \supset \Phi(y)$—cf. Convention 2.1,

A3.13 $(a)\ \alpha(\exists_1 x)\alpha \supset (\imath x)\alpha = x$, $(b) \sim (\exists_1 x)\alpha \supset (\imath x)\alpha = a^*$.

The next two axioms concern quasi extensionality and A3.16-18 concern existence of relations and functions. We assume that $x_1, \ldots, x_n$, R, S, f, and g are distinct variables of the respective types $t_1, \ldots, t_n$, $t = (t_1, \ldots, t_n)$, t, $\theta = (t_1, \ldots, t_n \colon t_0)$, and θ $(t_0 \neq 0)$, that $\Delta \in \mathcal{E}_{t_0}$, and that $R[f]$ is not free in $\alpha[\Delta]$.

A3.14 $\quad (\Box)\ R = S \equiv (\forall x_1, \ldots, x_n).R(x_1, \ldots, x_n) \equiv S(x_1, \ldots, x_n)\,,$

A3.15 $\quad (\Box)\ f = g \equiv (\forall x_1, \ldots, x_n)f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n)\,,$

A3.16 $\quad (\Box)\quad (\exists R)(\forall x_1, \ldots, x_n).R(x_1, \ldots, x_n) \equiv \alpha\,,$

A3.17 $\quad (\Box)\quad (\exists f)\ (\forall x_1, \ldots, x_n)f(x_1, \ldots, x_n) = \Delta\,,$

A3.18 $\quad (\Box)\quad (\exists R)R^{\cap} = R^{\cup} = S\,.$

That we are dealing with an effectively modal language, or that at last two Γ-cases exist, is substantially asserted by

A3.19 $\quad (\exists x, y).\Diamond x = y \Diamond x \neq y\,.$

In the following conventional axioms on the non-existing object x_i means $v_{t_i i+1}$ $(i = 0, \ldots, n)$.

A3.20 $\quad (\Box)\ a_t^*(x_1, \ldots, x_n) \supset \bigwedge_{i=1}^{n} x_i = a_{t_i}^*$ *for* $t = (t_1, \ldots, t_n)\,,$

A3.21 $\quad (\Box)\ a_\theta^*(x_1, \ldots, x_n) \supset x_0 = a_{t_0}^*$ *for* $\theta = (t_1, \ldots, t_n \colon t_0)$ *with* $t_0 \neq 0.$

The axiom of choice—where e.g. $(\forall F, G \in \Psi)\alpha$ stands for $(\forall F, G)(F \in \in \Psi \wedge G \in \Psi \supset \alpha)$—reads as in extensional logic:

A3.22 $\quad (\exists F)(F \in \Psi) \wedge (\forall F, G \in \Psi)(\exists x \in F)[(\exists y \in G) y \in F \supset F = G] \supset$

$$\supset (\exists G)(\forall F \in \Psi)(\exists_1 x)(x \in F \wedge x \in G)\,.$$

One can postulate the existence of infinitely many individuals—see AS 45.1 in [2]—or one can be contented with the following

A3.23 $\quad (\exists v_{t1})\,\Box\, v_{t1} \neq a_t^* \quad (t = 1, \ldots, \nu)\,.$

Before stating the next axiom, which is AS 25.1 in [2], we must remember the definitions of *modally constant relations* of type $t = (t_1, \ldots, t_n)$ (MConst_t or briefly MConst), *modally separated relations* (MSep), and *absolute relations* (Abs) of type t:

$$(3.1) \qquad R \in \mathrm{MConst} \equiv_D R^{\cup} = R^{\cap} \text{—cf. } (2.12)\,,$$

$$(3.2) \qquad R \in \mathrm{Msep} \quad \equiv_D (\forall x_1, y_1, \ldots, x_n, y_n).R(x_1, \ldots, x_n) \wedge \wedge R(y_1, \ldots, y_n) \wedge \Diamond \bigwedge_{=1}^{n} x_i = y_i \supset \Box \bigwedge_{i=1}^{n} x_i = y_i\,,$$

$$(3.3) \qquad R \in \mathrm{Abs} \equiv_D R \in \mathrm{MConst} \wedge R \in \mathrm{Msep}\,.$$

Mathematical classes are absolute concepts. In order to use them in connection with the physical world, it is important to define the *extensionalization* $R^{(e)}$ of any relation R (of tipe t), and the property of *estensionality* (Ext)

$$(3.4) \qquad R^{(e)} =_D (\lambda x_1, \ldots, x_n)(\exists y_1, \ldots, y_n).R(y_1, \ldots, y_n) \bigwedge_{i=1}^{n} x_i = y_i\,,$$

$$R \in \mathrm{Ext} \equiv_D R = R^{(e)}\,,$$

where $x_1, y_1, \ldots, x_n, y_n$ are suitable distinct variables.

A3.24 $\qquad (\exists F)\ F \in \mathrm{Abs} \wedge a^* \in F \wedge (x) x \in F^{(e)}\,.$

Those among the above axioms for MC^{ν}_{*}, that belong to ML^{ν} are the axioms for MC^{ν}. The only remaining axiom of MC^{ν}_{*} is the assertion (outside ML^{ν}) that two propositions coincide iff they are equivalent.

A3.25 (ML^{ν}_{*}) $\qquad (\alpha)(\beta)\alpha = \beta \equiv {}^{\cap}(\alpha \equiv \beta) \qquad (\alpha, \beta \in \mathcal{E}_0)\,.$

Remark that by essential uses of AA3.8, 23 the, so to say, quasi extensional axioms A3.16-17 on classes and functions can be strengthened into the following theorems—cf. [2, (46) on p. 166]—in both MC^{ν}_{*} and MC^{ν} (and in the same way)

$$(3.5) \qquad (\Box)\ (\exists R)(\forall x_1, \ldots, x_n).R(x_1, \ldots, x_n) \equiv {}^{\cap}\alpha\,,$$

$$(3.6) \qquad (\Box)\ (\exists f)\ (\forall x_1, \ldots, x_n) f(x_1, \ldots, x_n) = {}^{\cap}\Delta\,.$$

4. On the analogue El of elementary possible cases, defined in ML_*^ν itself, and indicators of propositions.

In [2], p. 203 the analogue El of the class Γ of elementary (possible) cases is defined in ML^ν itself. Hence this definition belongs to ML_*^ν too. The same holds for the matrix $|_u$ where u is a particular variable—cf. [2], p. 204—which substantially means: the elementary case u occurs. Let us remember from [2], pp. 203, 4, 6, 8 the following theorems on El and $|_u$, where α, β are matrices (and hence they may be variables if MC_*^ν is referred to) and where u is a variable that does not occur free in α or β (4).

(4.1) $$\vdash \mathrm{El} \in \mathrm{Abs}\,, \quad \vdash (\exists_1 u)|_u\,, \quad \vdash (\exists_1^\cap u)|_u\,, \quad \vdash u \in \mathrm{El} \equiv \Diamond\, |_u\,,$$

(4.2) $$\vdash \Diamond\, (\alpha|_u) \equiv^\cap u \in \mathrm{El}\, (|_u \supset^\cap \alpha)\,, \quad \vdash \Diamond\, [|_u(\alpha \sim \beta)] \equiv u \in \mathrm{El} \wedge \wedge \Diamond\, (|_u\alpha) \wedge \sim \Diamond\, (|_u\beta)\,,$$

(4.3) $$\vdash (u)[u \in \mathrm{El} \supset \Diamond\, (\alpha|_u)] \equiv \Box\, \alpha \equiv (u).|_u \supset^\cap \alpha\,,$$

(4.4) $$\vdash (\exists u)[u \in \mathrm{El} \wedge (|_u \supset^\cap \alpha) \equiv \Diamond\, \alpha \equiv (\exists u)\, \Diamond.\alpha|_u\,.$$

The extensionalization $\Phi(x)^{(ex)}$ of any matrix $\Phi(x)$ is defined—cf. [2], p. 36—by

(4.5) $$\Phi(x)^{(ex)} \equiv_D (\exists y).\Phi(x) x = y \text{ where } y \text{ is not free in } \Phi(x)\,.$$

The theorems below on $\imath$ will be used (5)—cf. [2], pp. 155, 164

(4.6) $$\vdash (\imath x)\alpha = (\imath x)\alpha^{(ex)}\,, \quad \Box\, (\exists_1 x)\alpha \supset (\imath x)\alpha =^\cap (\imath x)\, \Box\, \alpha^{(ex)}\,,$$

(4.7) $$\vdash (\exists_1 x)\, \Phi(x) \supset \Psi[(\imath x)\, \Phi(x)] \quad \text{where} \quad \Psi(x) \equiv_D \Phi(x)^{(ex)}\,,$$

(4.8) $$\vdash \Box\, (\exists_1 x)\, \Box\, \Phi(x) \supset \Box\, \Phi[(\imath x)\, \Box\, \Phi(x)]\,,$$
$$\vdash (\exists_1 x)\alpha \supset (x).x = (\imath x)\alpha \equiv \alpha^{(ex)}\,.$$

(4) Formulas (4.1) to (4.4) are $(82)_3$, $(83)_{3,6,4}$, $(\overline{89})$, $(\overline{92})$, (90), (91) (pp. 203-8 of [2]) respectively.

(5) Formulas (4.6) to (4.8) are (33), Ths. 39.5 (II) and 38.2, $(35)_4$, $(31)_1$, (pp. 155-164 of [2]), respectively.

* * *

After [4], we consider the *indicator* ind (α), briefly $\iota\alpha$, for any proposition α; substantially it is the range of α, i.e. the set of elementary possible cases in which α holds:

$$\iota\alpha =_D (\lambda u) \Diamond (\alpha|_u) \quad \textit{for } u \textit{ not free in the matrix } \alpha. \tag{4.9}$$

Set theoretic relations or operations such as $\subseteq, \subset, \cap$, and $\cup$ are understood to be introduced in ML^ν or ML^ν_* as usually—cf. [2], p. 66. Then by (4.1) and (4.9)

$$\vdash \iota\alpha \subseteq \mathrm{El}\,, \qquad \vdash \iota\alpha \in \mathrm{Abs}\,. \tag{4.10}$$

Now we briefly prove formally that

$$\vdash \iota(\sim\alpha) = \overline{\iota\alpha}, \qquad (\bar{\alpha} =_D \mathrm{El} - \alpha)\,, \qquad \vdash \iota(\alpha\vee\beta) = \iota\alpha \cup \iota\beta\,, \tag{4.11}$$
$$\vdash \iota(\alpha\wedge\beta) = \iota\alpha \cap \iota\beta\,.$$

$$\vdash (\beta \supset^\cap \gamma) \equiv \iota\beta \subseteq \iota\gamma\,, \qquad \vdash (\beta \equiv^\cap \gamma) \equiv \iota\beta = \iota\gamma \equiv \iota\beta =^\cap \iota\gamma\,. \tag{4.12}$$

By (4.9) and (4.2) $\vdash u \in \iota(\sim\alpha) \equiv u \in \mathrm{El}\,(|_u \supset^\cap \sim\alpha)$.

Furthermore we deduce likewise that $\vdash u \in \overline{\iota\alpha} \equiv u \in \mathrm{El}\,(|_u \supset^\cap \sim\alpha) \equiv$ $\equiv u \in \mathrm{El}. \sim \Diamond\,(|_u\alpha)$. Hence $\vdash u \in \mathrm{El} \supset . u \in \iota(\sim\alpha) \equiv u \notin \iota\alpha$ which by (4.9) and $(4.10)_2$ yields $(4.11)_1$.

By (4.9) $\vdash u \in \iota\alpha \cup \iota\beta \equiv u \in \mathrm{El}\wedge . \Diamond\,(\alpha|_u) \vee \Diamond\,(\beta|_u) \equiv u \in \mathrm{El}\wedge\Diamond.(\alpha\vee$ $\vee\beta)|_u \equiv u \in \iota(\alpha\vee\beta)$. Hence $(4.11)_2$ holds.

By $(4.11)_{1,2} \vdash \iota(\alpha\beta) = \iota \sim (\alpha\vee\beta) = \overline{\iota\alpha \cup \iota\beta} = \iota\alpha \cap \iota\beta$, whence $(4.11)_3$.

Let u not occur free in β or γ: then, by (4.9), (a) $(\iota\beta \subseteq \iota\gamma)$ is equivalent to (b) $(u).\Diamond\,(\beta|_u) \supset \Diamond\,(\gamma|_u)$, which is obviously implied by (c) $\beta \supset^\cap \gamma$. Furthermore $\sim(\beta \supset^\cap \gamma)$ yields $\Diamond\,(\beta \sim \gamma)$, whence by $(4.4)_2$ $\exists u \Diamond\,(\beta \sim \gamma|_u)$. By $(4.2)_2$ this yields $(\exists u).\Diamond\,(\beta|_u) \sim \Diamond\,(\gamma|_u)$ which contrasts to (b). Hence $\vdash (b) \supset (c)$. We conclude that $\vdash (a) \equiv (b) \equiv (c)$. Hence $(4.12)_1$ holds. This and $(4.10)_2$ imply $(4.12)_2$.

5. Notions and axioms of probability expressed in ML^ν and ML^ν_*.

To deal with Reinchenbach's implication $\ni$ in any $\mathfrak{L}$ of the languages ML^ν and ML^ν_* after [3], we consider (i) the constant $\mathbb{R}$—denoted by Real in [2]—that expresses the class of real numbers in $\mathfrak{L}$,

(ii) the types $t_{\mathbb{R}}$ and t_{El} of $\mathbb{R}$ and El, and (iii) the first constant $c^{\ni}$ $[c^{\ni}_{*}]$ of ML^{ν} $[ML^{\nu}_{*}]$ of type $(t_{\mathrm{El}}, t_{\mathrm{El}}, t_{\mathbb{R}})$ $[(0, 0, t_{\mathbb{R}})]$. Then we set

$$(5.1) \qquad \alpha \ni_p \beta \equiv_D \begin{cases} c^{\ni}(\iota\alpha, \iota\beta, p) & \text{in } ML^{\nu}, \\ c^{\ni}_{*}(\alpha, \beta, \mathrm{p}) & \text{in } ML^{\nu}_{*}, \end{cases}$$

and, following substantially Reichenbach, we accept

$$(5.2) \qquad \alpha \ni \beta \equiv_D (\exists p)\alpha \ni_p \beta$$

as a metalinguistic definition, so that (if α can be realized) $\alpha \ni \beta$ says that the probability of β relative to α exists.

By their interpretations it is natural to consider $\ni_p$ and $\ni$ as more cohesive than $\supset$ and less cohesive than $\wedge$ and $\vee$.

By A3.25 in ML^{*}_{ν} the substitution theorem

$$(5.3) \qquad \vdash (\alpha \equiv^{\cap} \gamma)(\beta \equiv^{\cap} \delta) \supset . \alpha \ni_p \beta \equiv \gamma \ni_p \delta$$

is a trivial consequences of A3.12. In ML^{ν} (5.3) holds by $(4.12)_2$ and A3.12. By (5.3)

$$(5.4) \qquad \beta \supset^{\cap} \alpha \supset \begin{cases} \alpha \ni_p \beta \equiv \alpha \ni_p \beta\gamma , \\ \beta \ni_p \alpha \equiv \beta\gamma \ni_p \beta . \end{cases}$$

Following substantially [3], we take as proper axioms (axiom schemes) of the probability calculus PC_{*} $[PC]$ based on MC^{ν}_{*} $[MC^{\nu}]$ the assertions A5.1-8 of ML^{ν}_{*} $[ML^{\nu}]$ below ([6]), where $\mathbb{R}$ is understood to express the natural concept of real numbers (denoted by Real in [2]) and to be defined in an obvious way in terms of N which is a constant of type t_{N} that expresses the natural concept of natural numbers in $\mathfrak{L}$ ([7]). One can prove

$$(5.5) \qquad \vdash \mathrm{N} \in \mathrm{Abs}, \qquad \vdash \mathbb{R} \in \mathrm{Abs}, \qquad \vdash a^{*} \notin \mathbb{R} .$$

CONVENTION 5.1. *The istance of* $(\ )\Delta$ *or* $(\Box)\Delta$ *with* $\Delta \in \mathcal{E}_0$, *that is the closure or modal closure of* Δ, *can be denoted by* $[\ ]\Delta$ *or* $[\Box]\Delta$

([6]) In [3] only A1 to A7 are stated.

([7]) In [2], NN. 27, 28, 45, N—denoted by *Nn*—is defined on purely logical grounds; and the basic theorems for it are stated.

respectively.

A5.1 *absoluteness* $[\ ]\Diamond(\alpha \ni_p \beta) \supset \Box\, \alpha \ni_p \beta$,

A5.2 *absoluteness* $[\ ]\alpha \ni_p \beta \supset p \in \mathbf{R}$,

A5.3 *uniqueness* $[\ ]p, q \in \mathbf{R} \wedge p \neq q \supset .(\alpha \ni_p \beta)(\alpha \ni_q \beta) \equiv \sim \Diamond \alpha$,

A5.4 *normalization* $[\ ]\Diamond\alpha(\alpha \supset^{\cap} \beta) \supset \alpha \ni_1 \beta$,

A5.5 *normalization* $[\Box]\Diamond\alpha(\alpha \ni_p \beta) \supset p \geqslant 0$,

A5.6 *sum* $(\alpha \ni_p \beta)(\alpha \ni_q \gamma)(\alpha\beta \supset^{\cap} \sim \gamma) \supset \alpha \ni_{p+q} \beta \vee \gamma$,

A5.7 *product* $[\ ](\alpha \ni_p \beta)(\alpha\beta \ni_q \gamma) \supset \alpha \ni_{pq} \beta\gamma$,

A5.8 $[\ ](\forall r \in \mathbf{N})(\alpha \ni_{p_r} B_r) \wedge (\alpha B_{r+1} \supset^{\cap} B_r)\, q =^{\cap} \sup\{p_r\} \supset \alpha \ni_q (\exists r \in \mathbf{N}) B_r$,

where $B[p]$ *is a variable of type* $(t_{\mathbf{N}}\!:\!0)$ $[(t_{\mathbf{N}}\!:\!t_{\mathbf{R}})]$ *and e.g.* B_r $[p_r]$ *stands for* $B(r)$ $[p(r)]$*—cf. the definition of* $(\forall x \in \Psi)$ *above* A3.22.

By reasonings that are essentially mathematic and substantially known—and hence omitted here—the conjunction of A5.6 and A5.8 is equivalent to

A5.9 $$[\ ](\forall r, s \in \mathbf{N})\Big[(\alpha \ni_{p_r} B_r)(r < s \wedge \alpha B_r \supset^{\cap} \sim B_s)\Big(q = \sum_{r=1}^{\infty} p_r\Big)\Big] \supset . \\ \supset . \alpha \ni_q (\exists r \in \mathbf{N}) B_r$$

where B *and* p *are as is* A5.8.

Incidentally A5.9 is strictly more powerful than any of A5.6 and A5.8; and by theorem $(6.1)_1$ below, A5.4 can be replaced with the equivalent axiom

A5.4′ $$\alpha \supset^{\cap} \beta \supset \alpha \ni_1 \beta \,.$$

In ML^{ν} propositional variables can be replaced with the variables of type $(t_{\mathbf{N}})$ that are restricted to the class PR_1:

(5.6) $$F \in PR_1 \equiv_D (x).x \in F \supset^{\cap} x =^{\cap} 1 \,.$$

Thus, briefly speaking, every proposition α is associated with the $F \in PR_1$ for which $\alpha \equiv^\cap F(1)$ holds; and this correspondence is a bijection. For istance the counterparts of A5.2 and A5.6 in MC^v are

$$(5.7) \qquad [\Box]\ F_1(1) \ni_p F_2(1) \supset p \in \mathbf{R}\,, \qquad [\Box]\ [F_1(1) \ni_p F_2(1)] \wedge$$
$$\wedge [F_1(1) \ni_q F_2(1)] \wedge [F_1(1) F_2(1) \supset^\cap \sim F_3(1)] \supset [F_1(1) \ni_{p+q} F_2(1) \vee F_3(1)]\,.$$

The metalinguistic formulation A5.2 and A5.6 can be used also for their counterparts $(5.7)_{1,2}$ provided α_1 to α_n are meant as metalinguistic variables standing for $F_1(1)$ to $F_n(1)$ respectively. The same holds for the others among axioms A5.1-7 and A5.4$'$. Obviously A5.8 and A5.9 are meaningful also in ML^v, so that they can be included into the PC.

6. Some theorems on $\ni$ in MC^v_* and MC^v.

The theorems (6.1) to (6.3) below on $\ni_p$ differ from their usual analogues in that most of them are explicitly modal and all of them belong to the modal calculus PC^* or PC. We think that to prove them explicitly is interesting especially because these proofs involve some modal axioms without any ordinary analogues.

$$(6.1) \qquad \vdash \sim \Diamond\, \alpha \supset (\forall p \in \mathbf{R}).\alpha \ni_p \beta\,, \qquad \vdash \alpha \ni_1 \alpha\,, \qquad \vdash \alpha \ni_1 \beta \vee \sim \beta\,.$$

$$(6.2) \qquad \vdash \sim \Diamond\, (\alpha\beta) \supset .\alpha \ni_1 \sim \beta\,, \qquad \vdash \alpha \supset^\cap \beta \supset .\alpha\gamma \ni_1 \beta\,, \qquad \vdash \Box\, \beta \supset .\alpha \ni_1 \beta\,.$$

$$(6.3) \qquad \vdash (\alpha \ni \beta) \Diamond\, \alpha \equiv (\exists_1 p).\alpha \ni_p \beta \equiv (\exists_1^\cap p).\alpha \ni_p \beta\,.$$

Proof. Assume $\sim \Diamond\, \alpha$ and $p \in \mathbf{R}$. Then, for $q = p + 1$ A5.3 easily yields $\alpha \ni_p \beta$. We conclude that $(6.1)_1$ holds; and hence A5.4 yields A5.4$'$. Since $\vdash \alpha \supset^\cap \alpha$ $[\alpha \supset^\cap \beta \vee \sim \beta]$, $(6.1)_2$ $[(6.1)_3]$ holds by A5.4. Since $\vdash \sim \Diamond\, (\alpha\beta) \equiv (\alpha \supset^\cap \sim \beta)$, A5.4 yields $(6.2)_1$. By A5.4 $(6.2)_{2,3}$ also hold.

Now we assume that (a) $(\alpha \ni \beta) \Diamond\, \alpha$ holds and (b) $(\exists_1 p)\alpha \ni_p \beta$ does not. By (a) and A5.2 $\alpha \ni_p \beta$ and $p \in \mathbf{R}$ for some p, or more explicitly by the choice rule; and if $\sim (b)$ holds, then for some $q \neq p$, $\alpha \ni_q \beta$, whence $q \in \mathbf{R}$ by A5.2. Then $\sim \Diamond\, \alpha$ by A5.3, which contrasts to (a). We conclude that $\vdash (a) \supset (b)$.

Assume (b). In order to deduce (c) $(\exists_1^\cap p)\alpha \ni_p \beta$, we also assume

$\alpha \ni_p \beta$ $(i = 1, 2)$, whence $p_1, p_2 \in \mathbb{R}$ and $p_1 = p_2$ by (b). Then $p_1 =^\cap p_2$ by $(5.5)_2$, (3.3), and (3.2). We conclude that (c) holds by $(2.8)_{2,4}$. Thus $\vdash (b) \supset (c)$; and by $(2.8)_{3,4}$, $\vdash (c) \supset (b)$.

By $(2.8)_3$, (b) and $\sim (a)$ yield $\sim \Diamond \alpha$ and (b') $(\exists^{(1)} p) \alpha \ni_p \beta$. By $\sim \Diamond \alpha$ and A5.3 $(\alpha \ni_0 \beta)(\alpha \ni_1 \beta)$ which contrasts to (b'). Hence $(b) \vdash (a)$. We conclude that $\vdash (a) \equiv (b) \equiv (c)$, i.e. (6.3) holds. q.e.d.

By A5.1 *for any* γ *of the members* (a) *to* (c) *of the equivalences* in (6.3), $\vdash \gamma \equiv \Box \gamma$.

7. The probability function $\mathcal{T}$ and some theorems on it in MC^ν_* and MC^ν.

The probability $\mathcal{T}(\alpha, \beta)$ of (the event) β relative to (the trial) α can be defined in both MC^ν_* and MC^ν—cf. def. (5.1)—by

$$(7.1) \qquad \mathcal{T}_{\alpha,\beta} =_D \mathcal{T}(\alpha, \beta) =_D (\iota p)(\alpha \ni_p \beta) .$$

However in ML^ν this definition must be regarded as metalinguistic, whereas in ML^ν_* it can be thought of as a contextual one that introduces a constant $\mathcal{T}$ of type $(0, 0 : t_{\mathbb{R}})$.

We now prove the following basic theorems on $\mathcal{T}$, which involve modalities and the non-existing objects.

$$(7.2) \quad \vdash (\exists_1 p)(\alpha \ni_p \beta) \equiv \mathcal{T}_{\alpha,\beta} \neq a^* , \qquad \vdash \mathcal{T}_{\alpha,\beta} \notin \mathbb{R} \equiv \mathcal{T}_{\alpha,\beta} = a^* \equiv \mathcal{T}_{\alpha,\beta} =^\cap a^*,$$

$$(7.3) \quad \vdash \mathcal{T}_{\alpha,\beta} \neq a^* \supset . p =^\cap \mathcal{T}_{\alpha,\beta} \equiv \alpha \ni_p \beta , \qquad \vdash \Diamond \alpha \supset . p =^\cap \mathcal{T}_{\alpha,\beta} \neq a^* \equiv \alpha \ni_p \beta,$$

$$(7.4) \quad \vdash \Diamond \alpha . \mathcal{T}_{\alpha\beta} = a^* \supset \sim (\alpha \ni \beta) , \qquad \vdash (\alpha \equiv^\cap \gamma)(\beta \equiv^\cap \delta) \supset \mathcal{T}_{\alpha,\beta} =^\cap \mathcal{T}_{\gamma,\delta} .$$

PROOF. Assume (a) $(\exists_1 p) \alpha \ni_p \beta$. Then by $(4.8)_1$ with $\Phi(x)$ identified with $\alpha \ni_x \beta$ and A5.1,

$$(7.5) \qquad \vdash \Box (\exists_1 p) \alpha \ni_p \beta \supset \Box \alpha \ni_q \beta \qquad \text{where} \qquad q =_D (\iota p) \alpha \ni_p \beta .$$

By the last remark in N. 6 (a) yields $\Box (a)$ and hence $\alpha \ni_q \beta$ by $(7.5)_1$. Then, by A5.2, $q \in \mathbb{R}$, which by $(7.5)_2$ is (b) $\mathcal{T}_{\alpha,\beta} \in \mathbb{R}$. By $(5.5)_3$ this yields (c) $\mathcal{T}_{\alpha,\beta} \neq a^*$.

On the other hand $\sim (a)$ yields $\mathcal{T}_{\alpha,\beta} = a^*$ by (7.1) and A3.13 (b). We conclude that $(7.2)_1$ holds.

We saw that $(a) \vdash (b)$ and $(b) \vdash (c)$; and $(c) \vdash (a)$ by (7.2). Then $\vdash (b) \equiv (c)$, i.e. the first equivalence in $(7.2)_2$ is a theorem. The second equivalence holds by $(5.5)_2$, (3.3) and (3.2).

Now assume (c), whence (a) follows by $(7.2)_1$: We showed on the basis of (7.5), that (a) yields $\alpha \ni_q \beta$. Furthermore, by the second equivalence in (6.3), (a) yields $(\exists_1^\cap p).\alpha \ni_p \beta$.

Then by $(2.8)_{2,4}$ and $\alpha \ni_q \beta$ we obtain $\alpha \ni_p \beta \supset p = q$; hence by A5.2 and $(7.5)_2$, $\alpha \ni_p \beta \supset p =^\cap \mathfrak{T}_{\alpha,\beta}$. The converse implication is a theorem by A3.12. We conclude that $(7.3)_1$ holds.

Now assume $\diamond \alpha$ and $\alpha \ni_p \beta$, whence (a) follows by (6.3). Then, by the instance $\alpha \ni_p \beta \wedge (a) \supset \mathfrak{T}_{\alpha,\beta} = p$ of A3.13 (a) and by A5.2, we obtain $p =^\cap \mathfrak{T}_{\alpha,\beta}$. Lastly (a) and $(7.2)_1$ yield $\mathfrak{T}_{\alpha,\beta} \neq a^*$. We conclude that

$$(7.6) \qquad \vdash \diamond \alpha \wedge (\alpha \ni_p \beta) \supset p =^\cap \mathfrak{T}_{\alpha,\beta} \neq a^* .$$

Let us conversely assume $\diamond \alpha \wedge p =^\cap \mathfrak{T}_{\alpha,\beta} \neq a^*$. Then (a) holds by $(7.2)_1$, which yields $\Box (a)$, as we saw. Then we have $\alpha \ni_q \beta$ by (7.5). This, $p =^\cap \mathfrak{T}_{\alpha,\beta}$, and A3.12 yield $\alpha \ni_p \beta$. We conclude that $\vdash \diamond \alpha \wedge p =^\cap$ $=^\cap \mathfrak{T}_{\alpha,\beta} \neq a^* \supset \alpha \ni_p \beta$, which by (7.6) yields $(7.3)_2$.

Assume $\diamond \alpha$ and $(\exists p).\alpha \ni_p \beta$, whence $\alpha \ni_p \beta$ by the choice rule. Then $\mathfrak{T}_{\beta,\beta} \neq a^*$ by $(7.5)_2$: Thus we can assert theorem $(7.4)_1$. It yields $(7.4)_2$ by (7.1) and (5.3). q.e.d.

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