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Otto H. Kegel

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Examples of Highly Transitive Permutation Groups.

OTTO H. KEGEL (*)

The permutation group (G, Ω) is called *highly transitive* if, for every natural number n, the group G acts n-fold transitively on the infinite set Ω. Every subgroup of the symmetric group on Ω which contains the group of all even finite permutations of Ω trivially provides an example of a highly transitive permutation group. In mainly unpublished discussions over the years other examples of this phenomenon have been fond; Mc Donough [5] shows that every non-abelian free group of at most countable rank admits a faithful permutation representation which is highly transitive. The aim of the present note is to point out that for very group G acting on a set Ω there is a permutation group (G^*, Ω^*) with $G \subseteq G^*$ and $\Omega \subseteq \Omega^*$ which is highly transitive and such that $\max(|G|, |\Omega|) = \max(|G^*|, |\Omega^*|)$. In fact, we shall see that we may endow (G^*, Ω^*) with the following property of *homogeneity*: let (A, Ω_A), (B, Ω_B) be finitely generated *subactions* of (G^*, Ω^*), i.e. finitely generated subgroups of G^* together with finitely many of their orbits in Ω^*, then, for every isomorphism $\varphi: (A, \Omega_A) \to (B, \Omega_B)$, there exists an element $g = g(\varphi)$ in G inducing this isomorphism. It is a remarkable—though in principle well-known—fact that a countable homogeneous permutation group is determined up to isomorphism by the isomorphism classes of its finitely generated subactions.

If (G, Ω) is any action of the group G on the set Ω there is no problem in finding a set Ω^* containing Ω and a group G^* containing G and acting faithfully and highly transitively on Ω^* such that (G, Ω) is a subaction of (G^*, Ω^*). To obtain Ω^* adjoin a copy of G to Ω on

Indirizzo dell'A.: Mathematisches Institut der Universität - Albertstrasse 23b - D-7800 Freiburg i. Br. (Germania Occidentale).

which G acts regulary from the right; now (G, Ω^*) is a permutation group. Let A denote the group of all even finite permutations of Ω^*, then $G^* = GA$ acts as a highly transitive group of permutations on Ω^*. If (G, Ω) is a permutation group and H is a group containing G then it is possible to find a set Ω^* containing Ω on which H acts faithfully so that (G, Ω) is a subaction of (H, Ω^*). For this let T be a right transversal from G to H with $1 \in T$; for every $t \in T$ choose a copy Ω^t of Ω and consider the disjoint union $\Omega^* = \dot{\bigcup}_{t \in T} \Omega^t$; identify Ω with Ω^1. If for $t \in T$ and $h \in H$ one has $th = gt'$ for some $g \in G$, then the action of H on Ω^* is described by

$$(\omega^*)^h = (\omega^t)^h = \omega^{th} = \omega^{gt'} = (\omega^g)^{t'} \quad \text{if } \omega^* \in \Omega^t.$$

Clearly (G, Ω) is a subaction of the permutation group (H, Ω^*) obtained by inducing up.

Assuming for the moment that every (abstract) group may be embedded into some (abstractly) homogeneous group without rasising the (infinite) cardinality, we now want to describe a way how to find for every infinite permutation group (G, Ω) a highly transitive and homogeneous permutation group (G^*, Ω^*) having (G, Ω) as a subaction and of the same cardinality. Realise (G, Ω) as a subaction (of the same cardinality) of the highly transitive permutation group (G_1, Ω_1). The group G_1 may be embedded into a homogeneous group G_2 of the same cardinality; inducing the action of G_1 on Ω_1 up to G_2 one obtains a set Ω_2 on which G_2 acts faithfully and such that (G_1, Ω_1) is a subaction of (G_2, Ω_2). Any isomorphism φ between two finitely generated subactions (A, Ω_A) and (B, Ω_B) of (G_2, Ω_2) consists of two parts $\varphi = (\varphi_1, \varphi_2)$ where φ_1 is the isomorphism of A onto B and φ_2 is a bijection of Ω_A onto Ω_B so that for $\omega \in \Omega_A$ one has $(\omega^a)\varphi = (\omega^a)\varphi_2 = (\omega\varphi_2)^{a\varphi_1}$. Since the group G_2 is homogeneous, one may assume that the isomorphism φ_1 is already the identity map. In the symmetric group on Ω_2 there is a element centralising A and inducing the map φ_2. Form the group G_3 by adjoining elements of the symmetric group on Ω_2 to G_2, one for every isomorphism between finitely generated subactions of (G_2, Ω_2). Put $\Omega_2 = \Omega_3$.

Thus, we have described the first three steps of a construction of larger and larger permutation groups, all of the same cardinality. Iterating this procedure, we obtain permutation groups (G_n, Ω_n) for all natural numbers n, all of the same cardinality and such that for $m < n$, the permutation group (G_m, Ω_m) is a subaction of (G_n, Ω_n);

if n is of the form $n = 3k + 1$ then (G_n, Ω_n) is highly transitive, and for every isomorphism between finitely generated subactions of (G_n, Ω_n), for n of the form $n = 3k + 2$, there is a element of G_{n+1} inducing this isomorphism. Thus, putting $G^* = \bigcup_{n \in \mathbf{N}} G_n$ and $\Omega^* = \bigcup_{n \in \mathbf{N}} \Omega_n$, one obtains a permutation group (G^*, Ω^*) with all the desired properties.

The embedding of every infinite abstract group into some homogeneous group probably was known to P. Hall, it is an immediate consequence of B. A. F. Wehrfritz's construction in [3], chapter VI, of the *constricted symmetric group* $Cs(G)$ of a group G (compare also [6], § 9.4). The group $Cs(G)$ is defined as follows:

$$Cs(G) = \{\alpha \in \text{symmetric group on the set } G; \text{ for some finitely generated subgroup } F = F(\alpha) \text{ of } G \text{ one has } (gF)^\alpha = gF \text{ for all } g \in G\} .$$

The group $Cs(G)$ contains all finite permutations on G as well as the multiplications from the right by elements of G; it is locally finite if and only if G is locally finite. Let G^ϱ be the group of all right multiplications with elements of G, then for every isomorphism between finitely generated subgroups of $G^\varrho(\simeq G)$ there is an element $\alpha \in Cs(G)$ inducing this isomorphism. Thus, by an obvious inductive procedure, one embeds the infinite group G into a homogeneous group H of the same cardinality.

Going through the preceding argument one sees that if in (G, Ω) the group G is locally finite all the steps may be performed so that G_n again is locally finite for every n, thus, G^* may be obtained locally finite. Summing up, one has

THEOREM 1. *Let (G, Ω) be an infinite (locally finite) permutation group. Then there is a (locally finite) highly transitive and homogeneous permutation group (G^*, Ω^*) of the same cardinality which has (G, Ω) as a subaction.*

One invariant of a permutation group (G, Ω) is its *skeleton* $sk(G, \Omega)$, i.e. the set of isomorphism types of finitely generated subactions of (G, Ω). The skeleton determines a countable homogeneous permutation group up to isomorphism. This is proved by the standard back-and-forth argument of Cantor.

THEOREM 2. – *Let (G, Ω) and (H, Σ) be two countable homogeneous permutation groups. (G, Ω) is isomorphic to (H, Σ) if and only if*

$$sk(G, \Omega) = sk(H, \Sigma) .$$

PROOF. Let (G_i, Ω_i) and (H_i, Σ_i) be ascending sequences of finitely generated subactions of (G, Ω) and (H, Σ), respectively, with $G = \bigcup_{i \in \mathbf{N}} G_i$, $\Omega = \bigcup_{i \in \mathbf{N}} \Omega_i$ and $H = \bigcup_{i \in \mathbf{N}} H_i$, $\Sigma = \bigcup_{i \in \mathbf{N}} \Sigma_i$. Since the skeletons are the same, there is an isomorphic embedding φ_1 of (G_1, Ω_1) into (H, Σ); choose $(H_{1'}, \Sigma_{1'})$ minimal among the (H_i, Σ_i) to contain $(G_1, \Omega_1)^{\varphi'}$ as well as (H_1, Σ_1). Choose an isomorphic embedding ψ_1 of $(H_{1'}, \Sigma_{1'})$ into (G, Ω) and $(G_{2'}, \Omega_{2'})$ minimal among the (G_i, Ω_i) to contain $(H_{1'}, \Sigma_{1'})^{\psi_1}$ as well as (G_2, Ω_2). By the homogeneity of (G, Ω) the map ψ_1 can be so chosen that the composed map $\varphi_1\psi_1$ is the identity map on (G_1, Ω_1). ... Now assume the maps $\varphi_n\colon (G_{n'}, \Omega_{n'}) \to (H, \Sigma)$ and $\psi_n\colon (H_{n'}, \Sigma_{n'}) \to (G, \Omega)$ so chosen that for $i < n'$ the map φ_{i+1} extends φ_i and ψ_{i+1} extends ψ_i and that the composite maps $\varphi_n\psi_n$ and $\psi_{n-1}\varphi_n$ are the identity maps on $(G_{n'}, \Omega_{n'})$ and $(H_{(n-1)'}, \Sigma_{(n-1)'})$, respectively. Choose $(G_{(n+1)'}, \Omega_{(n+1)'})$ minimal among the (G_i, Ω_i) to contain $(H_{n'}, \Sigma_{n'})^{\psi_n}$ as well as (G_{n+1}, Ω_{n+1}) and a map $\varphi_{n+1}\colon (G_{(n+1)'}, \Omega_{(n+1)'}) \to (H, \Sigma)$ extending φ_n such that $\psi_n\varphi_{n+1}$ is the identity map on $(H_{n'}, \Sigma_{n'})$. Choose $(H_{(n+1)}, \Sigma_{(n+1)'})$ minimal among the (H_i, Σ_i) to contain $(G_{(n+1)'}, \Omega_{n(+1)'})_{\varphi_{n+1}}$ as well as (H_{n+1}, Σ_{n+1}). Further choose $\psi_{n+1}\colon (H_{(n+1)'}, \Sigma_{(n+1)}) \to (G, \Omega)$ as an embedding such that $\varphi_{n+1}\psi_{n+1}$ is the identity map on $(G_{(n+1)'}, \Omega_{(n+1)'})$, ... Denote by φ the limit of the φ_n and put $\psi = \lim_n \psi_n$. Then φ is an isomorphic embedding of (G, Ω) into (H, Σ) and ψ is an isomorphic embedding of (H, Σ) into (G, Ω). But, as by constrution the composite maps $\varphi\psi$ and $\psi\varphi$ are the identity on (G, Ω) and (H, Σ), respectively, the maps φ and ψ must be onto. They are the required isomorphisms.

For $\alpha \in \Omega$ denote by G_α the stabliser of α in the group G acting on Ω. With this notation, one obtains the

COROLLARY. *Let (G, Ω) be a countable homogeneous permutation group such that every finitely generated subgroup of G leaves infinitely many points of Ω fixed, then $(G, \Omega) \simeq (G_\alpha, \Omega \setminus \alpha)$; if G acts transitively on Ω it acts highly transitively.*

PROOF. The last statement follows from the first by obvious induction. Let (F, Ω_F) be any finitely generated subaction of (G, Ω) then, by assumption, there is an element $g \in G$ making F^g a subgroup of G_α and thus (F, Ω_F) isomorphic to a subaction of $(G_\alpha, \Omega \setminus \alpha)$. Theorem 2 now yields the isomorphism.

REMARK. Given a countable homogeneous permutation group (G, Ω) and an uncountable cardinal $\varkappa$, one may ask the question: is there a homogeneous permutation group (H, Σ) of cardinality $\varkappa$ with $sk(G, \Omega) = sk(H, \Sigma)$? Sometimes this question may be answered affermatively using the technique of Shelah and Ziegler [7].

The permutation group (G, Ω) is called *existentially closed* in the class of all (locally finite) permutation groups if (it is locally finite and) every finite set of equations and inequalities (between elements) in the (first order) language of permutation groups which can be satisfied in some (locally finite) permutation group (H, Σ) containing (G, Ω) as a subaction, can already be satisfied in (G, Ω). It is easy to see (cf. Hirschfeld and Wheeler [2]) that every infinite (locally finite) permutation group (G, Ω) is contained in some existentially closed (locally finite) permutation group (G^*, Ω^*) of the same cardinality. We have same cardinality. We have seen in the proof of theorem 1 that one may enlarge the group G without paying too much attention to Ω; thus the group G^* will be existentially closed in the class of all (locally finite) groups. On the other hand, n-fold transitivity can be expressed in the (first order) language of permutation groups, and by theorem 1 every (locally finite) permutation group may be embedded into some homogeneous and higly transitive (locally finite) permutation group. Thus, *every permutation group* (G, Ω) *which is existentially closed in the class of all* (*locally fiite*) *permutation groups is homogeneous and highly transitive.* In this way, we have found for some existentially closed groups in the class of all (locally finite) groups a homogeneous and highly transitive permutation representation. Does every such group admit such a representation?

As there is only one countable locally finite group U which is existentially closed in the class of all locally finite groups, namely P. Hall's *universal losally finite group* (see P. Hall [1]; Kegel and Wehrfritz [3], Chapter VI; Mac Intyre and Shelah [4]), the preceding discussion shows that *U has a highly transitive permutation representation.* Are there many such highly transitive permutation representations of U, or is there only one?

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