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On the Euler Equations for Nonhomogeneous Fluids (I).

HUGO BEIRÃO DA VEIGA - ALBERTO VALLI (*)

1. - Introduction and main results.

In this paper we consider the motion of a non-homogeneous ideal incompressible fluid in a bounded connected open subset Ω of $\mathbb{R}^2$.

We denote by $v(t, x)$ the velocity field, by $\varrho(t, x)$ the mass density, and by $\pi(t, x)$ the pressure. The Euler equations of the motion are (see Sédov [18], chap. IV, § 1, p. 164)

$$
\text{(E)} \qquad \begin{cases}
\varrho\left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] = -\nabla\pi & \text{in } Q_{T_0} \equiv [0, T_0]\times\bar{\Omega}\,, \\
\operatorname{div} v = 0 & \text{in } Q_{T_0}\,, \\
v\cdot n = 0 & \text{in } [0, T_0]\times\Gamma\,, \\
\dfrac{\partial\varrho}{\partial t} + v\cdot\nabla\varrho = 0 & \text{in } Q_{T_0}\,, \\
\varrho|_{t=0} = \varrho_0 & \text{in } \bar{\Omega}\,, \\
v|_{t=0} = a & \text{in } \bar{\Omega}\,,
\end{cases}
$$

where $n = n(x)$ is the unit outward normal to the boundary Γ of Ω, $b = b(t, x)$ is the external force field, and $a = a(x)$, $\varrho_0 = \varrho_0(x)$ are the initial velocity field and the initial mass density, respectively. Non-homogeneous ideal incompressible fluids are considered by many

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authors; see for instance Sédov [18], Zeytounian [23], Yih [22]; see also LeBlond-Mysak [13].

For the case in which the fluid is homogeneous, i.e. the density ϱ_0 (and consequently ϱ) is constant, equations (E) have been studied by several authors (for some reference see [2]). For non homogeneous fluids, Marsden [15] has stated the existence of a local solution to problem (E), under the assumption that the external force field $b(t, x)$ is zero. Marsden claims that his proof can be extended to the case in which $b(t, x)$ is divergence free and tangential to the boundary, i.e. $\operatorname{div} b = 0$ in Q_{T_0} and $b \cdot n = 0$ on $[0, T_0] \times \Gamma$. However for non homogeneous fluids a general force field can not be reduced to this particular case (for homogeneous fluids this can be done by subtracting a gradient).

Marsden's proof relies on techniques of Riemannian geometry on infinite dimensional manifolds. Our proof is quite different and is related to those of Wolibner [21] and Kato [9]. However the generalization of the techniques used in these last papers gives the existence of a solution only under the additional assumption $\|\nabla \varrho_0/\varrho_0\|_\infty \leqslant K$, where K is an a priori fixed constant depending essentially on Ω (see our previous paper [2]). The aim of this paper is to drop this condition by introducing an essential device, the elliptic system consisting in the seventh, the eighth and the ninth equation of system (A), in § 4. The system (A) does not contain explicitly $\partial v/\partial t$ (compare with system (4.17) in [2]) and this allows us to drop the referred additional assumption.

We prove the following result:

THEOREM A. *Let Ω be of class $C^{3+\lambda}$, $0 < \lambda < 1$, and let $a \in C^{1+\lambda}(\bar{\Omega})$ with $\operatorname{div} a = 0$ in $\bar{\Omega}$ and $a \cdot n = 0$ on Γ, $\varrho_0 \in C^{1+\lambda}(\bar{\Omega})$ with $\varrho_0(x) > 0$ for each $x \in \bar{\Omega}$, and $b \in C^{0,1+\lambda}(Q_{T_0})$.*

Then there exists $T_1 \in]0, T_0]$, $v \in C^{1,1+\lambda}(Q_{T_1})$, $\varrho \in C^{1+\lambda,1+\lambda}(Q_{T_1})$, $\pi \in$ $\in C^{0,2+\lambda}(Q_{T_1})$ such that (v, ϱ, π) is a solution of (E) *in Q_{T_1}.*

A uniqueness theorem for problem (E) is proved by Graffi in [6]; see also [2].

For a mathematical study of non-homogeneous *viscous* incompressible fluids see Kazhikhov [10], Ladyženskaja-Solonnikov [11] and Antoncev-Kazhikhov [1]; see also Lions [14] and Simon [24].

In the forthcoming paper [3] we prove corresponding results for the three-dimensional case. Since the publication of this paper has had some delay, a new result of the authors has appeared in the meantime: we give an easier existence proof in Sobolev spaces, without the use of characteristics, and we prove a C^∞ regularity result (see [4]).

Finally the authors remark that the extension to the case when Ω is not simply connected follows an essentially well known argument ([9], [2]). However, for the sake of completeness a preprint containing all the computations is available (see *On the Euler equations for non-homogeneous fluids (I), Ω not simply connected*, Trento 1979).

2. – Notations.

Let Ω be a bounded connected open subset of $\mathbf{R}^2$. We denote by $C^{k+\lambda}(\bar{\Omega})$, with k a non negative integer and $0<\lambda<1$, the space of k-times continuously differentiable functions in $\bar{\Omega}$ with λ-Hölder continuous derivatives of order k. For each $T\in\,]0, T_0]$ we denote by $C^0(Q_T)$ the space of continuous functions in Q_T and by $C^1(Q_T)$ the space of continuously differentiable functions in Q_T.

We set

$$D_i\varphi \equiv \frac{\partial\varphi}{\partial x_i}\,, \qquad D^\alpha D_t^j\varphi \equiv \frac{\partial^{|\alpha|+j}\varphi}{\partial x_2^{\alpha_1}\partial x_2^{\alpha_2}\partial t^j}\,,$$

and

$$C^{k,h}(Q_T) \equiv \{\varphi\in C^0(Q_T)|D^\alpha D_t^j\varphi\in C^0(Q_T)$$
$$\text{if } 0\leqslant j\leqslant k,\ |\alpha|\leqslant h \text{ and } j+|\alpha|\leqslant\max(k,h)\}\,,$$

$$C^{\lambda,0}(Q_T) \equiv \{\varphi\in C^0(Q_T)|\varphi$$
$$\text{is } \lambda\text{-Hölder continuous in } t, \text{ uniformly with respect to } x\}\,,$$

$$C^{0,\lambda}(Q_T) \equiv \{\varphi\in C^0(Q_T)|\varphi$$
$$\text{is } \lambda\text{-Hölder continuous in } x, \text{ uniformly with respect to } t\}\,,$$

$$C^{k+\lambda,h}(Q_T) \equiv \{\varphi\in C^{k,h}(Q_T)|D^\alpha D_t^j\varphi\in C^{\lambda,0}(Q_T)$$
$$\text{if } j+|\alpha| = \max(k,h) \text{ or if } j=k\}\,,$$

$$C^{k,h+\lambda}(Q_T) \equiv \{\varphi\in C^{k,h}(Q_T)|D^\alpha D_t^j\varphi\in C^{0,\lambda}(Q_T)$$
$$\text{if } j+|\alpha| = \max(k,h) \text{ or if } |\alpha|=h\}\,,$$

$$C^{k+\lambda,h+\lambda}(Q_T) \equiv C^{k+\lambda,h}(Q_T)\cap C^{k,h+\lambda}(Q_T)\,.$$

We denote by $\|\cdot\|_\infty$ the supremum norm, either in $\bar{\Omega}$ or in Q_T, by $[\cdot]_\lambda$ the usual λ-Hölder seminorm in $\bar{\Omega}$, by $\|\cdot\|^{k+\lambda}$ the usual λ-Hölder norm in $C^{k+\lambda}(\bar{\Omega})$. Furthermore we define

$$[\varphi]_{\lambda,0} \equiv \sup_{\substack{t,s\in[0,T]\\ t\neq s\\ x\in\bar{\Omega}}} \frac{|\varphi(t,x)-\varphi(s,x)|}{|t-s|^\lambda},$$

$$[\varphi]_{0,\lambda} \equiv \sup_{\substack{x,y\in\bar{\Omega}\\ x\neq y\\ t\in[0,T]}} \frac{|\varphi(t,x)-\varphi(t,y)|}{|x-y|^\lambda},$$

$$[\varphi]_{\mathrm{lip},0} \equiv \sup_{\substack{t,s\in[0,T]\\ t\neq s\\ x\in\bar{\Omega}}} \frac{|\varphi(t,x)-\varphi(s,x)|}{|t-s|},$$

$$[\varphi]_{0,\mathrm{lip}} \equiv \sup_{\substack{x,y\in\bar{\Omega}\\ x\neq y\\ t\in[0,T]}} \frac{|\varphi(t,x)-\varphi(t,y)|}{|x-y|}.$$

Corresponding definitions, with the same notations, are given for vector fields $u=(u_1,u_2)$. Every norm and seminorm is computed as in the following example:

$$[u]_\lambda \equiv \sup_{\substack{x,y\in\bar{\Omega}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\lambda}.$$

Moreover we set

$$|Du(t,x)| \equiv \left[\sum_{i,j=1}^{2} |D_i u_j(t,x)|^2\right]^{\frac{1}{2}}, \qquad \|Du\|_\infty \equiv \sup_{(t,x)\in Q_T} |Du(t,x)|,$$

and analogously for all other norms or seminorms. We put

$$\mathrm{Rot}\,\varphi \equiv \left(\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_1}\right),$$

$$\mathrm{rot}\,u \equiv \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},$$

where φ is a scalar function and $u=(u_1,u_2)$ is a vector function.

3. – Preliminaries.

In the following, $\varphi(t, x) \in C^{0,\lambda}(Q_T)$ will be a generic element of the sphere

$$\|\varphi\|_{0,\lambda} \leqslant A\ , \tag{3.1}$$

where the radius A is a positive constant, which we will specify below (see (4.11)).

We denote by $c, c_1, c_2, \ldots$, positive constants depending at most on λ and Ω.

Let ψ be the solution of the problem

$$\begin{cases} -\Delta\psi(t, x) = \varphi(t, x) & \text{in } \Omega\ , \\ \psi|_\Gamma = 0\ , \end{cases} \tag{3.2}$$

for each $t \in [0, T]$. We put

$$v = \operatorname{Rot}\psi\ , \tag{3.3}$$

and we write $v = F^1[\varphi]$.

One has

LEMMA 3.1. *Let* $v = F^1[\varphi]$. *Then* $v \in C^{0,1+\lambda}(Q_T)$ *and*

$$\|v\|_{0,1+\lambda} \leqslant c\|\varphi\|_{0,\lambda} \leqslant cA\ . \tag{3.4}$$

Moreover

$$\operatorname{div} v = 0\ , \qquad \operatorname{rot} v = \varphi \text{ in } Q_T\ , \qquad v\cdot n = 0 \text{ in } [0, T]\times\Gamma\ . \tag{3.5}$$

For the proof see [2], Lemma 3.1.

We now construct the streamlines of the vector field $v(t, x)$. We set $U(\sigma, t, x) \equiv y(\sigma)$, $\sigma, t \in [0, T]$, $x \in \bar\Omega$, where $y(\sigma)$ is the solution of the ordinary differential equation

$$\begin{cases} \dfrac{dy}{d\sigma} = v(\sigma, y(\sigma)) & \text{in } [0, T]\ , \\ y(t) = x\ . \end{cases} \tag{3.6}$$

This solution is global since $v\cdot n = 0$ on $[0, T]\times\Gamma$. Moreover $U \in C^1 \cdot ([0, T]\times Q_T)$ since $v \in C^{0,1}(Q_T)$ (see for instance Hartman [7], chap. V, Theor. 3.1, p. 95). We put $\|D_i U\|_\infty \equiv \sup_{\sigma\in[0,T]} \|D_i U(\sigma, \cdot, \cdot)\|_\infty$; an analogous convention holds for all norms and seminorms concerning U and its derivatives.

We have

LEMMA 3.2. *The vector function* $U(\sigma, t, x)$ *satisfies the following estimates*

$$
(3.7)\qquad
\begin{aligned}
&[U]_{0,\mathrm{lip}} \leqslant \exp\,[T[v]_{0,\mathrm{lip}}] \leqslant \exp\,[cTA]\,,\\
&[U]_{\mathrm{lip},0} \leqslant \|v\|_\infty \exp\,[T[v]_{0,\mathrm{lip}}] \leqslant cA\,\exp\,[cTA]\,,\\
&\|Di\,U\|_\infty \leqslant \exp\,[T\|Dv\|_\infty] \leqslant \exp\,[cTA]\,,\\
&[Di\,U]_{0,\lambda} \leqslant T[Dv]_{0,\lambda} \exp\,[T(2\|Dv\|_\infty + \lambda[v]_{0,\mathrm{lip}})] \leqslant cTA\,\exp\,[cTA]\,,\\
&[Di\,U]_{\lambda,0} \leqslant (T^{1-\lambda}\|Dv\|_\infty + T[Dv]_{0,\lambda}\,\|v\|_\infty^{\lambda}\exp\,[\lambda T[v]_{0,\mathrm{lip}}])\,\cdot\\
&\qquad\qquad \cdot \exp\,[2T\|Dv\|^\infty] \leqslant cT^{1-\lambda}A(1 + T^\lambda A^\lambda)\,\exp\,[cTA]\,.
\end{aligned}
$$

PROOF. From the resolving formula

$$
(3.8)\qquad U(\sigma, t, x) = x + \int_t^\sigma v(\tau, U(\tau, t, x))\,d\tau
$$

we obtain

$$
|U(\sigma, t, x) - U(\sigma, t, y)| \leqslant |x - y| + [v]_{0,\mathrm{lip}}\left|\int_t^\sigma |U(\tau, t, x) - U(\tau, t, y)|\,d\tau\right|.
$$

Hence from Gronwall's lemma

$$
|U(\sigma, t, x) - U(\sigma, t, y)| \leqslant |x - y|\,\exp\,[T[v]_{0,\mathrm{lip}}]\,,
$$

i.e. estimate $(3.7)_1$.

Analogously

$$
\begin{aligned}
|U(\sigma, t, x) - U(\sigma, s, x)| \leqslant &\left|\int_t^s |v(\tau, U(\tau, t, x))|\,d\tau\right| +\\
&+\left|\int_s^\sigma |v(\tau, U(\tau, t, x)) - v(\tau, U(\tau, s, x))|\,d\tau\right| \leqslant\\
&\leqslant \|v\|_\infty |t - s| + [v]_{0,\mathrm{lip}}\left|\int_s^\sigma |U(\tau, t, x) - U(\tau, s, x)|\,d\tau\right|.
\end{aligned}
$$

From Gronwall's lemma we have $(3.7)_2$.

On the other hand (3.8) yields

$$D_i U(\sigma, t, x) = e_i + \int_t^\sigma \sum_h D_h v(\tau, U(\tau, t, x)) D_i U_h(\tau, t, x)\, d\tau \,,$$

where e_i is the unit vector corresponding to the i-th axis.

By using Gronwall's lemma to estimate $|D_i U(\sigma, t, x)|$, $|D_i U\cdot(\sigma, t, x) - D_i U(\sigma, t, y)|$ and $|D_i U(\sigma, t, x) - D_i U(\sigma, s, x)|$ we obtain respectively $(3.7)_3$,

$$[D_i U]_{0,\lambda} \leqslant T[Dv]_{0,\lambda}[U]^{\lambda}_{0,\text{lip}} \|D_i U\|_\infty \exp[T\|Dv\|_\infty]$$

and

$$[D_i U]_{\lambda,0} \leqslant \left(T^{1-\lambda}\|Dv\|_\infty + T[Dv]_{0,\lambda}[U]^{\lambda}_{\text{lip},0}\right)\|D_i U\|_\infty \exp[T\|Dv\|_\infty] \,. \quad \square$$

Given a velocity $v(t, x)$, we denote by $\varrho = F^2[v]$ the solution of the problem

$$\begin{cases} \dfrac{\partial\varrho}{\partial t} + v\cdot\nabla\varrho = 0 & \text{in } Q_T\,, \\ \varrho|_{t=0} = \varrho_0 & \text{in } \bar{\Omega}\,. \end{cases} \tag{3.9}$$

We denote by $\bar{c}, \bar{c}_1, \bar{c}_2, \ldots,$ positive constants depending at most on $\lambda, \Omega, \varrho_0, b$.

The following result holds

LEEMA 3.3. *Let $\varrho_0 \in C^{1+\lambda}(\bar{\Omega})$ with $\varrho_0(x) > 0$ for each $x \in \bar{\Omega}$. Then the solution of* (3.9) *is given by*

$$\varrho(t, x) = \varrho_0(U(0, t, x)) \,. \tag{3.10}$$

Moreover $\varrho \in C^{1,1+\lambda}(Q_T)$, $\nabla\varrho \in C^{\lambda,0}(Q_T)$ *and*

$$\begin{aligned} &\left\|\frac{D_i\varrho}{\varrho^2}\right\|_\infty \leqslant \left\|\frac{\nabla\varrho_0}{\varrho_0^2}\right\|_\infty \|D_i U\|_\infty \leqslant \bar{c}\exp[cTA]\,, \\ &\left[\frac{D_i\varrho}{\varrho^2}\right]_{0,\lambda} \leqslant \left[\frac{\nabla\varrho_0}{\varrho_0^2}\right]_\lambda [U]^{\lambda}_{0,\text{lip}}\|D_i U\|_\infty + \\ &\qquad + \left\|\frac{\nabla\varrho_0}{\varrho_0^2}\right\|_\infty [D_i U]_{0,\lambda} \leqslant \bar{c}(1 + TA)\exp[cTA]\,, \\ &[D_i\varrho]_{\lambda,0} \leqslant [\nabla\varrho_0]_\lambda [U]^{\lambda}_{\text{lip},0}\|D_i U\|_\infty + \|\nabla\varrho_0\|_\infty [D_i U]_{\lambda,0} \leqslant \\ &\qquad \leqslant \bar{c}A^\lambda(1 + TA)\exp[cTA]\,. \end{aligned} \tag{3.11}$$

PROOF. By using the method of characteristics one easily obtains (3.10). From this last formula it follows that

$$D_i \varrho(t, x) = \sum_h D_h \varrho_0\big(U(0, t, x)\big) D_i U_h(0, t, x) ,$$

and we prove (3.11) by direct computation. □

Now we wish to study the following equation, which will be useful in the next section:

$$\text{(3.12)} \qquad \begin{cases} \dfrac{\partial \zeta}{\partial t} + v \cdot \nabla \zeta = \gamma & \text{in } Q_T , \\ \zeta|_{t=0} = \alpha & \text{in } \bar{\Omega} . \end{cases}$$

We easily obtain the formal solution of (3.12) by using the method of characteristics:

$$\text{(3.13)} \qquad \zeta(t, x) = \alpha\big(U(0, t, x)\big) + \int_0^t \gamma(\tau, U(\tau, t, x))\, d\tau .$$

By direct computation of this formula we obtain

LEMMA 3.4. *Let* $\alpha \in C^\lambda(\bar{\Omega})$, $\gamma \in C^{0,\lambda}(Q_T)$ *and let* ζ *be defined by* (3.13). *Then* $\zeta \in C^{\lambda,\lambda}(Q_T)$ *and*

$$\text{(3.14)} \qquad \begin{aligned} &\|\zeta\|_\infty \leqslant \|\alpha\|_\infty + T\|\gamma\|_\infty , \\ &[\zeta]_{0,\lambda} \leqslant [\alpha]_\lambda [U]^\lambda_{0,\mathrm{lip}} + T[\gamma]_{0,\lambda}[U]^\lambda_{0,\mathrm{lip}} , \\ &[\zeta]_{\lambda,0} \leqslant [\alpha]_\lambda [U]^\lambda_{\mathrm{lip},0} + T[\gamma]_{0,\lambda}[U]^\lambda_{\mathrm{lip},0} + T^{1-\lambda}\|\gamma\|_\infty . \end{aligned}$$

4. – Existence of a local solution of the auxiliary system (A) when Ω is simply-connected.

We wish to prove the existence of a local solution of the following system

$$
\text{(A)}\quad \begin{cases}
\dfrac{\partial\zeta}{\partial t} + v\cdot\nabla\zeta = \beta + \dfrac{\operatorname{Rot}\varrho}{\varrho^2}\cdot w & \text{in } Q_T\,,\\
\operatorname{rot} v = \zeta & \text{in } Q_T\,,\\
\operatorname{div} v = 0 & \text{in } Q_T\,,\\
v\cdot n = 0 & \text{on } [0,T]\times\Gamma\,,\\
\dfrac{\partial\varrho}{\partial t} + v\cdot\nabla\varrho = 0 & \text{in } Q_T\,,\\
\varrho|_{t=0} = \varrho_0 & \text{in } \bar{\Omega}\,,\\
\operatorname{rot} w = 0 & \text{in } Q_T\,,\\
\operatorname{div} w = \dfrac{\nabla\varrho}{\varrho}\cdot w + \varrho\sum\limits_{i,j}(D_j v_i)(D_i v_j) - \varrho\operatorname{div} b & \text{in } Q_T\,,\\
w\cdot n = -\varrho\sum\limits_{i,j}(D_i n_j)v_i v_j - \varrho b\cdot n & \text{on } [0,T]\times\Gamma\,,\\
\zeta|_{t=0} = \alpha & \text{in } \bar{\Omega}\,,
\end{cases}
$$

where $\alpha(x)\equiv\operatorname{rot} a(x)$, $\beta(t,x)\equiv\operatorname{rot} b(t,x)$, and we have extended the outward normal vector $n(x)$ to a neighbourhood of Γ.

First of all we study the system

$$
(4.1)\quad \begin{cases}
\operatorname{rot} w = 0 & \text{in } \Omega\,,\\
\operatorname{div} w - \dfrac{\nabla\varrho}{\varrho}\cdot w = \varrho\sum\limits_{i,j}(D_i v_j)(D_j v_i) - \varrho\operatorname{div} b \equiv f & \text{in } \Omega\,,\\
w\cdot n = -\varrho\sum\limits_{i,j}(D_i n_j)v_i v_j - \varrho b\cdot n \equiv g & \text{on } \Gamma\,,
\end{cases}
$$

where v is divergence free and tangential to the boundary, i.e. $\operatorname{div} v = 0$ in Ω and $v\cdot n = 0$ on Γ. Since Ω is simply-connected, $(4.1)_1$ is equivalent to

$$
(4.2)\qquad w = -\nabla\pi\,;
$$

hence (4.1) is equivalent to

$$(4.3)\qquad \begin{cases} -\Delta\pi + \dfrac{\nabla\varrho}{\varrho}\cdot\nabla\pi = \varrho\sum\limits_{i,j}(D_i v_j)(D_j v_i) - \varrho \operatorname{div} b & \text{in } \Omega\,, \\ \dfrac{\partial\pi}{\partial n} = \varrho\sum\limits_{i,j}(D_i n_j)\, v_i v_j + \varrho b\cdot n & \text{on } \Gamma\,. \end{cases}$$

LEMMA 4.1. *Let* $v \in C(\bar\Omega)^{1+\lambda}$, $\varrho \in C^{1+\lambda}(\bar\Omega)$ *with* $\min\limits_{\bar\Omega} \varrho > 0$, *and* $b \in C^{1+\lambda}(\bar\Omega)$. *Then* $f \in C^{\lambda}(\bar\Omega)$, $g \in C^{1+\lambda}(\Gamma)$ *and problem* (4.1) *has a unique solution* w. *Moreover* $w \in C^{1+\lambda}(\bar\Omega)$ *and*

$$(4.4)\qquad \|w\|_{1+\lambda} \leqslant K\left(\lambda, \Omega, \left\|\frac{\nabla\varrho}{\varrho}\right\|_{\lambda}\right)\{\|f\|_{\lambda} + \|g\|_{1+\lambda;\Gamma}\}\,,$$

where K *is a non-decreasing function in the variable* $\|\nabla\varrho/\varrho\|_{\lambda}$.

PROOF. The existence and uniqueness follow from classical Fredholm alternative arguments (see for instance Miranda [17], Theorems 22.I and 22.III, p. 84); in fact the adjoint homogeneous problem of (4.3), i.e.

$$\begin{cases} \Delta\pi^* + \operatorname{div}\left(\dfrac{\nabla\varrho}{\varrho}\pi^*\right) = 0 & \text{in } \Omega\,, \\ \dfrac{\partial\pi^*}{\partial n} + \left(\dfrac{\nabla\varrho}{\varrho}\cdot n\right)\pi^* = 0 & \text{on } \Gamma\,, \end{cases}$$

has a unique linearly independent solution (since the same holds for the homogeneous equation (4.3)). By direct computation one verifies that this solution is $1/\varrho$ and hence the compatibility condition

$$\int\limits_{\Omega}\frac{f}{\varrho}\,dx = \int\limits_{\Gamma}\frac{g}{\varrho}\,d\sigma$$

is satisfied (see Lemma 5.2).

Moreover the solution π belongs to $C^{2+\lambda}(\bar\Omega)$ and is unique up to a constant. Furthermore (see Miranda [16], Theor. 5.1, or Ladyženskaja-Ural'ceva [12], chap. III, Theor. 3.1, p. 126)

$$\|\pi\|_{2+\lambda} \leqslant K\{\|f\|_{\lambda} + \|g\|_{1+\lambda;\Gamma} + \|\pi\|_{\infty}\}\,,$$

where $K = K(\lambda, \Omega, \|\nabla\varrho/\varrho\|_\lambda)$ is a non-decreasing function in the variable $\|\nabla\varrho/\varrho\|_\lambda$.

One easily sees that the particular solution π of (4.3) such that $\pi(x_0) = 0$, where $x_0 \in \bar{\Omega}$ is fixed, satisfies

$$\|\pi\|_{2+\lambda} \leqslant K\{\|f\|_\lambda + \|g\|_{1+\lambda;\Gamma}\},$$

where K is as before; hence (4.4) holds.

The functional $H[\pi] \equiv \pi(x_0)$ can be replaced by any other bounded linear functional in the uniform topology. □

Lemma 4.2. *Let $v = F^1[\varphi]$, $\varrho = F^2[v]$. Then problem* (4.1) *has a unique solution $w(t, \cdot)$ for each $t \in [0, T]$. Moreover $w \in C^{0,1+\lambda}(Q_T)$ and*

$$\|w\|_{0,1+\lambda} \leqslant \bar{c}(A, T), \tag{4.5}$$

where $\bar{c}$ is non-decreasing in the variables A and T. We denote this unique solution by $w = F^3[v, \varrho]$.

Proof. We have only to see that $w \in C^{0,\lambda}(Q_T)$ and that (4.5) holds. Since $f \in C^{0,\lambda}(Q_T) \subset C^0([0, T]; C^{\lambda'}(\bar{\Omega}))$ and $g \in C^{0,1+\lambda}([0, T] \times \Gamma) \subset C^0([0, T]; C^{1+\lambda'}(\Gamma))$ for each $\lambda' < \lambda$ (see Kato [9], Lemma 1.2), it follows easily from estimate (4.4) (with λ replaced by λ') that $w \in C^0([0, T]; C^{1+\lambda'}(\bar{\Omega})) \subset C^{0,1}(Q_T)$. Finally (4.5) follows from (4.4), from

$$\left\|\frac{\nabla\varrho}{\varrho}\right\|_{0,\lambda} \leqslant c(1 + TA)\left\|\frac{\nabla\varrho_0}{\varrho_0}\right\|_\lambda \exp[cTA], \tag{4.6}$$

and from (3.4) and (3.10). Estimate (4.6) follows from (3.10). □

Now we want to study the vorticity equation

$$\begin{cases} \dfrac{\partial\zeta}{\partial t} + v\cdot\nabla\zeta = \beta + \dfrac{\operatorname{Rot}\varrho}{\varrho^2}\cdot w & \text{in } Q_T, \\ \zeta|_{t=0} = \alpha & \text{in } \bar{\Omega}, \end{cases} \tag{4.7}$$

i.e. equation (3.12) with

$$\gamma \equiv \beta + \frac{\operatorname{Rot}\varrho}{\varrho^2}\cdot w. \tag{4.8}$$

From Lemma 3.4, (3.11), (3.7) and (4.5) one gets easily the following result:

LEMMA 4.3. *Let* $v = F^1[\varphi]$, $\varrho = F^2[v]$, $w = F^3[v, \varrho]$ *and let* $\zeta = F^4\cdot$ $\cdot[v, \varrho, w]$ *be defined by* (3.13) *and* (4.8).
Then $\zeta \in C^{\lambda,\lambda}(Q_T)$ *and*

$$\begin{aligned} &\|\zeta\|_\infty \leqslant \|\alpha\|_\infty + T\bar{c}(A, T)\,, \\ (4.9) \qquad &[\zeta]_{0,\lambda} \leqslant [\alpha]_\lambda \exp[cTA] + T\bar{c}(A, T)\,, \\ &[\zeta]_{\lambda,0} \leqslant c_1 A^\lambda [\alpha]_\lambda \exp[cTA] + T\bar{c}(A, T) + T^{1-\lambda}\bar{c}(A, T)\,. \end{aligned}$$

The function ζ of Lemma 4.3 satisfies $(4.7)_2$ trivially; moreover ζ is a solution of $(4.7)_1$ in the following weak sense:

LEMMA 4.4. *For each* $\phi \in C^1(\bar{\Omega})$ *one has*

$$(4.10) \qquad \frac{d}{dt}(\zeta, \phi) = (\zeta, v\cdot\nabla\phi) + \left(\beta + \frac{\operatorname{Rot}\varrho}{\varrho^2}\cdot w, \phi\right),$$

where (,) *is the scalar product in* $L^2(\Omega)$.

For the proof see Kato [9], Lemma 2.4.

We now define a map F as follows. The main of F is the sphere of $C^{0,\lambda}(Q_T)$ defined by (3.1) with A such that

$$(4.11) \qquad A > \|\alpha\|_\lambda\,.$$

We put

$$\zeta = F[\varphi] = F^4[v, \varrho, w]\,,$$

where successively $v = F^1[\varphi]$, $\varrho = F^2[v]$ and $w = F^3[v, \varrho]$.

It follows from estimates (4.9) that there exists $T_1 \in]0, T_0]$ such that the set

$$(4.12) \qquad S \equiv \{\varphi \in C^{\lambda,\lambda}(Q_T) \mid \|\varphi\|_{0,\lambda} \leqslant A,\ [\varphi]_{\lambda,0} \leqslant c_1 A^{1+\lambda}\}$$

satisfies $F[S] \subset S$, where F, the norms, and the seminorms correspond to the interval $[0, T_1]$.

S is a convex set and by the Ascoli-Arzelà theorem it follows that S is compact in $C^0(Q_{T_1})$.

Moreover

LEMMA 4.5. *The map* $F\colon S \to S$ *has a fixed point.*

PROOF. By Schauder's fixed point theorem we have only to prove that F is continuous from S in S in the $C^0(Q_{T_1})$-topology. Assume that $\varphi_n \to \varphi$ in $C^0(Q_{T_1})$, $\varphi_n \in S$. Then $\varphi_n \to \varphi$ in $C^\varepsilon([0, T_1]; C^\varepsilon(\bar{\Omega}))$, since the immersion (see Kato [9], Lemma 1.2)

$$C^{\lambda,\lambda}(Q_{T_1}) \hookrightarrow C^\varepsilon([0, T_1]; C^\varepsilon(\bar{\Omega}))$$

is compact for $\varepsilon > 0$ small enough.
Consequently from Schauder's estimates

$$v^n \to v \qquad \text{in } C^\varepsilon([0, T_1]; C^{1+\varepsilon}(\bar{\Omega})) . \tag{4.13}$$

By estimating $|U^n(\sigma, t, x) - U(\sigma, t, x)|$ by Gronwall's lemma, one obtains as in [2], Lemma 4.3

$$\| U^n - U \|_\infty \leqslant T_1 \| v^n - v \|_\infty \exp \left[T_1 [v]_{0,\mathrm{lip}} \right] \tag{4.14}$$

and

$$\begin{aligned} \| D_i U^n - D_i U \|_\infty \leqslant T_1 ([Dv]_{0,\lambda} \| D_i U \|_\infty \| U^n - U \|^\lambda + \\ + \| D_i U^n \|_\infty \| Dv^n - Dv \|_\infty) \exp \left[T_1 \| Dv \|_\infty \right]) . \end{aligned} \tag{4.15}$$

Consequently

$$\begin{aligned} &\varrho_n \to \varrho && \text{in } C^0(Q_{T_1}) , \\ &\mathrm{Rot}\, \varrho_n \to \mathrm{Rot}\, \varrho && \text{in } C^0(Q_{T_1}) . \end{aligned} \tag{4.16}$$

On the other hand from the formula $\varrho_n(t, x) = \varrho_0(U^n(0, t, x))$ it follows that $\{\varrho_n\}$ is bounded in $C^{1,1+\lambda}(Q_{T_1})$ and $\{D_i \varrho_n\}$ is bounded in $C^{\lambda,\lambda}(Q_{T_1})$; hence

$$\begin{cases} \varrho_n \to \varrho & \text{in } C^\varepsilon([0, T_1]; C^{1+\varepsilon}(\bar{\Omega})) , \\ \dfrac{\nabla \varrho_n}{\varrho_n} \to \dfrac{\nabla \varrho}{\varrho} & \text{in } C^\varepsilon([0, T_1]; C^\varepsilon(\bar{\Omega})) , \end{cases} \tag{4.17}$$

where $\varepsilon > 0$ is small enough.

Now it follows from (4.13), (4.17) and (4.4) (with λ replaced by ε) that

$$w^n \to w \qquad \text{in } C^0([0, T_1]; C^{1+\varepsilon}(\bar{\Omega})) .$$

Hence $\zeta_n \to \zeta$ in $C^0(Q_{T_1})$. $\square$

The fixed point $\varphi = \zeta = F[\varphi]$ so obtained, together with $v = F^1[\varphi]$, $\varrho = F^2F^1[\varphi]$ and $w = F^3[F^1[\varphi], F^2F^1[\varphi]]$, is a solution of auxiliary system (A) in Q_{T_1}, since from (3.5) $\operatorname{rot} v = \varphi = \zeta$.

Equation $(\mathrm{A})_1$ is satisfied in the sense described in Lemma 4.4.

5. – Existence of a solution of system (E) when Ω is simply-connected.

First of all we prove that $D_t v$ exists in the classical sense and belongs to $C^{0,\lambda}(Q_{T_1})$. Define

$$(G\varphi)(t, x) \equiv \int_\Omega G(x, y)\varphi(t, y)\, dy\,, \tag{5.1}$$

where $G(x, y)$ is the Greenls function for the operator $-\Delta$ with zero boundary condition. Recall that $G\varphi$ is the solution of problem (3.2).

LEMMA 5.1. *Put*

$$\omega(t, x) \equiv -\int_\Omega \nabla_y G(x, y)\cdot(\zeta v)(t, y)\, dy\,. \tag{5.2}$$

Then

$$D_t v = \operatorname{Rot} G\left(\beta + \frac{\operatorname{Rot}\varrho}{\varrho^2}\cdot w\right) - \operatorname{Rot}\omega \qquad \textit{in } Q_{T_1}; \tag{5.3}$$

moreover $\omega \in C^{0,1+\lambda}(Q_{T_1})$, *hence* $D_t v \in C^{0,\lambda}(Q_{T_1})$.

PROOF. For (5.3) see Kato [9], Lemma 3.2. For the regularity of ω see [9], Lemma 1.5, using in this lemma a result of Widman [20] (see also Gilbarg-Trudinger [5], pp. 105-106) instead of a result of Kellogg. $\square$

The following two known results will be useful for proving (5.6) below.

LEMMA 5.2. *If* $v \in C^1(\bar\Omega)$, $\operatorname{div} v = 0$ *in* Ω *and* $v\cdot n = 0$ *on* Γ, *then*

$$\begin{cases} \operatorname{div}[(v\cdot\nabla)v] = \sum\limits_{i,j}(D_i v_j)(D_j v_i) & \textit{in } \Omega\,, \\ [(v\cdot\nabla)v]\cdot n = -\sum\limits_{i,j}(D_i n_j) v_i v_j & \textit{on } \Gamma\,, \end{cases} \tag{5.4}$$

where the operator div *is to be intended in the sense of distributions in* Ω.

For the proof see for instance Temam [19], Lemma 1.1.

LEMMA 5.3. *If* $v \in C^1 \in (\bar{\Omega})$, *then*

$$\operatorname{rot}[(v\cdot\nabla)v] = \operatorname{div}(v \operatorname{rot} v) \qquad \textit{in } \Omega \tag{5.5}$$

in the sense of distributions.

For the proof see Kato [9], Lemma 1.1.

LEMMA 5.4. *The solution* w *of system* (4.1) *is given by*

$$w = \varrho\left[\frac{\partial v}{\partial t} + v(v\cdot\nabla) - b\right] \quad \textit{in } Q_{T_1}. \tag{5.6}$$

PROOF. Set

$$w^* \equiv \frac{w}{\varrho} - \left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] \in C^{0,\lambda}(Q_{T_1}).$$

From $(\mathrm{A})_3$, $(\mathrm{A})_8$ and $(5.4)_1$ it follows that for each $t \in [0, T_1]$

$$\operatorname{div} w^* = 0 \qquad \text{in } \Omega \tag{5.7}$$

in the sense of distributions.

On the other hand from $(\mathrm{A})_9$ and $(5.4)_2$ one has for each $t \in [0, T_1]$

$$w^* \cdot n = 0 \qquad \text{on } \Gamma. \tag{5.8}$$

Finally from (4.10), (5.5) and $(\mathrm{A})_2$ one obtains for each $t \in [0, T_1]$

$$\operatorname{rot}\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right] = \beta + \frac{\operatorname{Rot}\varrho}{\varrho^2}\cdot w \qquad \text{in } \Omega$$

in the sense of distributions.

Hence by $(\mathrm{A})_7$ one obtains for each $t \in [0, T_1]$

$$\operatorname{rot} w^* = 0 \qquad \text{in } \Omega \tag{5.9}$$

in the sense of distributions.

From (5.9) it follows that $w^* = \nabla q$ for $q \in C^{0,1+\lambda}(Q_{T_1})$ (see for instance Kato [9], Lemma 1.6, or Hopf [8]); by using now (5.7), (5.8) it follows that $w^* = 0$ in Q_{T_1}. $\square$

From (4.2) it follows that

$$\varrho\left[\frac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] = -\nabla\pi \quad \text{in } Q_{T_1},$$

i.e. $(\mathrm{E})_1$ holds, with $\pi \in C^{0,2+\lambda}(Q_{T_1})$.

Furthermore

$$\begin{cases} \operatorname{rot}\left(v|_{t=0} - a\right) = \zeta|_{t=0} - \alpha = 0 & \text{in } \bar{\Omega}, \\ \operatorname{div}\left(v|_{t=0} - a\right) = 0 & \text{in } \bar{\Omega}, \\ \left(v|_{t=0} - a\right)\cdot n = 0 & \text{on } \Gamma, \end{cases}$$

and consequently $(\mathrm{E})_6$ holds.

REMARK 5.5. To complete the proof of Theorem A, we observe that from Lemma 3.1 and Lemma 5.1 it follows that $v \in C^{1,1+\lambda}(Q_{T_1})$.

Consequently, from $(\mathrm{E})_4$ and $(3.11)_3$, $D_t\varrho \in C^{\lambda,0}(Q_{T_1})$, i.e.

$$\varrho \in C^{1+\lambda,1+\lambda}(Q_{T_1}) .$$

REMARK 5.6. If estimate (3.7) of [12], chap. III, p. 127, holds with $|u|_{2,\alpha,\Omega}$ and $|Bu|_{1,\alpha,S}$ replaced by $|u|_{1,\alpha,\Omega}$ and $|Bu|_{0,\alpha,S}$ respectively, then in our result it is sufficient to assume that Ω is of class $C^{2+\lambda}$. In this case $\pi \in C^{0,1+\lambda}(Q_{T_1})$ and the function w defined by (4.1) belongs to $C^{0,\lambda}(Q_{T_1})$. The estimates for w in $C^{0,\lambda}(Q_{T_1})$ are sufficient for our method to be applied.

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