Rendiconti
del
Seminario Matematico
della
Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 63 (1980), p. 127-143

<http://www.numdam.org/item?id=RSMUP_1980__63__127_0>

Rend. Sem. Mat. Univ. Padova, Vol. 63 (1980)

Minimal Equations, Global Reducibility of Holomorphic Functions, and Relative Rationality of Analytic Sets.

Chia-Chi Tung (*)

Summary - Given a complex space Y and an analytic subset S of $Y \times \mathbf{C}^k$ two natural questions may be asked: (1) When is S locally given by the common zeroes of holomorphic pseudopolynomials defined over open sets in Y? (2) If f is a holomorphic function on $Y \times \mathbf{C}^k$ with zero set S, when can f be factorized in terms of the minimal equations defining the branches of S? The main results of this paper provide sufficient conditions which assure positive answers to (1) and (2).

Let X be a (reduced) complex space of pure dimension $m > 0$. An analytic subset S of X is called an (*analytic*) *hypersurface* if S has pure codimension 1. Let $S \subseteq X$ be a hypersurface. A holomorphic equation $g = 0$ on an open subset U of X is called a ***minimal equation*** of S on U iff (i) $S \cap U = Z(g)$ (the zero set of g); (ii) g has 0-multiplicity 1 at every simple point of S in U. If U consists of normal points only, the above definition amounts to requiring that the germs of g generate all stalks of the (nullstellen) ideal sheaf of S over U (cf. Theorem 1.3).

Let Y be an irreducible complex space of dimension $p > 0$ and Z a hypersurface in $Y \times \mathbf{P}$. Assume the projection map $\psi\colon Z \to Y$ has strict rank p ([1, p. 17]) and the set $Z_\infty = Z \cap (Y \times \{\infty\})$ is thin on Z. Set $Y^* = Y_{\text{reg}} - \psi(Z_\infty)$, $Z_0 = Z \cap \psi^{-1}(Y^*)$. Stoll [1, 5.1] showed

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that on $Y^*\times\mathbf{C}$, the fiber product

$$(0.1)\qquad g(y,t)=\prod\{(t-\xi)^{\nu(\psi;(y,\xi))}|(y,\xi)\in Z_0\}$$

(where $\nu(\psi;w)$ denotes the multiplicity of ψ at w) is holomorphic with zero set Z_0. Geometrically (0.1) implies, when restricted to the fibres of the projection π: $Y^*\times\mathbf{C}\to Y^*$, the function g attains the minimal 0-multiplicity at every $w\in Z_0$, that is,

$$(0.2)\qquad \nu^0(g_y;w)=\nu(\psi;w)\quad(\text{where } y=\pi(w))\,.$$

It seems reasonable, therefore, to expect that (0.2) would give a characterization for a holomorphic equation $g=0$ (on a product space) to be the minimal equation defining its zero set. The precise (in fact somewhat weakened) conditions are given in Theorem 2.3 and Lemma 2.1. Furthermore, with Z as above, for every point in $Y_{\text{norm}}-\{y\in Y_{\text{sing}}|\dim Z_y=1\}$, a neighborhood V and a minimal equations $g=0$ on $V\times\mathbf{C}$ of $Z\cap(V\times\mathbf{C})$ can be constructed, and (0.2) remain valid on all fibres $Z(g)_y$ where $g_y\not\equiv 0$ (see Theorem 4.5).

Let Y be a complex space of dimension $p\geqslant 0$. The above considerations give rise to two natural questions: (1) When is an analytic set S in $Y\times\mathbf{C}^k$ locally given by the common zeros of holomorphic pseudo-polynomials defined over an open subset of Y? (Such an S is said to be *rational over* Y). (2) If f is a holomorphic function on $Y\times\mathbf{C}^k$ whose zero set is rational over Y, can f be factorized in terms of the minimal equations defining the branches of $Z(f)$ over an open subset of Y?

The rationality question is treated in § 3. If S is a hypersurface and if Y is pure dimensional, the following conditions are equivalent (Theorem 3.2): Let $\bar{S}$ denote the closure of S in $Y\times\mathbf{P}_k(\mathbf{C})$ and $s=\dim S$.

(i) $\bar{S}$ is analytic in $Y\times\mathbf{P}_k(\mathbf{C})$.

(ii) S is rational over Y.

(iii) The slice S_y is properly algebraic or empty for all y in a set of positive (Hausdorff) $2p$-measure in every branch of Y.

(iv) $\bar{S}_\infty$ has zero $2s$-measures in $Y\times\mathbf{P}_k(\mathbf{C})$.

If S has positive codimension, then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii), and (iv) $\Rightarrow$ (i) provided S is pure dimensional. The assertion « (i) $\Rightarrow$ (ii) » is a consequence of the relative Chow's Theorem (see Fischer [3, 1.3]). The proof given in Theorem 3.1 is, however, based on the relative G.A.G.A. theory of Grauert-Remmert [4]; this proof also yields the Chow's Theorem for the reduced case.

Possible answers to question (2) are provided by Theorem 4.2, Corollaries 4.3 and 4.4. To mention the main result, let Y be a connected Stein manifold of dimension p with $H^2(Y; \mathbf{Z}) = 0$; assume S is a hypersurface in $Y \times \mathbf{C}^k$ satisfying condition (iii) above with $\operatorname{rank} \pi_Y | S = p$ (π_Y being the projection: $Y \times \mathbf{C}^k \to Y$). Then the set $\operatorname{id}(S) = \{f \in \mathcal{O}(Y \times \mathbf{C}^k) | f | S = 0\}$ is a principal ideal. It follows that if $f \in \mathcal{O}(Y \times \mathbf{C}^k)$ with $Z(f) = S$, then f can be factorized up to a unit in the form $(\pi_Y^* \varphi) g_1^{\lambda_1} \dots g_r^{\lambda_r}$, where $\varphi \in \mathcal{O}(Y)$, every g_j is a pseudo-polynomial defined over Y, and the set $\{Z(g_1) \dots, Z(g_r)\}$ gives all branches of S on which π_Y attains maximal rank p. If f is a generator of id (S), then all $\lambda_j = 1$ and $(\pi_y^* \varphi) g_1 \dots g_r = 0$ gives a minimal equation of S.

The author wishes to thank Professor O. Forster for suggestion the use of [4] in proving the relative Chow's Theorem, and Professor J. Becker for observing that « $(a) \Rightarrow (d)$ » in Theorem 3.1.

1. – Divisibility of holomorphic functions.

If U is an open set in X, let $\mathcal{O}(U) = H^0(U, \mathcal{O}_X)$. Two functions $f, g \in \mathcal{O}(X)$ are called equivalent ($f \sim g$) on X if $f/g \in \mathcal{O}^*(X)$. If $\pi: X \to Y$, $f: X \to \mathbf{C}$, $M \subseteq X$ and $y \in Y$, set $M_y = M \cap \pi^{-1}(y)$, $f_y = f | X_y$ (provided $X_y \neq \emptyset$); also define

$$d(f) = \{y \in \operatorname{Im}(\pi) | f_y \equiv 0\}, \qquad Y_f = Y - d(f).$$

Some general assumptions shall be stated here for later reference.

(G_1) $\pi: X \to Y$ is a holomorphic fiber space with pure fiber dimension $m - p \geqslant 0$ over a complex space Y of pure dimension $p \geqslant 0$.

(G_2) Y is a locally irreducible complex space.

(G_3) $\pi: X \to Y$ is a locally trivial holomorphic fiber space over a complex space Y of pure dimension $p \geqslant 0$ with irreducible fibers.

(G_4) Y is a connected Stein mainfold of dimension p with $H^2(Y; \mathbf{Z}) = 0$.

Unless otherwise stated, assume (G_1). Call $f \in \mathcal{O}(X)$ *π-simple* iff $d(f)$ is either empty or almost thin of codimension 2 in Y ([1, p. 14]). If (G_3) holds, the set $d(f)$ is analytic in Y by [12, Lemma 1.1] and, for $f = f_1 f_2$ with $f_j \in \mathcal{O}(X)$, f is π-simple iff so are f_1 and f_2.

Let $f \in \mathcal{O}(X) - \{0\}$. A pair $(g, \varphi) \in \mathcal{O}(X) \times \mathcal{O}(Y)$ is called a *reduction* of f (over Y) iff $f = (\pi^* \varphi) g$ on X. Two reductions (g, φ), $(\tilde{g}, \tilde{\varphi})$ of f are called *equivalent* iff there exists $u \in \mathcal{O}^*(Y)$ such that $\varphi = u\tilde{\varphi}$ and $\tilde{g} = (\pi^* u) g$. A reduction (g, φ) of f over Y is called a *simplification* of f(w.r.t. π) iff g is π-simple.

Let S be a hypersurface in X and $\psi = \pi | S$. Define

$$\hat{S} = \{w \in S | \operatorname{rank}_w \psi = p\}$$

$$E(\psi) = S - \hat{S}\,, \qquad d(S) = \psi(E(\psi))\,.$$

If π has irreducible fibers, then

$$d(S) = \{y \in \operatorname{Im}(\pi) | S_y = X_y\}\,. \tag{1.1}$$

The hypersurface S is called *π-strict* if ψ has strict rank p.

LEMMA 1.1. Assume (G_3). Let $f \in \mathcal{O}(X) - \{0\}$ with $Z = Z(f) \neq \emptyset$.

(1) Z is a π-strict hypersurface in X iff f is π-simple.

(2) Assume $\operatorname{codim} Z > 0$, $\operatorname{rank} \pi | Z = p$, and f has a simplification (g, φ). Then $Z(g)$ is the largest π-strict hypersurface in X on which f vanishes.

PROOF. If Z has pure codimension 1, (1.1) implies that $d(f) = d(Z)$. Therefore, if Z is a π-strict hypersurface, f is π-simple by [1, 1.24]. Conversely, if f is π-simple, then $\operatorname{codim} d(f) \geqslant 2$, which implies that Z is a hypersurface with $\operatorname{codim} \pi^{-1}(d(f)) \geqslant 2$, thus the restriction map $\psi = \pi\colon Z \to Y$ has strict rank p. Now assertion (2) also follows. Q.E.D.

Define

$$Y_S = \{y \in Y | \dim S_y < \dim X_y\}\,.$$

Then

$$Y_S = \operatorname{Im}(\pi) - d(S)\,. \tag{1.2}$$

Let D be a divisor on X and $S(D) =$ the support of D. For $y \in Y_{S(D)}$, a divisor D_y on X_y can be defined as follows: If f is a meromorphic function on an open set $U \subseteq X$ with divisor $(f) = D|U$, then

$$D_y|U_y = (f|U_y)\,.$$

D is called *effective* iff locally D is represented by holomorphic functions. If X is normal, then D is effective iff $D|X_{\mathrm{reg}}$ is effective.

Let $g \in \mathcal{O}(X) - \{0\}$. Denote by $\nu(g; w)$ (resp. $\nu_0(g; w)$) the multiplicity (resp. 0-multiplicity) of g at $w \in X$ (see [10], [11]).

THEOREM 1.2. Assume (G_1)-(G_2) and X is normal. Let D be a divisor on X. Assume (i) $S = S(D)$ is π-strict; (ii) Y_S contains a dense subset Q such that for each $y \in Q$, D_y is effective. Then D is effective.

PROOF. Let W be a non-empty, open subset of $\hat{S}$. Then $V = \psi(W)$ is open in Y. It follows from (1.2) that $V \cap Y_S \neq \emptyset$. Hence

$$(1.3) \qquad \big(V - d(S)\big) \cap Q \neq \emptyset\,.$$

Take $w_0 \in \hat{S}_{\mathrm{reg}}$. W.o.l.g. assume X is non-singular. Choose a neighborhood U of w_0 in $X - E(\psi)$ such that U is biholomorphic to an open ball in $\mathbf{C}^m$ and $S \cap U = S_{\mathrm{reg}} \cap U$ is connected. There exist relatively prime holomorphic functions f, g on U with $(f/g) = D|U$. Then $\operatorname{codim} Z(f) \cap Z(g) \geqslant 2$. Suppose $g(w_0) = 0$. Then f is non-vanishing on a neighborhood $U' \subseteq U$ of w_0. By (1.3), $S \cap U'$ contains a point w with $\psi(w) = u \in Q$. For such (w, y),

$$D_y(w) = -\nu^0(g_y;\, w) < 0\,,$$

hence D_y is not effective, a contradiction! Therefore, $f/g \in \mathcal{O}(U)$. It follows that D is effective on X. Q.E.D.

THEOREM 1.3. Assume (G_1), where X is a normal space, and π has normal irreducible fibers. Let $f, g \in \mathcal{O}(X)$. Assume (i) g is π-simple; (ii) there exists a dense subset Q of $Y_{\mathrm{reg}} - d(g)$ such that if $y \in Q$,

$$\nu^0(g_y;\, w) \leqslant \nu^0(f_y;\, w) < +\infty$$

for all $w \in X_{\mathrm{reg}} \cap Z(g_y)_{\mathrm{reg}}$: Then $f/g \in \mathcal{O}(X)$.

PROOF. Observe that by Lemma 1.1-(1), $\operatorname{codim} Z(g) \cap \pi^{-1}\big(d(f)\big) \geqslant 2$. Therefore it suffices to prove the holomorphy of f/g on $\pi^{-1}(Y_f) \cup$ $\cup \big(X - Z(g)\big)$. In view of Theorem 1.2, one needs only show that f_y/g_y

is holomorphic for each $y \in Q$. This follows from Stoll [1, p. 267] (or Whitney, [13, 9J, p. 29]). Here a simple proof of this lemma (for a normal space) using the Weierstrass division theorem and results of [11] shall be given.

LEMMA. Let U be a normal space and $f, g \in \mathcal{O}(U)$. Assume $Z = Z(g)$ has positive codimension and for all $w \in Z_{\text{reg}} \cap U_{\text{reg}}$,

$$\nu^0(g; w) \leqslant \nu^0(f; w)$$

Then $f/g \in \mathcal{O}(U)$.

PROOF. Assume $Z \neq \emptyset$ and $a \in Z \cap U_{\text{reg}}$. In terms of a local patch of U, one can identify a neighborhood N of a with an open set $G \times H \subseteq \mathbf{C}^{m-1} \times \mathbf{C}$ such that (i) $a = 0 \in \mathbf{C}^m$; (ii) for some open polydiscc $H_0 \subset\subset H$, $Z \cap (G \times \bar{H}_0) = Z \cap N$. Then $s = \sum \{\nu^0(g_{(z)}; \zeta) | \zeta \in H\} =$ const. for all $z \in G$ (where $g_{(z)}(\zeta) = g(z; \zeta)$) ([10, Thm. 3.7]). By the division theorem, there exist $h \in \mathcal{O}(G \times H)$ and $r \in \mathcal{O}(G)[t]$ with $\deg r < s$ (if $r \not\equiv 0$) such that

$$f = gh + r \quad \text{on } G \times H . \tag{1.4}$$

There exist thin analytic subsets A of $Z \cap N$ and A' of G such that the projection $P: (Z \cap N) - A \to G - A'$ is locally biholomorphic. Take $(z, b) \in (Z \cap N) - A$. Then P is a minimal slicing map of g at (z, b) ([11, 2.1.7]): Thus

$$\begin{aligned} \nu^0(g; (z, b)) &= \nu^0((g, P); (z, b)) && ([11, 2.2.1]) \\ &= \nu^0(g_{(z)}; b) && ([11, 1.2.2]) . \end{aligned}$$

Now

$$\begin{aligned} \nu^0(f_{(z)}; b) &\geqslant \nu^0(f; (z, b)) && ([1, 13.1]) \\ &\geqslant \nu^0(g; (z, b)) = \nu^0(g_{(z)}; b) . \end{aligned}$$

If $r_{(z)} \not\equiv 0$, (1.4) implies that $\deg r \geqslant s$, a contradiction! Thus $h = f/g \in \mathcal{O}(N - A)$. Since codim $A \geqslant 2$, h is holomorphic on N. It follows that $h \in \mathcal{O}(U)$. Q.E.D.

LEMMA 1.4. Assume (G_3) and the fibers of π are normal. Let $f \in \mathcal{O}(X) - \{0\}$.

(1) If Y is normal and codim $Z(f) > 0$, then any two simplifications of f w.r.t. π are equivalent.

(2) If (G_4) holds, then f admits a simplification w.r.t. π.

PROOF. (1) By Grauert-Remmert [5, p. 197, Satz 14], X is a normal space. Let (g, φ), $(\tilde{g}, \tilde{\varphi})$ be any two simplifications of f w.r.t. π. If $y \in Y_f$, $g_y \sim \tilde{g}_y$ on X_y. Hence Theorem 1.3 asserts that the quotient $h = \tilde{g}/g \in \mathcal{O}^*(X)$. Clearly on $\pi^{-1}(Y_f)$, $h = \pi^*(\tilde{\varphi}/\varphi)$. Therefore, $\varphi/\tilde{\varphi}$ extends to a function $u \in \mathcal{O}^*(Y)$, thus proving the equivalence.

(2) By [12, Lemma 3], w.r.t. the map $\pi_0 = \pi: X_{\text{reg}} \to Y$, $f|X_{\text{reg}}$ has a simplification (g, φ). Since X is normal, (g, φ) extends to a simplification of f. Q.E.D.

REMARK. Assume Y is a normal space and $g \in \mathcal{O}(Y)[t_1, \dots, t_k]$. Then lemma 1.4-(2) implies that g is π_Y-simple iff the coefficients a_μ $(\mu \in \mathbf{Z}^k)$ of g induce coprime germs in $\mathcal{O}_{Y,y}$, for every $y \in Y_{\text{reg}}$.

2. – Minimal equations.

Throughout this section assume (G_1)-(G_2) with $p > 0$. Let $\mathcal{T}(\pi)$ be the set of all $w \in X$ at which π is locally equivalent to a projection ([11, § 1.2]). Let S be a hypersurface in X. If $f \in \mathcal{O}(X)$ with $Z(f) = S$ and if $w \in \hat{S}$, there exists a neighborhood U of w in X such that the junction $(\pi, f): U \to Y \times \mathbf{C}$ is q-fibering with $q = m - p - 1$, hence the multiplicities $\nu((\pi, f); w)$, $\nu(\psi; w)$ are defined ([11]).

LEMMA 2.1. Assume $S_{\text{reg}} \subseteq \mathcal{T}(\pi)$, and $g = 0$ is a minimal equation of S. If $q > 0$, assume X, Y are non-singular. Then for all $w \in \hat{S}_{\text{reg}}$,

$$\nu(\psi; w) = \nu(g_y; w) \quad \big(\text{where } y = \pi(w)\big). \tag{2.1}$$

PROOF. By [11, 2.2.2], g is projective at every simple point of S. Let $w \in \hat{S}_{\text{reg}}$ and $y = \pi(w)$. Then [11, 1.2.4] (if $q = 0$) or [11, 2.2.7] (if $q > 0$) imply that

$$\nu(\psi; w) = \nu\big((\pi, g); w\big) = \nu(g_y; w). \quad \text{Q.E.D.}$$

LEMMA 2.2. Assume $S_{\text{reg}} \subseteq \mathcal{T}(\pi)$. Let $f \in \mathcal{O}(X)$ with $f|S = 0$.

(1) If $q = 0$, then for all $w \in \hat{S}$,

$$\nu(\psi; w) \leqslant \nu\big((\pi, f); w\big) = \nu(f_y; w) \quad \big(\text{where } y = \pi(w)\big). \tag{2.2}$$

(2) If $q > 0$, assume X, Y are non-singular. Then (2.2) remains valid for all $w \in \hat{S}_{\text{reg}}$.

PROOF. If $q = 0$, (2.2) follows from [11, 1.1.4 and 1.2.4]. Now assume $q > 0$ and X, Y are non-singular. Let $v \in \hat{S}_{\text{reg}}$. Choose an (open) neighborhood $U \subseteq X$ of w biholomorphic to an open ball in $\mathbf{C}^m$. Then $S \cap U$ admits a minimal defining equation $g = 0$ on U. By (2.1), Theorem 1.3 and [11, 2.2.7].

$$\begin{aligned} \nu(\psi; w) = \nu(g_v; w) &\leqslant \nu(f_v; w) \qquad \big(y = \pi(w)\big) \\ &= \nu\big((\pi, f); w\big). \qquad \text{Q.E.D.} \end{aligned}$$

THEOREM 2.3. *Let D be a divisor on X such that the support S of D is π-strict and $S_{\text{reg}} \subseteq \mathfrak{T}(\pi)$. If $q = 0$, assume X is normal; if $q > 0$, assume X, Y are non-singular. Assume there exists a dense subset Q of Y_S such that for each $y \in Q$, D_y is effective and*

$$D_y(w) = \nu(\psi; w) \qquad \text{for } w \in \hat{S}_{\text{reg},y}. \tag{2.3}$$

Then D is the minimal effective divisor with $S(D) = S$.

PROOF. Taking $W = X_{\text{reg}} \cap \hat{S}_{\text{reg}}$, (1.3) shows there is a point $w \in W$ with $y = \psi(w) \in Q$. Let U be a neighborhood of w in $X_0 = X_{\text{reg}} - E(\psi)$ such that $S \cap U$ is defined by a minimal equation $g = 0$ on U and $D|U = (f)$ for some $f \in \mathcal{O}(U)$. Let $h = f/g$. Then

$$\nu^0(h_y; z) = \nu^0(f_y; z) - \nu^0(g_y; z) \qquad (z \in U_y).$$

By (2.1) and (2.3), $\nu_0(h_y; w) = 0$. Hence f/g is invertible holomorphic on a neighborhood of w. Then $D|X_0$ gives minimal local equations defining $S \cap X_0$. Since $E(\psi) \cup X_{\text{sing}}$ has codimension $\geqslant 2$, D gives minimal local equations defining S. Q.E.D.

3. – Relative rationality of an analytic set.

Let Y be a complex space of dimension p. A subset M of $Y \times \mathbf{C}^k$ is called *relatively rational* iff M is the solution set of a system of pseudo-polynomial equations

$$g_j(y, t) = \sum_{|\mu| \geqslant r_j} a_{j\mu}(y)\, t_1^{\mu_1} \dots t_k^{\mu_k} = 0 \qquad (j = 1, \dots, l) \tag{3.1}$$

where $a_{j\mu} \in \mathcal{O}(Y)$; such a g_j is called *rational over* Y.

Let $E \to Y$ be a holomorphic $\mathbf{C}^k$-bundle over Y, and M an analytic subset of E. Then M is said to be *rational over* Y iff, in terms of local trivializations of E, M is (locally) relatively rational. Observe that the local relative rationality of M does not depend on the choice of the local trivialization of E.

Define $\tilde{E} = (Y \times \mathbf{C}) \oplus E$ and $\bar{E} = \mathbf{P}(\tilde{E})$. The fiber bundle $\bar{E} \xrightarrow{\bar{\pi}} Y$ is called the *projective completion* of E. There exist natural biholomorphic imbeddings of E, resp. $\mathbf{P}(E)$, into $\bar{E}$, such that $\bar{E} = E \cup \mathbf{P}(E)$ and $E \cap \mathbf{P}(E) = \emptyset$. If $W \subseteq \bar{E}$, set $W_\infty = W \cap \mathbf{P}(E)$. Let $\bar{M}$ be the closure of M in $\bar{E}$, and $s = \dim M$.

Consider the following conditions:

(a) M is rational over Y.

(b) $A(M) = \{y \in Y | M_y$ is properly algebraic or empty$\}$ is of positive $2p$-measure ([12, p. 399]) in every branch of Y.

(c) $\bar{M}_\infty$ has zero $2s$-measure in $\bar{E}$.

(d) $\bar{M}$ is analytic in $\bar{E}$.

THEOREM 3.1. Assume $s < p + k$. Then (d) $\Leftrightarrow$ (a) $\Rightarrow$ (b); moreover, if M has pure dimension s, then (c) $\Rightarrow$ (d).

PROOF. Let $\bar{E}(U) = \bar{\pi}^{-1}(U) \xrightarrow{\sim} U \times \mathbf{P}_k$ be a local trivialization of $\bar{E}$ such that (under identifications) $E(U) = U \times \mathbf{C}^k \hookrightarrow \bar{E}(U) = U \times \mathbf{P}_k$ by $(y_1; t_1, \ldots, t_k) \hookrightarrow (y_1; [1, t_1, \ldots, t_k])$. The hyperplane section bundle over $\mathbf{P}_k$ lifts to a line bundle $H \to U \times \mathbf{P}_k$ by the projection: $U \times \mathbf{P}_k \to \mathbf{P}_k$ Let $\mathfrak{F}$ denote the sheaf of germs of holomorphic sections in H and set $\mathfrak{F}^n = \overset{n}{\otimes} \mathfrak{F}$.

Assume $\bar{M}$ is analytic in $\bar{E}$. Let $\mathfrak{S}$ be the ideal sheaf of $\bar{M}_U = \bar{M} \cap \bar{E}(U)$ over $U \times \mathbf{P}_k$. Since $\mathfrak{S} \subseteq \mathcal{O}_{U \times \mathbf{P}_k}$, the space $H^0(U \times \mathbf{P}_k, \mathfrak{S} \otimes \mathfrak{F}^n)$ corresponds to the subspace of those sections of $H^0(U \times \mathbf{P}_k, \mathfrak{F}^n)$ which vanish on $\bar{M}_U$. Thus to each $\sigma \in H^0(U \times \mathrm{Pk}, \mathfrak{S} \otimes \mathfrak{F}^n)$, there is associated a homogeneous pseudo-polynomial

$$H(y; t_0, \ldots, t_k) = \sum_{|\mu| \leqslant n} a_\mu(y) t_0^{n-|\mu|} t_1^{\mu_1} \ldots t_k^{\mu_k}$$

vanisshing on $\bar{M}_U$ with coefficients $a_\mu \in \mathcal{O}(U)$.

Let $y_0 \in U$ and V_0 be an open neighborhood of y_0 with compact closure in U. According to Grauert-Remmert [4, Satz I_n], for sufficiently large $n \in \mathbf{Z}(0, \infty)$, the coherent sheaf $\mathfrak{S} \otimes \mathfrak{F}^n(V_0 \times \mathbf{P}_k)$ is simple w.r.t. the projection $\tau: V_0 \times \mathbf{P}_k \to V_0$. Thus a neighborhood $V \subseteq V_0$ of y_0 and finitely many sections $\sigma_j \in H^0(V \times \mathbf{P}_k, \mathfrak{S} \otimes \mathfrak{F}^n)$ $(j = 1, \dots l)$ can be selected so that $\sigma_1, \dots, \sigma_l$ together generate every stalk of $\mathfrak{S} \otimes \mathfrak{F}^n$ over $V \times \mathbf{P}_k$.

Take $w \in U \times \mathbf{P}_k$. Let $\mathfrak{J}_w$ be the ideal sheaf of w over $U \times \mathbf{P}_k$. There is an exact sequence

$$0 \to \mathfrak{J}_w \otimes \mathfrak{S} \otimes \mathfrak{F}^n \to \mathfrak{S} \otimes \mathfrak{F}^n \to \mathfrak{S}_w \otimes \mathfrak{F}^n_w \to 0 .$$

Taking n sufficiently large, the coherent sheaf $\mathfrak{J}_w \otimes \mathfrak{S} \otimes \mathfrak{F}^n$ is B-simple w.r.t. τ ([4, p. 416]). Then there exists a neighborhood basis $\{W\}$ of w such that for every W,

$$H^0(W \times \mathbf{P}_k, \mathfrak{S} \otimes \mathfrak{F}^n) \to H^0(W \times \mathbf{P}_k, \mathfrak{S}_w \otimes \mathfrak{F}^n_w) \to 0 .$$

Hence, given $w \in (V \times \mathbf{P}_k) - \bar{M}_V$, at least one σ_j $(1 \leqslant j \leqslant l)$ does not vanish at w. Let H_j be the homogeneous pseudo-polynomial associated to σ_j. Then on $V \times \mathbf{C}^k$ the solution set of the system of equations

$$H_j(y; 1, t_1, \dots, t_k) = 0 \qquad (j = 1, \dots, l) .$$

is precisely M_V. This proves the assertion $(d) \Rightarrow (a)$.

Assume M is rational over Y. Let M_U be defined on $E(U)$ by the equations (3.1) with $a_{j,u} \in \mathcal{O}(U)$. Then the set

$$U - A(M) = \bigcap_{j=1}^{l} \bigcap_{|\mu| \leqslant r_j} Z(a_{j\mu})$$

in thin analytic in U. Hence (b) follows. To prove (d), observe that the homogenized system of (3.1) defines an analytic subset Z of $\bar{E}(U)$ with $Z - \bar{E}_\infty = M_U$. Hence by Narasimham [7, Prop. 4′, p. 71], $\bar{M}_u$ is analytic in $\bar{E}(U)$. This yields the analyticity of $\bar{M}$.

Now assume M is pure s-dimensional and condition (c) holds. Take $w \in \bar{E}_\infty$. In terms of a local patch of $\bar{E}$ at w, the extension theorem of $\bar{E}$. Bishop [2, Lemma 9] shows that $\bar{M}$ is analytic at w. Therefore (d) follows. Q.E.D.

THEOREM 3.2. *Assume Y has pure dimension p. Let M be a hypersurface in E. Then the above conditions (a)-(d) are equivalent.*

PROOF. It remains to prove that $(b) \Rightarrow (c)$. This follows from [12, Thm. 1.2]. Q.E.D.

For later use, some general properties of the counting function of light holomorphic maps shall be provided here.

Let M, Y, Q be complex spaces of pure dimension m, p, q resp. with $m = p + q$. Let $\tau\colon M \to Q$ be a holomorphic p-fibering and $f\colon M \to Y$ be continuous. If $G \subseteq M$, $t \in Q$ and $y \in Y$, set $G_t = G \cap \cap \tau^{-1}(t)$. $G_t^y = G_t \cap f^{-1}(y)$. Assume Y is locally irreducible and, for each $t \in \tau(M)$, the map $f_t = f|M_t$ is holomorphic.

Take $(a, b) \in Y \times Q$. Let $G \subseteq M$ be an open set. If $\bar{G}$ is compact with $\bar{G}_b^a = \emptyset$. Then $\bar{G}_t^y = \emptyset$ for all (y, t) in a neighborhood of (a, b) ([11, 1.1.5]). Now assume dim $M_b^a \leqslant 0$. Define

$$n_f(G_b;\, a) = \begin{cases} 0\,, & \text{if } G_b^a = \emptyset\,. \\ \sum \{\nu(f_b;\, w) | w \in G_b^a\}\,, & \text{if } G_b^a \neq \emptyset\,. \end{cases}$$

$$n_f(a, b) = n_f(M_b;\, a)\,.$$

Assume also: 1) $\bar{G}$ is compact with $G_b^a \neq \emptyset$; 2) $\partial G \cap M_b^a = \emptyset$; 3) $M_b^a \subseteq \subseteq \mathcal{S}(\tau)$. Then there exist open sets G', G'' in G such that $\bar{G}_b^a \subseteq G'' \subset\subset G'$ and $f|G'$ is light along fibers of $\tau|G'$ ([11, 1.1.1]). One can choose open connected neighborhoods A of a and B of b with $(\bar{G} - G'') \cap f^{-1}(A) \cap \cap \tau^{-1}(B) = \emptyset$ ([11, 1.1.5]). Then by [11, 1.2.17 and 1.2.4],

$$n_f(G_t;\, y) = \text{const.} > 0 \tag{3.2}$$

for all $(y, t) \in A \times B$. Therefore

$$n_f(a, b) < +\infty \Rightarrow n_f \text{ attains a local min. at } (a, b)\,. \tag{3.3}$$

LEMMA 3.3. Assume f_t is a finite map for all $t \in \tau(M)$ and $M = \mathcal{S}(\tau)$. Then the set

$$U = \{(y, t) \in Y \times Q | n_f \text{ is locally const. at } (y, t)\}$$

is open, dense in $Y \times Q$. Also, U is the largest open set in $Y \times Q$ such that setting $U' = \{w \in M | (f(w), \tau(w)) \in U\}$, the map $(f, \tau) = U' \to U$ is proper.

PROOF. Let $W \neq \emptyset$ be an open subset of $Y \times Q$. Define $W(r) = \{(y, t) \in W | n_f(y, t) \leq r\}$. Then $W = \bigcup W(r)$. Since, for some r, $W(r)$ has non-void interior, (3.3) shows that $U \cap W \neq \emptyset$. Hence U is dense in $Y \times Q$. Using (3.2), it can be shown that a point (y, t) of $Y \times Q$ is in U iff it has a neighborhood $J \times H \subseteq Y \times Q$ such that $f^{-1}(J) \cap \cap\, \tau^{-1}(H)$ has compact closure in M. From this the second assertion follows. Q.E.D.

REMARK. 1) It follows from (3.3) and the denseness of U that $\sup n_f | U = \sup n_f | Y \times Q$. 2) If $f\colon M \to Y$ is a light holomorphic map with $n_f(M; y)$ locally constant on Y, then f is proper.

4. – Global reducibility of a holomorphic function.

Let $\pi\colon E \to Y$ be a holomorphic $\mathbf{C}^k$-bundle over a complex space Y. The first part of the following result generalizes a theorem of Ronkin [9].

THEOREM 4.1. Assume Y is normal, irreducible, $g_j \in \mathcal{O}(E) - \{0\}$ $(j = 1, 2)$ and $f = g_1 g_2$: Assume f is (locally) rational over Y. Then (1) there exist $P_j \in \mathcal{O}(E)$ $(j = 1, 2)$ with $P_j \sim g_j$ such that $f = P_1 P_2$ and each P_j is rational over Y; (2) if g_1 is rational over Y, so is g_2.

PROOF. (1) Since $Z(f) = Z(g_1) \cup Z(g_2)$, it follows from [12, Thm. 1.1] that there exist $P_j \in \mathcal{O}(E)$ $(j = 1, 2)$ with P_j rational over Y and $u_j = g_j / P_j \in \mathcal{O}(E)$. Let $\tilde{u} = u_1 u_2$. The set

$$c(\tilde{u}) = \{y \in Y | \tilde{u}_y = \text{const.}\}$$

is analytic in Y ([12, Lemma 1]) and contains Y_f. Hence $c(\tilde{u}) = Y$. Then $\tilde{u} = \pi^* u$ for some $u \in \mathcal{O}^*(Y)$, from which the conclusion follows.

(2) Write $f = P_1 P_2$ as above. Then $c(P_1/g) = Y$. Hence $g_2 = P_1 P_2 / g_1 = (\pi^* u) P_2$ for some $u \in \mathcal{O}^*(Y)$. Q.E.D.

In Theorem 4.2 and Corollary 4.3, assume (G_4) and let S be a hypersurface in E. Define $\operatorname{id}(S) = \{f \in \mathcal{O}(E) | f | S = 0\}$.

THEOREM 4.2. Assume $A(S)$ has positive measure in Y. The $\operatorname{id}(S)$ is a principal ideal with a generator g having the following properties:

(1) $g = 0$ is a minimal equation of S, and g admits a simplication $(\tilde{g}, \tilde{\varphi})$ with $\tilde{g} = h_0 + \ldots + h_n (h_n \not\equiv 0)$, the h_j being homogeneous of degree j on E and $(h_j, \tilde{\varphi})$ uniquely determined up to equivalence.

(2) Assume $\pi|S$ has rank P. Then *a*) $\tilde{g} = g_1 \dots g_r$, where each $g_j \in \mathcal{O}(E)$ is π-simple, irreducible, and rational over Y of positive degree, and the set $\{Z(g_1), \dots, Z(g_r)\}$ gives all branches of $Z(\tilde{g})$; *b*) if $f \in \mathcal{O}(E)$ with $Z(f) = S$, there exist $\varphi \in \mathcal{O}(Y)$, $u \in \mathcal{O}^*(E)$ and integers $\lambda_j > 0$ $(j = 1, \dots, r)$ such that

$$f = (\pi^*\varphi) g_1^{\lambda_1} \dots g_r^{\lambda_r} \cdot u \tag{4.1}$$

PROOF. (1) Since E is Stein with $H^2(E; \mathbf{Z}) = 0$, S has a minimal equation $g = 0$ on E. By [12, Thm. 1.1], one can choose g to be rational over Y. Let $(\tilde{g}, \tilde{\varphi})$ be a simplification of g w.r.t. π. Then $\tilde{g}$ is rational over Y (Thm 4.1-(2)); hence [12, Thm 1.1] yields an expansion $\tilde{g} = \sum_{j=0}^{n} h_j$, where h_j is homogeneous of degree j on E and $h_n \not\equiv 0$. Now Lemma 1.4-(1) implies the uniqueness of the $(h_j, \tilde{\varphi})$.

(2) Let $\tilde{S} = Z(\tilde{g})$. Every branch Z_j of $\tilde{S}$ has a minimal equation $f_j = 0$ on E. Since the quotient $\tilde{g}/f_j \in \mathcal{O}(E)$, Theorem 4.1-(1) shows that $A(Z_j)$ has non-void interior. Then (1) gives a rational minimal equation $g_j = 0$ of Z_j. Clearly g_j in π-simple, irreducible on E of positive degree. Also, $\tilde{S}$ consists of finitely many branches, say $Z_1, \dots, Z_r$, where $Z_j = Z(g_j)$. The quotient $\omega = \tilde{g}/g_1 \dots g_r \in \mathcal{O}(E)$ and $Z(\omega) \cap S_{\text{reg}} = \emptyset$. Hence $\omega = \pi^* v$ for some $v \in \mathcal{O}^*(Y)$. Replacing g_1 by ωg_1, the assertion *a*) follows.

Now assume $f \in \mathcal{O}(E)$ with $Z(f) = S$. Let $(\tilde{f}, \varphi)$ be a simplication of f. By Lemma 1.1-(2), $Z(\tilde{f}) = \tilde{S}$. Let λ_j be the 0-multiplicity of $\tilde{f}$ at a point of $Z_j \cap \tilde{S}_{\text{reg}}$. Then $f/g_1^{\lambda_1} \dots g_r^{\lambda_1} \in \mathcal{O}^*(E)$. This proves *b*). Q.E.D.

COROLLARY 4.3. Let $f \in \mathcal{O}(E) - \{0\}$ with $\operatorname{rank} \pi|Z(f) = p$. Assume the set $R(f) = \{b \in Y | f_b \text{ is rational}\}$ has positive measure in Y. Then f has a simplification (4.1) with $u = 1$.

PROOF. According to [12, Thm 2.2], f is rational over Y. Theorem 4.1-(2) asserts that in (4.1) the unit u is expressible in the form $u = \pi^* v$ for some $v \in \mathcal{O}^*(Y)$ Replacing φ by φv the conclusion follows. Q.E.D.

COROLLARY 4.4. Let Y be a complex manifold, $f \in \mathcal{O}(Y)[t_1, \dots, t_k]$ and $S = Z(f)$. Assume $b \in Y_f$. Then there exist a neighborhood V of b and $g_j \in \mathcal{O}(V)[t_1, \dots, t_k]$ with $\deg(g_j) > 0$ for $j = 1, \dots, r$ such that (i) $g_1 \dots g_r = 0$ is a minimal equation of $S^k = S \cap (V \times \mathbf{C}^k)$; (ii) $\{Z(g_1), \dots, Z(g_r)\}$ gives all branches of S_V; (iii) (by taking germs of coefficients) every g_j induces an irreducible germ $g_{j[b]} \in \mathcal{O}_{Y,b}[t_1, \dots, t_k]$, and the germ $f_{[b]}$ is given by

$$f_{[b]} = (g_{1[b]})^{\lambda_1} \dots (g_{r[b]})^{\lambda r} \tag{4.2}$$

where $\lambda_j =$ the multiplicity of f at a point of $Z(g_j) \cap S_{\mathrm{reg}}$.

PROOF. Let V be a neighborhood of b biholomorphic to an open ball $\mathbf{C}^p(\varrho)$. Then $f|V \times \mathbf{C}^k$ admits a simplification (4.1) with $u = 1$. Taking ϱ sufficiently small, it may be assumed that $Z(\varphi) = \emptyset$ and each g_j induces an irreducible germ in $\mathcal{O}_{Y,b}[t_1, \dots, t_k]$. It follows that the g_j are irreducible on $V \times \mathbf{C}^k$, hence $\{Z(g_1), \dots, Z(g_r)\}$ gives all of branches of S_V: Q.E.D.

THEOREM 4.5. Let Y be a normal, irreducible complex space of dimension $p \geqslant 0$. Assume S is a hupersurface in $Y \times \mathbf{C}$ such that $A(S)$ has positive $2p$-measure in Y. Let $\bar{\pi}$: $Y \times \mathbf{P} \to Y$ be the projection and $\psi = \bar{\pi}|S$. Let $Y_0 = Y - d(S)$. $T = \bar{\pi}(\bar{S}_\infty)$. Then (1) $d(S)$ and T are thin analytic in Y. (2) Assume ψ has rank p. Then *a*) $\lambda = n_\psi(S; y) = \text{const.} > 0$ for all $y \in Y - T$; *b*) $Y_0 \cap T = \{y \in Y_0 | n_\psi(S; y) < \lambda\}$; *c*) if $y_0 \in Y - d(S) \cap Y_{\mathrm{sing}}$, there exist a neighborhood V of y_0 and a rational minimal equation $g = 0$ of S_V on $V \times \mathbf{C}$ with the following properties:

(i) For all $w \in \hat{S}_V$ and $y = \psi(w)$,

$$\nu(\psi; w) = \nu(g_y; w) \tag{4.3}$$

(ii) There is a function $\eta_V \in \mathcal{O}(V)$ with $Z(\eta_V) = T \cap V$, and setting $V^0 = V - T$.

$$g(y, t) = \eta_V(y) \prod \{(t - \xi)^{\nu(\psi; (y, \xi))} | (y, \xi) \in S_{V_0}\} \tag{4.4}$$

on $V^0 \times \mathbf{C}$.

PROOF. Assume $p > 0$. Theorem 3.2 implies $A(S) = Y_0$ is open, dense in Y. Also, $\bar{S}_\infty$ is thin analytic in $\bar{S}$, hence codim $T > 1$. Assume rank $\psi = p$. Let $Y_0 = Y - T$, $W = Y_0 \times \mathbf{C}$. The map $\bar{\pi}$: $\bar{S} \cap \cap (Y_0 \times \mathbf{P}) \to Y_0$ defines an analytic covering with sheet number $\lambda > 0$.

The same holds for the restriction $\psi^0 = \psi: S \cap W \to Y^0$. Define

$$U = \{y \in Y_0 | n_\psi(S; \square) \text{ is locally const. at } y\}$$

By Lemma 3.3, $U =$ the largest open set in Y_0 such that $\psi|S_U: S_U \to U$ is a proper map. Hence (3.2) implies $T \cap U = \emptyset$. Therefore, $Y^0 = U$, so that $Y^0 \cap T = Y^0 - U$; from this and (3.3) assertion *b*) follows.

If $y_0 \in Y_{\text{reg}}$, let V be a neighborhood of y_0 biholomorphic to an open ball in $\mathbf{C}^p$. Then Theorem 4.2 yields a rational minimal equation $g = 0$ of S_V on $V \times \mathbf{C}$.

Assume $y_0 \in Y_0$. One can choose a neighborhood $V \subseteq Y_0$ of y_0 and an open set $G \subset\subset \hat{S}$ such that

$$\psi^{-1}(V) \cap \partial G = \emptyset . \tag{4.5}$$

Then the fiber product

$$\Psi_G(y, t) = \prod \{(t - \xi)^{\nu(\psi;(y,\xi))} | (y, \xi) \in G\} \tag{4.6}$$

is holomorphic on $V \times \mathbf{C}$ ([11, 1.2.20]). If $y_0 \in Y^0$, the pair (V, G) can be chosen so that $V \subseteq Y^0$ and $S_{y_0} \subseteq G$. Therefore, a holomorphic function Ψ is well-defined on W by setting $\Psi | V \times \mathbf{C} = \Psi_G$ for every such pair (V, G).

With (V, G) as in (4.5) and setting $V^0 = V - T$, the quotient $H_G = \Psi / \Psi_G$ is defined on $V^0 \times \mathbf{C}$. Moreover,

$$H_G(y, t) = a_0(y) + a_1(y) t + \ldots + a_\lambda(y) t^\lambda ,$$

where $a_j \in \mathcal{O}(V^0)$ (with $a_\lambda = 1$). Riemann's extension theorem shows that there exist $q_j \in \mathcal{O}(V)$ $(j = 0, 1, \ldots, \lambda)$ with $q_j = a_j / a_0$ on V^0. Then the product

$$g(y, t) = q_\lambda(y) \Psi(y, t) \qquad ((y, t) \in V^0 \times \mathbf{C})$$

extends holomorphically to $V \times \mathbf{C}$. Since $T \cap U = \emptyset$, it follows from (3.2) that $V \cap T = Z(q_\lambda)$. Therefore, $Z(g) = S_V$. By (4.6) and Theorem 2.3, $g = 0$ is a minimal equation of S_V. Define $\eta_V = q_\lambda$.

Take $w = (y, z) \in \hat{S}$. With (V, G) selected for y as in (4.5), Lemma 2.2 and (4.6) imply that

$$\nu^0(g_y; w) \geqslant \nu(\psi; w) = \nu(\Psi_{G,y}; w)\,.$$

Since Ψ_G/g is holomorphic at w, (4.3) follows.

It remains to prove assertion c)-(ii) for the case $y_0 \in Y_{\text{reg}}$. Define $\eta_V : V \to \mathbf{C}$ by

$$\eta_V(y)\,t^n = \tilde{\varphi}(y) h_n(y, t) \qquad \big((y, t) \in V \times \mathbf{C}\big) \tag{4.7}$$

(where $(j_n, \tilde{\varphi})$ is given in Theorem 4.2). Then $Z(\eta_V) = T \cap V$. By (4.3), (4.6) and Theorem 1.3, the quotient $g/\Psi \in \mathcal{O}(V_0 \times \mathbf{C})$. Hence

$$g = [(\pi_{V^0})^* u]\Psi \quad \text{ on } V^0 \times \mathbf{C}$$

for some $u \in \mathcal{O}^*(V^0)$. It follows that $u = \eta_V$ on V^0, which proves (4.4). Q.E.D.

REMARK. Assume (G_4) and S satisfies the same conditions as in Theorem 4.5 (with rank $\psi = p$). Then Theorem 4.2 gives a rational minimal equation $g = 0$ of S on $Y \times \mathbf{C}$. If η_Y is defined by (4.7), the preceding proof shows that $Y^0 = Y - Z(\eta \mathrm{I})$ and (4.4) holds with $V = Y$.

REFERENCES

[1] A. ANDREOTTI - W. STOLL, *Analytic and algebraic dependence of meromorphic functions*, Lectures Notes in Math., No. 234, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[2] E. BISHOP, *Conditions for the analyticity of certain sets*, Michigan Math. J., **11** (1964), pp. 289-304.

[3] G. FISCHER, *Ein relativer Satz von Chow und die Elimination von Unbestimmtheitsstellen meromorpher Funktionen*, Math. Ann., **217** (1975), pp. 145-152.

[4] H. GRAUERT - R. REMMERT, *Bilder and Urbilder analytischer Garben*, Ann. of Math. (2), **68** (1958), pp. 393-443.

[5] H. GRAUERT - R. REMMERT, *Analytische Stellenalgebren*, Grundl. Math. Wiss. Band, **176**, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[6] R. C. GUNNING - H. ROSSI, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

[7] R. Narasimhan, *Introduction to the theory of analytic spaces*, Lecture Notes in Math., No. 25, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[8] W. F. Osgood, *Lehrbuch der Funktionentheorie - II*, part 1, 2-nd ed., Teubner, Leipzig-Berlin, 1929.

[9] L. I. Ronkin, *The global reducibility of pseudopolynomials* (Russian), Mat. Fiz. i Funktional Anal. Vyp., **2**, 238 (1971), pp. 117-121, 238, MR 56, 15966.

[10] W. Stoll, *The multiplicity of a holomorphic map*, Invent. Math., **2** (1966), pp. 15-38.

[11] C. Tung, *The first main theorem of value distribution on complex spaces*, Memorie dell'Accademia Nazionale dei Lincei, Serie VIII, Vol. XV, Sez. 1, Fasc. (1979), pp. 91-263.

[12] C. Tung, *On the rationality of divisors and meromoprhic functions*, Trans. Amer. Math. Soc., **239** (1978), pp. 399-406.

[13] H. Whitney, *Complex analytic varieties*, Addison-Wesley, Reading, Mass., 1972.

Manoscritto pervenuto in redazione l'1 settembre 1979.