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G. D'ESTE

Torsion-free abelian groups and completely decomposable p-adic modules

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Torsion-Free Abelian Groups and Completely Decomposable p-adic Modules.

G. D'ESTE (*)

Let $\mathcal{C}$ be the class of all torsion-free abelian groups G such that $J_p \otimes_{\mathbf{Z}} G$ is a completely decomposable J_p-module for every prime p where J_p is the ring of p-adic integers. This class has been studied by Procházka in [5], where it is proved that $\mathcal{C}$ is quite large. In fact, by ([5] Theorem 4*), $\mathcal{C}$ contains all torsion-free groups which belong to some Baer class Γ_σ.

In section 1 of this note we show that, if G is torsion-free, then $G \in \mathcal{C}$ if and only if, for every prime p, the group $G/p^\omega G$ is a p-pure subgroup with divisible cokernel of a free J_p-module whose rank coincides with that of G/pG.

In section 2 we give some examples of groups that are not in $\mathcal{C}$. Indeed, since $\mathcal{C}$ is not closed under direct products, very few reduced torsion-free separable or cotorsion groups belong to $\mathcal{C}$.

In section 3 we describe the behaviour of $\mathcal{C}$ with respect to pure subgroups and extensions.

I am indebted to Prof. L. Procházka for the advice that the original proof of theorem 1 was incorrect.

§ 1. All groups considered in the following are abelian groups. For all unexplained terminology and notation we refer to [1]. In particular, $\mathbf{P}$ is the set of prime numbers, $\mathbf{N}$ the set of natural numbers; $\mathbf{Z}$ and J_p are respectively the groups (or rings) of integers and p-adic

(*) Indirizzo dell'A.: Seminario Matematico, Università di Padova - Via Belzoni 7 - 35100 Padova (Italy).

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integers. If G is a torsion-free group and X is a subset of G, then $\langle X \rangle_*$ denotes the pure subgroup of G generated by X. If H is a pure subgroup of G, then we write $H \leqslant_* G$. Throughout the paper, the symbol $\otimes$ always stands for $\otimes_{\mathbb{Z}}$. For every reduced torsion-free group G, we view G as a subgroup of its cotorsion completion $G^\bullet = \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ and we identify $G^\bullet$ with $\prod_{p \in \mathbf{P}} G_p^\bullet$ where $G_p^\bullet = \operatorname{Ext}(\mathbb{Z}(p^\infty), G)$ for all p. If G is torsion-free and p is a prime, then we regard $G_p = G/p^\omega G$ as subgroup of $G_p^\bullet$.

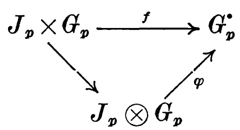
We can now determine the groups that belong to $\mathbf{C}$.

Theorem 1. *Let G be a torsion-free group and let $p \in \mathbf{P}$. The following statements are equivalent:*

(i) *$J_p \otimes G$ is a completely decomposable J_p-module.*

(ii) *$G_p = G/p^\omega G$ is a p-pure subgroup with divisible cokernel of a free J_p-module.*

Proof. (i) $\Rightarrow$ (ii). Since $J_p \otimes p^\omega G$ is divisible, $J_p \otimes G$ is isomorphic to $(J_p \otimes p^\omega G) \oplus (J_p \otimes G/p^\omega G)$. Hence, by (i), $J_p \otimes G_p = J_p \otimes G/p^\omega G = F \oplus D$ where F is a free J_p-module and D is divisible. Let $f\colon J_p \times G_p \to G_p^\bullet$ be the map defined by $f((\alpha, x)) = \alpha x$ for all $\alpha \in J_p$, $x \in G_p$. Then there is a homomorphism φ that makes the following diagram commute:



Let $\pi\colon J_p \otimes G_p = F \oplus D \to F$ be the natural projection and identify G_p with the subgroup $\mathbb{Z} \otimes G_p$ of $J_p \otimes G_p$. Thus G_p is p-pure in $J_p \otimes G_p$ and $J_p \otimes G_p/G_p$ is divisible. Since $\operatorname{Ker}\pi \leqslant \operatorname{Ker}\varphi$, it is easy to check that $G_p \cap \operatorname{Ker}\pi = 0$. Hence G_p is isomorphic to $\pi(G_p)$ and (ii) holds.

(ii) $\Rightarrow$ (i). By the remark made in the first part of the proof, it is not restrictive to assume $G \simeq G_p$. Consequently, by (ii), there is a free J_p-module F such that G is a p-pure subgroup of F and $H = F/G$ is divisible; so $H = A \oplus T$ where A is torsion-free and T is a torsion-group with $t_p(T) = 0$. Since $J_p \otimes H \simeq J_p \otimes F/J_p \otimes G$ and $J_p \otimes H \simeq J_p \otimes A$ is torsion-free, we conclude that $J_p \otimes G$ is a pure subgroup of $J_p \otimes F$. Let $J_p \otimes F = D \oplus L$, $J_p \otimes G = D' \oplus R'$ with D, D' divisible, R' reduced and L a free J_p-module. Since $R' \leqslant J_p \otimes F =$

$= D \oplus L$, one obtains $R' \cap D = 0$. Therefore R' is isomorphic to a submodule of L under the canonical projection $D \oplus L \to L$. Thus R' is a free J_p-module ([1] Theorems 14.5 and 14.7) and $J_p \otimes G$ is a completely decomposable J_p-module. □

The above characterization of the class $\mathcal{C}$ is the analogue of a result ([3] Theorem 4.1) concerning torsion-free modules over discrete valuation rings. Also note that, if $G \in \mathcal{C}$ and $p \in \mathbb{P}$, then the rank r of G and the p-rank r_p of G, i.e. the dimension of G/pG over the field with p elements, uniquely determine the structure of $J_p \otimes G$. Indeed let $J_p \otimes G = F \oplus D$ with F free and D divisible; then, as in ([3] Lemma 1.2), $J_p \otimes G$ and F are J_p-module of rank r and r_p respectively.

§ 2. We now look at the behaviour of $\mathcal{C}$ with respect to direct products.

PROPOSITION 2. *Let* $G = \prod_{i \in I} G_i$ *where* $G_i \in \mathcal{C}$ *for every* $i \in I$ *and let* $p \in \mathbb{P}$. *The following are equivalent*:

(i) $pG_i = G_i$ *for almost all* $i \in I$.

(ii) $J_p \otimes G$ *is a completely decomposable* J_p*-module.*

PROOF. (i) $\Rightarrow$ (ii). Let $I' = \{i \in I : pG_i \neq G_i\}$ and let $I'' = I \setminus I'$. Since $J_p \otimes G = \big(J_p \otimes \prod_{i \in I'} G_i\big) \oplus \big(J_p \otimes \prod_{i \in I''} G_i\big)$ where the second summand is divisible, I' is finite and $G_i \in \mathcal{C}$ for all $i \in I'$, evidently (ii) holds.

(ii) $\Rightarrow$ (i). As before, let $I' = \{i \in I : pG_i \neq G_i\}$. We want to prove that I' is finite. Assume the contrary. To see that this is impossible, choose, for every $i \in I'$, an element $x_i \in G_i \setminus pG_i$ and let $L_i = \langle x_i \rangle_* \leqslant G_i$. Let now $L = \oplus L_i$ and $H = \prod_{i \in I'} G_i$. Then, by (ii), $J_p \otimes H$ has a decomposition of the form $F \oplus D$ where F is a free J_p-module and D is divisible. Write $F = \bigoplus_{k \in K} F_k$ with $F_k \simeq J_p$ for all $k \in K$. Let π and π_k, for every k, be the canonical projections of $J_p \otimes H$ onto F and F_k respectively. Observe now that there exists a commutative diagram

$$\begin{array}{ccc} J_p \times H & \xrightarrow{\;f\;} & H_p^{\bullet} \\ & \searrow \quad \nearrow_{\varphi} & \\ & J_p \otimes H & \end{array}$$

where φ is a homomorphism and f sends (α, x) to $\alpha\bar{x} = \alpha(x + p^{\omega}H)$ for all $\alpha \in J_p$, $x \in H$. Hence the same arguments used in theorem 1 and the definition of L tell us that L is isomorphic to $\pi(L)$. Since L is p-pure in $J_p \otimes H$, it follows that $\pi(L)$ is p-pure in F; consequently $L/pL \cong \pi(L)/p\pi(L) \cong \pi(L) + pF/pF$. From this remark and the hypothesis that L/pL is not finite, we deduce that the set $S = \{k \in K: \pi_k(L) \neq 0\}$ is not finite; thus we may assume $\mathbf{N} \subseteq S$. For every $n \in \mathbf{N}$, let $S_n = \{k \in K: \pi_k(x_n) \neq 0\}$ and let $T_n = \bigcup_{m=0}^{n} S_m$. Define by induction a sequence $\{y_n\}_{n\in\mathbf{N}}$ in L with the following properties: if $n = 0$, then $y_n = x_n$; if y_m is defined for all $m \leqslant n$ and $h_n = \max\{h(\pi_k(y_n)): k \in T_n\}$ where h is the p-height function on F, then $y_{n+1} = y_n + p^{h_n+n+1}x_{n+1}$. Obviously $\{y_n\}_{n\in\mathbf{N}}$ has a limit $y \in H$ with respect to the p-adic topology. On the other hand, the support of $\pi(y_n)$ is T_n for every $n \in \mathbf{N}$ and, by the choice of $\{y_n\}_{n\in\mathbf{N}}$, $\bigcup_{n\in\mathbf{N}} T_n$ is not finite. Therefore $\{\pi(y_n)\}_{n\in\mathbf{N}}$ cannot converge in F equipped with the p-adic topology. This contradiction shows that I' is finite, as claimed. $\square$

While completely decomposable torsion-free groups clearly belong to $\mathcal{C}$, we have the following

COROLLARY 3. *The class $\mathcal{C}$ does not contain the class of torsion-free separable groups.*

PROOF. Let $G = \mathbf{Z}^{\mathbf{N}}$; then, by ([1] Proposition 87.4), G is separable and, by proposition 2, $G \notin \mathcal{C}$. $\square$

We next prove that very few reduced cotorsion groups belong to $\mathcal{C}$.

COROLLARY 4. *Let G be a reduced torsion-free group and let $G^{\bullet} = \prod_{p\in\mathbf{P}} G^{\bullet}_p$ be its cotorsion completion. The following facts hold:*

(i) *$G^{\bullet} \in \mathcal{C}$ if and only if, for every prime p, $G^{\bullet}_p$ is a J_p-module of finite rank.*

(ii) *If $G^{\bullet} \in \mathcal{C}$, then $G \in \mathcal{C}$.*

PROOF. (i) Fix $p \in \mathbf{P}$. Then

$$J_p \otimes G^{\bullet} = (J_p \otimes G^{\bullet}_p) \oplus \Big(J_p \otimes \prod_{q\neq p} G^{\bullet}_q\Big)$$

where the second summand is divisible. Let $\mathbf{Z}_p$ be the group of rational numbers whose denominators are prime to p. Since the sequence

$$0 \to G_p^\bullet \simeq \mathbb{Z}_p \otimes G_p^\bullet \to J_p \otimes G_p^\bullet \to J_p \otimes G_p^\bullet/G_p^\bullet \to 0$$

is exact and $J_p \otimes G_p^\bullet/G_p^\bullet$ is torsion-free, we get

$$J_p \otimes G_p^\bullet = G_p^\bullet \otimes (J_p \otimes G_p^\bullet)/G_p^\bullet \quad \text{with} \quad J_p \otimes G_p^\bullet/G_p^\bullet$$

divisible. Consequently $J_p \oplus G_p^\bullet$ is a completely decomposable J_p-module if and only if the same applies to $G_p^\bullet$, i.e. if and only if $G_p^\bullet$ is a J_p-module of finite rank.

(ii) Assume $G^\bullet \in \mathcal{C}$ and let p be a prime. Then, by (i), $G_p^\bullet \simeq J_p^n$ for some $n \in \mathbf{N}$, while, ([4] Ch. II § 5.5 Theorem 1), G_p is a p-pure subgroup with divisible cokernel of $G_p^\bullet$. This remark and theorem 1 assure that $J_p \otimes G$ is a completely decomposable J_p-module. Thus $G \in \mathcal{C}$ and (ii) follows. □

The above proof indicates that, if G is a reduced torsion-free cotorsion group and $p \in \mathbf{P}$, then $J_p \otimes G$ has a decomposition of the form $C \oplus D$ where $C \simeq G_p^\bullet \simeq G/p^\omega G$ and D is divisible. Therefore $J_p \otimes G$ is generally very far from being completely decomposable. As the following statement shows, also the larger class of locally cotorsion groups, introduced in [6], does not contain many groups of $\mathcal{C}$.

Corollary 5. *Let G be a reduced group and let $G \in \mathcal{C}$. Then G is locally cotorsion if and if $G/p^\omega G$ is a J_p-module of finite rank for every $p \in \mathbf{P}$.*

Proof. Recall that, if G is reduced and torsion-free, then, by ([6] Theorem 4.2), G is locally cotorsion if and only if $G/p^\omega G \simeq G_p^\bullet$ for all prime p. Hence our claim is an immediate consequence of corollary 4. □

§ 3. The next result gives a closure property of the class $\mathcal{C}$.

Proposition 6. *The class $\mathcal{C}$ is closed under pure subgroups.*

Proof. Let $G \in \mathcal{C}$ and let H be a pure subgroup of G. To see that $H \in \mathcal{C}$, fix $p \in \mathbf{P}$. Since $H_p = H/p^\omega H = H/p^\omega G \cap H \simeq H + p^\omega G/p^\omega G \leqslant \leqslant G/p^\omega G = G_p$ and $G \in \mathcal{C}$, theorem 1 implies that H_p is a p-pure subgroup of a free J_p-module F. Let R be the submodule of F generated by H_p; then R is free and clearly R/H_p is divisible. Applying theorem 1, we conclude that $J_p \otimes H$ is a completely decomposable J_p-module. Therefore $H \in \mathcal{C}$, as required. □

Let us note that $\mathcal{C}$ is the class of all torsion-free groups G such that, for every prime p, the $\mathbf{Z}_p$-module $\mathbf{Z}_p \otimes G/p^\omega G$ admits $\mathbf{Q}_p$, the field of p-adic numbers, as a splitting field in the sense of [3]. Hence proposition 6 is similar to half of the first assertion of ([3] Corollary 2.2). Indeed, since free groups belong to $\mathcal{C}$, the class $\mathcal{C}$ is not closed with respect to torsion-free homomorphic images.

Observe now that there exists a group G such that $G \notin \mathcal{C}$, but $G \leqslant H$ where H is reduced, $H \in \mathcal{C}$ and H/G is a torsion group. For instance, let $G = \mathbf{Z}^{\mathbf{N}}$ and let $H = \langle G \rangle_* \leqslant \prod_{p \in \mathbf{P}} \mathbf{Z}_p$. Then, by proposition 2, $G \notin \mathcal{C}$ and $\prod_{p \in \mathbf{P}} \mathbf{Z}_p \in \mathcal{C}$; thus, by proposition 6, $H \in \mathcal{C}$ and the rest is obvious.

Nevertheless, it is easy to find examples of reduced torsion-free groups which are not subgroups of a reduced group of $\mathcal{C}$. Indeed we can prove the following

PROPOSITION 7. *Let G be a reduced torsion-free cotorsion group and let R be a reduced group of $\mathcal{C}$. If $G \leqslant R$, then $G \in \mathcal{C}$.*

PROOF. Let G and R be as in the hypothesis and let $G \leqslant R$; we claim that $G \in \mathcal{C}$. By corollary 4, it is enough to prove that, if p is any prime and G is a J_p-module as in the hypotheses, then G is a J_p-module of finite rank. Indeed, by theorem 1, there is a free J_p-module F such that $R/p^\omega R \leqslant F$. Since $G \cap p^\omega R = 0$, we deduce that G is isomorphic to a submodule of F. Consequently G must be a J_p-module of finite rank and the proof is complete. □

Recall that ([5] Lemma 4), if G is a torsion-free group such that $G/H \in \mathcal{C}$ where H is of finite rank, then $G \in \mathcal{C}$. We shall now see that the restriction on H cannot be omitted.

PROPOSITION 8. *The class $\mathcal{C}$ is not closed under extensions.*

PROOF. Let $G = J_p^{\mathbf{N}}$ and let B be a free J_p-module such that $B \leqslant_* G$ and G/B is divisible. Evidently B, $G/B \in \mathcal{C}$, while, by proposition 2, $G \notin \mathcal{C}$. □

Let G be a torsion-free group and let $H \leqslant_* G$. Then the following are obvious consequences of the preceding results.

(i) $H, G/H \in \mathcal{C} \nRightarrow G \in \mathcal{C}$;

(ii) $H, G \in \mathcal{C} \quad \nRightarrow G/H \in \mathcal{C}$;

(iii) $G,\ G/H \in \mathcal{C} \Rightarrow H \in \mathcal{C}$.

A lot of slender groups belong to $\mathcal{C}$; however, we have the following

PROPOSITION 9. *There exists a slender group G such that $G \notin \mathcal{C}$.*

PROOF. Fix $p \in \mathbb{P}$ and let G be a $\mathbb{Z}_p$-module of J_p such that $1 \in G$; $p^n J_p \nleqslant G$ for every $n \in \mathbb{N}$ and G is not p-pure in J_p. Since G is slender ([4] Ch. V § 2.4 Theorem), it remains only to check that $G \notin \mathcal{C}$. Suppose the contrary. Then, by theorem 1, there is an embedding $\psi\colon G \to F$ where F is a free J_p-module and $\psi(G)$ is p-pure in F. If $\bar{\psi}$ is the extension of ψ to the cotorsion completions of G and F, evidently $\psi(g) = \bar{\psi}(g) = g\bar{\psi}(1) = g\psi(1)$ for all $g \in G$. Hence $\psi(G) \leqslant J_p\psi(1)$ and this enables us to assume $F = J_p^n$ for some $n \in \mathbb{N}$. Let now $\psi(1) = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_i \in J_p$ for every i. Then the hypothesis that $\psi(G)$ is p-pure in F guarantees that $\alpha_k \notin pJ_p$ for some k. Let $\tau\colon G \to J_p$ be the map that takes g to $\alpha_k g$ for all $g \in G$. To see that $\tau(G)$ is p-pure in J_p, suppose $p^n x = \tau(g)$ with $x \in J_p$, $n \in \mathbb{N}$ and $g \in G$. Since $\psi(G)$ is p-pure in F and $h(\alpha_i g) \geqslant h(\alpha_k g) \geqslant n$ for every i, there exists $g' \in G$ such that $\psi(g) = p^n \psi(g')$. Therefore $p^n x = \tau(g) = p^n \tau(g')$ and so $\tau(G)$ is p-pure in J_p. On the other hand, since G is not p-pure in J_p, we can choose $\alpha \in J_p$, $m \in \mathbb{N}$ such that $\alpha \notin G$ and $p^m \alpha \in G$. Consequently $\alpha_k \alpha \notin \tau(G)$, while $p^m \alpha_k \alpha \in \tau(G)$. This contradiction implies that $G \notin \mathcal{C}$, as claimed. $\square$

By the preceding result, $\mathcal{C}$ does not contain the class of all $\mathbb{Z}_p$-modules that are subgroups with divisible cokernel of a free J_p-module.

Let p be a prime and let $R = K \cap J_p$ where, as in [3], K is a field such that $\mathbb{Q} < K \leqslant \mathbb{Q}_p$. If $G \in \mathcal{C}$, then it is natural to ask if $R \otimes G$ is a completely decomposable R-module. We shall prove that, if $R \otimes G$ is a completely decomposable R-module, then R contains the subring of J_p generated by the set S, where

$$S = \{\alpha \in J_p : \alpha x \in G_p = G/p^\omega G \text{ for some } x \in G_p \setminus pG_p\}\,.$$

Indeed, with the same arguments used in the first part of theorem 1, it is easy to show that there is an injective homomorphism $\psi\colon G_p \to F$ such that F is a free R-module and $\psi(G_p)$ is p-pure in F. Let now $\alpha x \in G_p$ where $\alpha \in J_p$ and $x \in G_p \setminus pG_p$. Since F has a decomposition of the form $F = F' \oplus F''$ with $\psi(x) \in F'$ and $F' \cong R^n$ for some $n \in \mathbf{N}$, we may write $\varphi(x) = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in R$ for all i. Moreover, since $x \notin pG_p$ and $\psi(G_p)$ is p-pure in F, there exists some k such that $\alpha_k \notin pR$; hence α_k is a unit of R. Since $\alpha x \in G_p$ and $\psi(\alpha x) = \alpha\psi(x)$, it immediately follows that $\alpha \in R$. This completes the proof.

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