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Infinite Soluble Groups with no Outer Automorphisms.

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1. Introduction.

If G is a group, Aut G will denote the full automorphism group of G and Inn G the normal subgroup of all inner automorphisms. Out $G = \text{Aut}\, G/\text{Inn}\, G$ is the outer automorphism group. The group G is said to be *complete* if both the outer automorphism group and the centre $\zeta(G)$ are trivial. If g^τ is the inner automorphism induced by an element g, then of course $\tau\colon G \to \text{Aut}\, G$ is a homomorphism, and G is complete precisely when τ is an isomorphism.

Complete groups were introduced over eighty years ago by Hölder [9] in connection with the theory of group extensions, which was then in an early stage of development. Hölder noticed that there were certain groups G which admitted only trivial extensions in the sense that $G \lhd H$ always implied that $H = G \times K$ for some subgroup K. It is now well-known that this property characterizes complete groups.

After a long period of neglect complete groups are beginning to receive attention once more, especially finite soluble complete groups —see [3], [4], [8]. Here we shall consider infinite soluble complete groups with particular reference to polycyclic groups or, more generally, to soluble groups of finite total rank. Recall that a soluble group is said to have *finite total rank* if the sum of the p-ranks for all p

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(including $p = 0$) is finite when taken over all the factors of some abelian series.

In studying metanilpotent complete groups one encounters two quite different types of problem.

(i) A cohomological problem: given that Q is a nilpotent group and A a Q-module, can one deduce from $H^0(Q, A) = 0 = H^1(Q, A)$ that $H^2(Q, A) = 0$?

(ii) A Carter subgroup problem; identify the Carter (i.e. self normalizing nilpotent) subgroups of the outer automorphism group of a nilpotent group N.

Of course (i) has a positive answer when both group and module are finite in view of Gaschütz's well-known theorem [5]. However in the infinite case one cannot always draw this conclusion. The best result we know of is

THEOREM 1. Let Q be a nilpotent group and let A be a Q-module which has finite total rank as an abelian group. If $H^0(Q, A) = 0 = H^1(Q, A)$, then $H^n(Q, A) = 0 = H_n(Q, A)$ for all n.

This theorem, which is the basic result of the paper, depends on certain (near) splitting theorems established in [14].

The second problem has been completely solved when N is a finite abelian group [3]. However, as Theorem 1 of [8] indicates, such a complete solution is not to be expected even when N is a finite p-group. On the other hand, if one could obtain information about say Carter subgroups of $GL(n, \mathbb{Z})$, this would have repercussions for polycyclic complete groups that are abelian by nilpotent.

We establish a characterization of metanilpotent complete groups with finite total rank which reduces their classification to that of certain Carter subgroups of outer automorphism groups of nilpotent groups. This is Theorem 2.

Subsequently it is shown that various types of infinite soluble group with finite total rank cannot be complete, among them groups of Hirsch length 1, supersoluble groups, torsion-free abelian by nilpotent groups.

In the final section we construct an example of a torsion-free polycyclic group of Hirsch length 7 which is complete, thus answering a question raised during the 1978 Warwick Symposium. This is rather similar to the question of the existence of complete groups of odd order that was first mentioned by Miller [12] and answered positively by Dark [1] and Horoševskiĭ [10]: see also [8].

2. Techniques.

Almost all theorems about outer automorphisms of soluble groups depend on the well-known relation between derivations and automorphisms stabilizing a series. For convenience we state the necessary facts as a lemma: for a proof see [7] (§ 3.5) or [11].

LEMMA 1. Let N be a normal subgroup of a group G such that $C_G(N) = \zeta(N)$. Denote by A_N the group of automorphisms of G that stabilize the series $1 \lhd N \lhd G$. If $Q = G/N$ and $A = \zeta(N)$, then

$$A_N/A_N \cap \operatorname{Inn} G \cong \operatorname{Der}(Q, A)/\operatorname{Inn}(Q, A) \cong H^1(Q, A) .$$

Here $\operatorname{Der}(Q, A)$ and $\operatorname{Inn}(Q, A)$ are the group of derivations and the subgroup of inner derivations respectively. Also A is a Q-module via conjugation.

It is Lemma 1 which makes the entry of homological methods inevitable. We consider now the consequences of the vanishing of $H^1(Q, A)$, our main object being Theorem 1.

PROOF OF THEOREM 1. Let T denote the ($\mathbb{Z}$-)torsion-subgroup of A; then T satisfies the minimal condition. Since $0 = H^0(Q, A) \equiv A^Q$, we have $T^Q = 0$. It now follows from [14], Theorem B, that $H^n(Q, T) = 0 = H_n(Q, T)$ for all n. Applying the cohomology sequence to $T \rightarrowtail A \twoheadrightarrow A/T$ we obtain

$$0 \to T^Q \to A^Q \to (A/T)^Q \to H^1(Q, T) \to H^1(Q, A) \to H^1(Q, A/T) \to H^2(Q, T),$$

from which it follows that $H^0(Q, A/T) = 0 = H^1(Q, A/T)$. It therefore suffices to prove the theorem for the module A/T. Assume from now on that A is torsion-free.

Since $A^Q = 0$, we deduce via [14], Lemma 5.12, that A_Q is finite. Now if $A_Q = 0$, the result follows at once from [14], Corollary CD. We therefore suppose that $A_Q \neq 0$ and choose a prime p which divides $|A_Q|$. Thus $(A/pA)_Q \neq 0$. An application of the cohomology sequence to the exact sequence $A \overset{p}{\rightarrowtail} A \twoheadrightarrow A/pA$, where the first mapping is multiplication by p, reveals that $(A/pA)^Q = 0$. But this implies that $(A/pA)_Q = 0$ since A/pA is finite and Q is nilpotent, and we reach a contradiction.

While Theorem 1 is not valid if Q fails to be nilpotent, there are some special situations where something can be salvaged. We record one such result for future reference.

LEMMA 2. Let Q be an extension of a cyclic group by a cyclic p-group and let A be a Q-module which is a finitely generated free abelian group. If $H^0(Q, A) = 0 = H^1(Q, A)$, then $H^2(Q, A) = 0$.

PROOF. There is a cyclic normal subgroup N such that $Q/N = \langle xN \rangle$ is a cyclic p-group, say of order p^k. Consider the five term sequence

$$0 \to H^1(Q/N, A^N) \to H^1(Q, A) \to H^1(N, A)^Q \to H^2(Q/N, A^N) \to H^2(Q, A) .$$

We see at once that $H^1(Q/N, B) = 0$ where $B = A^N$. Thus, if ν is the endomorphism $b \mapsto b(1 + x + x^2 + \ldots + x^{p^k - 1})$ of B, we have $\operatorname{Ker} \nu = B(x-1)$. But $B\nu \leqslant A^Q = 0$, so $B = B(x-1)$. Now x acts on B/pB as an element of p-power order. Consequently $B = pB$, which shows that $B = 0$. We deduce from the sequence that

$$H^1(N, A)^Q = 0 .$$

Now N is cyclic, $A^N = 0$ and A is finitely generated: thus the group $H^1(N, A)$ is finite. The above equation therefore implies that $H^1(Q/N, H^1(N, A)) = 0$. Finally $H^2(N, A) = 0$ because N is cyclic and $A^N = 0$. That $H^2(Q, A) = 0$ is now a consequence of the Lyndon-Hochschild-Serre spectral sequence.

To make the transition to the problem of self-normalizing subgroups it is necessary to note

LEMMA 3 (Wells [17], Schmid [15]). Let N be a normal subgroup of a group G such that $C_G(N) = \zeta(N)$. Let $Q = G/N$ and assume that $H^2(Q, \zeta(N)) = 0$. Then if α in $\operatorname{Aut} N$ normalizes Q, regarded as a subgroup of $\operatorname{Out} N$, there is an automorphism of G that induces α in N.

This machinery can now be applied to give criteria for a group to be complete.

LEMMA 4. Let N be a normal subgroup of a group G such that $C_G(N) = \zeta(N) = A$ has finite total rank and $Q = G/N$ is nilpotent. If G is complete, then

(i) $H^n(Q, A) = 0 = H_n(Q, A)$ for all n.

(ii) Q is a Carter subgroup of $\operatorname{Out} N$.

(iii) $C_A(Q) = 1$ and $A = [A, Q]$.

Conversely if (ii) and (iii) hold, every automorphism of G that leaves N invariant is inner. If N is characteristic, then G is complete.

PROOF. Suppose that G is complete. Then (i) is true by Lemma 1 and Theorem 1, while (iii) is a special case of (i). If $\alpha(\mathrm{Inn}\,N)$ normalizes Q, then, according to Lemma 3, the automorphism α extends to a—necessarily inner—automorphism of G, which implies that $\alpha(\mathrm{Inn}\,G) \in Q$. Thus Q is a Carter subgroup of $\mathrm{Out}\,G$.

Conversely suppose that (ii) and (iii) are valid. Assume that γ is an outer automorphism of G that leaves N fixed. Then γ induces an automorphism α in A that normalizes G/A as a subgroup of $\mathrm{Aut}\,N$, and hence Q as a subgroup of $\mathrm{Out}\,N$: thus $\alpha(\mathrm{Inn}\,N) \in Q$. Since we may modify γ by an inner automorphism, it is permissible to assume that γ acts trivially on N. Hence γ also acts trivially on Q. To complete the proof we need to show that $H^1(Q, A) = 0$.

Let T be the torsion-subgroup of A. Since $T^Q = C_T(Q) = 1$, we deduce from [14], Theorem B, that $H^1(Q, T) = 0$. Also $(A/T)_Q = A/T[A, Q] = 1$, so it follows via Theorem D of [14] that $H^1(Q, A/T) = 0$. Consequently $H^1(Q, A) = 0$.

Lemma 4 can be applied with N equal to the Fitting subgroup of a metanilpotent group. Alternatively A can, in suitable circumstances, be a maximal normal abelian subgroup.

3. Metanilpotent complete groups.

We can now prove a classification theorem for metanilpotent complete groups with finite total rank. First some terminology. Suppose that N is a nilpotent group and Q a subgroup of $\mathrm{Out}\,N$. An element $\alpha(\mathrm{Inn}\,N)$ of Q will be called *unipotent* if there is a positive integer n such that $[a, {}_n\alpha] = 1$ for all $a \in N$. It is easy to prove that this concept is well-defined by using the nilpotence of N.

THEOREM 2.

(i) Let G be a metanilpotent complete group with finite total rank. Let $F = \mathrm{Fit}\,G$ be the Fitting subgroup, A the centre of F and $Q = G/F$. Then F is nilpotent and Q is a Carter subgroup of $\mathrm{Out}\,F$ with no non-trivial unipotent elements; also $A = [A, Q]$ and $C_A(Q) = 1$.

(ii) Conversely, let F be a nilpotent group of finite total rank whose outer automorphism group contains a Carter subgroup Q with no non-trivial unipotent elements. If $A = [A, Q]$ and $C_A(Q) = 1$ where A is the centre of F, then there exists a complete group $G = G(F, Q)$ such that $\operatorname{Fit} G \simeq F$ and $G/\operatorname{Fit} G \simeq Q$: moreover the coupling associated with the extension $F \rightarrowtail G \twoheadrightarrow Q$ is the inclusion $Q \to \operatorname{Out} F$.

(iii) The groups $G(F, Q)$ and $G(\bar{F}, \bar{Q})$ are isomorphic if and only if there is an isomorphism $\alpha: F \to \bar{F}$ inducing α': $\operatorname{Out} F \to \operatorname{Out} \bar{F}$ such that $Q^{\alpha'}$ and $\bar{Q}$ are conjugate in $\operatorname{Out} \bar{F}$. Thus $G(F, Q)$ is determined to within isomorphism by the isomorphism class of F and the conjugacy class of Q in $\operatorname{Out} F$.

PROOF.

(i) That F is nilpotent has been proved in [6], Theorem 1.3. Suppose that xF is a unipotent element of Q. Then it is easy to see that $\langle x, F\rangle$ is nilpotent. But $\langle x, F\rangle$ is also subnormal in G, so it is contained in the Baer radical. It follows from [6], Theorem 1.3 once again that $x \in F$. Therefore Q contains no non-trivial unipotent elements. The remaining statements in (i) follow from Lemma 4.

(ii) By Lemma 4 and Theorem 1 we have $H^3(Q, A) = 0$. Hence there is an extension $F \rightarrowtail G \twoheadrightarrow Q$ which realizes the coupling $Q \to \operatorname{Out} F$. The unipotent condition guarantees that the image of F equals $\operatorname{Fit} G$. That G is complete is now a consequence of Lemma 4.

(iii) If $G(F, Q) \simeq G(\bar{F}, \bar{Q})$, then certainly $F \simeq \bar{F}$. It is clearly permissible to identify F and $\bar{F}$. Then plainly Q and $\bar{Q}$ are conjugate in $\operatorname{Out} F$.

Conversely suppose that Q and $\bar{Q}$ are conjugate in $\operatorname{Out} F$. Changing our point of view slightly let us suppose that there are injective couplings $\chi: Q \to \operatorname{Out} F$ and $\bar{\chi}: Q \to \operatorname{Out} F$ whose images are conjugate Carter subgroups of the allowed type. We shall prove that $G = G(F, Q^{\chi})$ and $\bar{G} = G(F, Q^{\bar{\chi}})$ are isomorphic.

By hypothesis there exists $\alpha \in \operatorname{Aut} F$ such that

$$Q^{\bar{\chi}} = (\alpha \operatorname{Inn} F)^{-1} Q^{\chi} (\alpha \operatorname{Inn} F) .$$

If α' is the inner automorphism of $\operatorname{Out} F$ induced by α, then $\bar{\chi} = \varkappa\chi\alpha'$ for some $\varkappa \in \operatorname{Aut} Q$.

Now form successive push-out and pull-back diagrams

$$\begin{tikzcd} F \arrow[r, tail] \arrow[d, "\alpha"'] & G \arrow[r, two heads] \arrow[d] & Q \arrow[d, equal] \\ F \arrow[r, tail] & G^* \arrow[r, two heads] & Q \\ F \arrow[r, tail] \arrow[u, equal] & G^+ \arrow[r, two heads] \arrow[u] & Q \arrow[u, "\varkappa"'] \end{tikzcd}$$

The couplings of these three extensions are respectively χ, $\chi\alpha'$ and $\varkappa\chi\alpha' = \bar{\chi}$. Since $H^2(Q, A) = 0$ by Lemma 4, two extensions with coupling $\bar{\chi}$ are equivalent; thus $G^+ \simeq \bar{G}$. On the other hand it is clear that $G \simeq G^* \simeq G^+$. Hence $G \simeq \bar{G}$.

4. Incomplete groups.

We consider next some types of infinite soluble group that cannot be complete.

THEOREM 3. Let G be a non-trivial abelian by nilpotent group with finite total rank. If G contains no elements of order 2, then G is not complete. In particular a non-trivial torsion-free abelian by nilpotent group cannot be complete if it has finite torsion-free rank.

PROOF. Let A be a maximal normal abelian subgroup such that $Q = G/A$ is nilpotent. Then clearly $A = C_G(A)$. If G is complete, we deduce from Lemma 4 that G splits over A; thus $G = XA$, $X \cap A = 1$ for some $X \leqslant G$. But X must be self-normalizing in Aut A, so it contains an element of order 2, a contradiction.

On the other hand there are infinite metabelian groups with finite total rank that are complete, the following being a simple example.

Let π be a set of primes containing 2 and define G to be the holomorph of the additive group of all rationals whose denominators are π-numbers. Then it is not difficult to prove that G is a complete metabelian group: of course G contains an element of order 2. If $|\pi| = r < \infty$, then G is a finitely generated minimax group with Hirsch length $r + 1$. Taking $\pi = \{2\}$ we get a group of Hirsch length 2, which is the least possible in view of the next result.

THEOREM 4. Let G be a soluble group with finite total rank whose Hirsch length is 1. Then G is not complete.

PROOF. Assume that G is in fact complete. By Lemma 9.34 and Theorem 9.39.3 of [13] there is a normal series $1 \leqslant T \leqslant N \leqslant G$ such that T is a divisible abelian group with the minimal condition, N/T is a torsion-free abelian group of rank 1 and G/N is finite.

We claim that $H^2(G/T, T)$ has finite exponent. Since G/N is finite, it is enough to prove that $H^2(N/T, T)$ has finite exponent. Now T satisfies the minimal condition, so there is an integer k such that $[T, {}_kN] = [T, {}_{k+1}N] = S$ say. Clearly a central extension of a divisible group by a group of rank 1 always splits; from this it is straightforward to prove that $H^2(N/T, T/S) = 0$. Also $S = [S, N]$, so Theorem C of [14] shows that $H^2(N/T, S)$ has finite exponent. Therefore $H^2(N/T, T)$ has finite exponent and our claim is established.

Suppose that $T \neq 1$; then for some prime p the p-component T_p of T is non-trivial. Let Δ be the cohomology class of the extension $T \rightarrowtail G \twoheadrightarrow G/T$; by the previous paragraph Δ has finite order, say e. It is easy to see that there exist infinitely many multiplicatively independent p-adic integers $\theta_1, \theta_2, \ldots$ such that $\theta_i \equiv 1 \bmod p^2 e$. For each i we define a G-automorphism α_i of T by means of the assignments $x \mapsto x^{\theta_i}$, $(x \in T_p)$, and $x \mapsto x$, $(x \in T_q, q \neq p)$. Now $\Delta(\alpha_i)_* = \Delta$ since $e\Delta = 0$. It follows from [16], Proposition 4.1 that α_i is induced by a necessarily inner automorphism of G. However $\langle \alpha_1, \alpha_2, \ldots \rangle$ is a free abelian group of infinite rank and such a group cannot be isomorphic with a factor of G. By this contradiction $T = 1$.

Let F be the Fitting subgroup of G. Then $N \leqslant F$ since N is abelian. Now N has rank 1 and F is nilpotent, so $[N, F] = 1$ and $F \leqslant C_G(N) = C$ say. Hence G/C is finite. Since N cannot be central in G, we must have $|G : C| = 2$ and $G = \langle x, C \rangle$ where $a^x = a^{-1}$ for all a in N.

Let $A = \zeta(F)$. By Lemma 1 we have $H^1(G/F, A) = 0$, which implies that $H^1(G/C, B) = 0$ where $B = C_A(C)$. If ν is the endomorphism of B in which $b \mapsto b^{1+x}$, then $\operatorname{Ker} \nu = [B, x]$. But $B^\nu \leqslant \zeta(G) = 1$, so in fact $B = [B, x]$. Also $1 \neq B^m \leqslant N$ for some $m > 0$: therefore $N = N^2$.

Next $H^2(G/N, N)$ has finite exponent, say e, since G/N is finite. Choose $k > 0$ so that $2^k \equiv 1 \bmod e$ and let α be the automorphism $a \mapsto a^{2^k}$ of N. Then $\Delta\alpha_* = \Delta$; thus α is induced by conjugation by some element of G. However this means that α has finite order, which is certainly not the case.

We turn now to polycyclic groups.

THEOREM 5. *An infinite supersoluble group G cannot be complete.*

PROOF. Suppose that G is in fact complete. Let A be a maximal normal abelian subgroup of G: then it is well-known that $A = C_G(A)$, so that $H^1(Q, A) = 0$ where $Q = G/A$. Notice that A cannot contain elements of order 2 because $\zeta(G) = 1$. Therefore we may form the exact sequence $A \overset{2}{\rightarrowtail} A \twoheadrightarrow A/A^2$, where in the first mapping $a \mapsto a^2$. The cohomology sequence yields

$$1 \to A^Q \to A^Q \to (A/A^2)^Q \to H^1(Q, A) ,$$

which shows that $(A/A^2)^Q = 1$. Consequently $A = A^2$ and A is finite. But this implies that G is finite.

It is known that there are numerous finite supersoluble complete groups but a classification of them does exist: for information about these groups see [8]. Note that infinite supersoluble groups may well fail to have outer automorphisms, as is demonstrated by the group

$$\langle x, y \mid x^{-1}yx = y^2, y^5 = 1\rangle .$$

Finally a result about abelian by metacyclic groups.

THEOREM 6. Let G be a non-trivial torsion-free polycyclic group. Assume that G is an extension of an abelian group by a cyclic by cyclic p-group for some prime p. Then G is not complete.

PROOF. Supposing G to be complete we choose a maximal normal abelian subgroup A such that $Q = G/A$ is cyclic by cyclic-p. Then $H^0(Q, A) = 0 = H^1(Q, A)$. Applying Lemma 2 we conclude that G splits over A. However this forces G to contain an element of order 2.

5. Polycyclic complete groups.

We have seen that a supersoluble complete group is finite. Polycyclic groups, however, behave quite differently and can easily be complete and infinite, as a simple example shows.

Let Q be the subgroup of $GL(2, \mathbb{Z})$ generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Thus Q is a dihedral group of order 12. Let A be the natural module. It is easy to prove that Q is self-normalizing in $\operatorname{Aut} A \simeq GL(2, \mathbb{Z})$. Also $A = [A, x]$ where x is an element of order 6 in Q; therefore $H^1(Q, A) = 0$. It now follows (as in the proof of Lemma 4) that the semidirect product $G = AQ$ is complete.

Of course this group is not metabelian. To construct an infinite polycyclic group which is complete and metabelian one looks for a self-normalizing abelian subgroup of $GL(n, \mathbb{Z})$. These can only be found if $n > 2$. The existence of such subgroups may be deduced from the following lemma.

Lemma 5. *Suppose that F is an algebraic number field with trivial Galois group. Let U be the group of algebraic units of F and let A be the additive group generated by U. If U acts on A as a group of automorphisms via the field multiplication, then $U = N_{\operatorname{Aut} A}(U)$.*

Proof. Let $\tau\colon U \to \operatorname{Aut} A$ be the natural embedding and suppose that $\alpha \in N_{\operatorname{Aut} A}(U^\tau)$. Choose u from U and let f be its minimal polynomial. Now $(u^\tau)^\alpha = v^\tau$ for some v in U. Also u^τ, and hence v^τ, is a root of f; consequently v is a root of f. We deduce that there is a field automorphism mapping u to v. Of course this means that $u = v$ and $\alpha \in C_{\operatorname{Aut} A}(U^\tau)$.

For any $a \in A$ and $u \in U$ we have $(au)\alpha = (a\alpha)u$. Setting $a = 1$ we obtain $u\alpha = vu$ where $v = 1\alpha$. Since α^{-1} exists, $v \in U$. Therefore $\alpha = v^\tau \in U^\tau$.

For example consider the irreducible polynomial

$$t^3 + t - 1\,,$$

which has exactly one real root, say a. Let F be the field $\mathbb{Q}(a)$; then F has trivial Galois group. Hence the group of units U is self-normalizing in $\operatorname{Aut} A$. Note that A has a basis $\{1, a, a^2\}$ and by the Dirichlet Units Theorem $U \simeq \mathbb{Z} \times \mathbb{Z}_2$. In fact a is a fundamental unit of F (see [2], p. 3), so that

$$U = \langle a \rangle \times \langle -1 \rangle\,.$$

Consider the semidirect product

$$G = AU\,.$$

Since $a - 1 = -a^3 \in U$, we see that $H^1(U, A) = 0$. It follows as in Lemma 4 that G is complete. Thus

THEOREM 7. There exists a metabelian polycyclic group which is complete and has Hirsch length 4.

6. A torsion-free polycyclic complete group.

We shall employ the example just discussed to construct a non-trivial torsion-free polycyclic complete group. Notice that such a group cannot be abelian by nilpotent in view of Theorem 3.

THEOREM 8. There exists a torsion-free polycyclic group of Hirsch length 7 that is complete.

PROOF. Let N be a free nilpotent group of class 2 and rank 3 generated by a_1, a_2, a_3. Then $N_{ab} = N/N'$ and N' are free abelian groups with respective bases

$$\{a_1 N', a_2 N', a_3 N'\}$$

and

$$\{[a_2, a_3], [a_3, a_1], [a_1, a_2]\}.$$

We define an automorphism ξ of N by the rules

$$a_1^\xi = a_2, \quad a_2^\xi = a_3, \quad a_3^\xi = a_1 a_2^{-1}.$$

Thus the effect of ξ on N_{ab} with respect to the given basis is described by the matrix

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

The point of this choice is that the normalizer of $\langle \xi \rangle$ considered as a subgroup of $\operatorname{Aut} N_{ab}$ is just $\langle \xi \rangle \times \langle -1 \rangle$, as we see from the action of a on A in the number field F above.

The effect of ξ on N' is given by the matrix $(\operatorname{adj} X)^T$, that is, by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The two minimal polynomials of ξ are, then,

$$t^3 + t - 1 \text{ on } N_{ab}, \quad \text{and } t^3 - t^2 - 1 \text{ on } N'.$$

Now define G to be the semidirect product

$$N\langle\xi\rangle .$$

Obviously G is a torsion-free polycyclic group with Hirsch length 7 and it has trivial centre. We shall show that $\operatorname{Out} G = 1$.

Assume that α is an outer automorphism of G. Now N is the Fitting subgroup of G, so it is characteristic. Also $C_G(N) = \zeta(N) = N'$. Clearly α normalizes $\langle\xi\rangle$ as a subgroup of $\operatorname{Aut} N_{ab}$. Consequently α must induce in N_{ab} an automorphism of the form $\pm\,\xi^r$. Therefore we may assume that α induces in N_{ab} either the identity or the mapping $x \mapsto x^{-1}$. In both cases $\xi^\alpha = \xi x$ for some $x \in N$.

Now $[\xi, N_{ab}] = N_{ab}$ because $\det(X - 1) = -1$. Thus we can find $x_1 \in N$ and $z \in N'$ such that $x = [\xi, x_1]z$; then $\xi^\alpha = \xi x = \xi^{x_1} z$. Modifying α by the inner automorphism induced by x_1 we can assume that $\xi^\alpha = \xi z$. In addition $[\xi, N'] = N'$, so by a similar argument we can suppose that

$$\xi^\alpha = \xi .$$

Thus $(x^\xi)^\alpha = (x^\alpha)^\xi$ for all $x \in N$.

Suppose that α acts non-trivially on N_{ab}. Then there exist elements $d_i \in N'$ such that

$$a_i^\alpha = a_i^{-1} d_i .$$

Now $(a_i^\xi)^\alpha = (a_i^\alpha)^\xi$. From this one obtains the equations

$$\left[\begin{array}{lll} d_1^{-\xi} d_2 & & = 1 \\ & d_2^{-\xi} d_3 & = 1 \\ d_1 & d_2^{-1} d_3^{-\xi} & = [a_1, a_2] . \end{array}\right.$$

Solving for d_1 one finds that

$$d_1^{-\xi^3-\xi+1} = [a_1, a_2] .$$

Recall that $\xi^3 = \xi^2 + 1$ on N'; thus the above becomes $(d_1^{-\xi})^{1+\xi} = [a_1, a_2]$. Writing $d_1^{-\xi} = (u, v, w)$, we have $(u, v, w)(\xi + 1) = (0, 0, 1)$ in additive notation, which yields the contradiction $3u + 1 = 0$.

We conclude that α acts trivially on N_{ab}, so that there are eleents d_i of N' such that

$$a_i^\alpha = a_i d_i .$$

Now $(a_i^\xi)^\alpha = (a_i^\alpha)^\xi$ yields the linear system

$$\left[\begin{array}{lll} d_1^{-\xi} d_2 & & = 1 \\ & d_2^{-\xi} d_3 & = 1 \\ d_1 & d_2^{-1} d_3^{-\xi} & = 1 . \end{array}\right.$$

The determinant of the coefficient matrix acts on N' like

$$-\xi^3 - \xi + 1 = -\xi(\xi + 1) .$$

Since $\det(\xi + 1) = 3$, it follows that $d_1 = d_2 = d_3 = 1$. Thus we arrive at the contradiction $\alpha = 1$.

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