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John C. Lennox

Derek J. S. Robinson

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Soluble Products of Nilpotent Groups.

JOHN C. LENNOX - DEREK J. S. ROBINSON (*) (**)

1. Introduction.

It is a famous theorem of P. Hall that every finite soluble group can be expressed as a product of pairwise permutable nilpotent subgroups, for example the members of a Sylow basis. Here we are concerned with the question: to what extent do infinite soluble groups have this property? We shall be particularly interested in polycyclic groups or more generally soluble groups with finite total rank. Here a soluble group is said to have *finite total rank* if the sum of the p-ranks (including $p = 0$) is finite when taken over all the factors of some abelian series. (Groups of this kind are sometime called soluble groups of type A_3 or $\mathfrak{S}_1$-groups.)

THEOREM A. A soluble group of finite total rank which has Hirsch length $\leqslant 1$ is expressible as a product of finitely many pairwise permutable nilpotent subgroups.

On the other hand

EXAMPLE 1 There exists a polycyclic group of Hirsch length 2 that is not a product of pairwise permutable nilpotent subgroups.

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(**) Indirizzo degli AA.: Derek J. S. Robinson, Dept. of Math., University of Illinois, Urbana, Ill. 61801, U.S.A.; John C. Lennox, University College, Cardiff, Wales, G.B.

Supersoluble groups exhibit somewhat better behaviour, as a more elaborate analysis of their nilpotent subgroups reveals.

THEOREM B. A supersoluble group of Hirsch length $\leqslant 2$ is expressible as a product of finitely many pairwise permutable nilpotent subgroups.

Again this is the best that can be done in this direction.

EXAMPLE 2. There exists a supersoluble group of Hirsch length 3 that is not a product of pairwise permutable nilpotent subgroups.

Nor is it possible to extend Theorem B to soluble groups of finite total rank in the sense indicated by the following example.

EXAMPLE 3. There exists a soluble group with finite total rank and Hirsch length 2, having a normal series whose factors are locally cyclic, but which is not a product of pairwise permutable nilpotent subgroups.

In spite of these examples soluble groups of finite total rank that are expressible as products of permutable nilpotent subgroups are numerous, and they can have arbitrarily complicated subgroup structure.

THEOREM E. Any soluble group of finite total rank can be embedded in a soluble group of finite total rank that is expressible as a product of finitely many pairwise permutable nilpotent subgroups. The same is true of polycyclic groups.

The proof of Theorem E depends on a useful result of Zaičev [8] to the effect that a soluble group with finite total rank has a subgroup of finite index that is the product of two nilpotent groups, one of them normal. Taken together with Theorem E this shows that soluble groups of finite total rank are in a sense « sandwiched » between products of permutable nilpotent subgroups.

We shall prove a generalization of Zaičev's result by a simpler method, using some of the « near splitting » techniques developed in [6] and [7]: this is Theorem C.

Under certain circumstances it is possible to embed the group of Theorem E as a subgroup of finite index.

THEOREM F. Let G be a soluble group of finite total rank which is abelian by finite by nilpotent. Then there exist a finite normal subgroup F and a soluble group $\bar{G}$ of finite total rank which is a product of finitely many pairwise permutable nilpotent subgroups such that G/F is isomorphic to a subgroup of finite index in $\bar{G}$. If G has no quasicyclic subgroups, F can be taken to be 1.

The proof of Theorem F depends on certain results about modules over finite by nilpotent groups which, we feel, are of independent interest. The crucial fact needed is

THEOREM G. Suppose that Q is a finite by nilpotent group and A is a Q-module which has finite total rank as an abelian group. Then the following conditions are equivalent:

(i) The largest Q-trivial image A_Q of A is finite.

(ii) The largest Q-trivial subgroup A^Q of A is finite.

This leads to

THEOREM H. Let Q be a finite by nilpotent group and let A be a Q-module which has finite total rank as an abelian group. Assume that A_Q (or equivalently A^Q) is finite. Then $H^n(Q, A)$ and $H_n(Q, A)$ have finite exponent for all n.

We mention a consequence of this theorem that may be compared with Theorem C (and also with Theorem D below).

COROLLARY. Suppose that G is a group with a normal abelian subgroup B of finite total rank such that G/B is finite by nilpotent. Then there exist a normal subgroup A of G, contained in B, and a nilpotent subgroup X such that $|G: XA|$ and $|X \cap A|$ are finite.

In the context of the Corollary (and also of Theorem D) it is important to observe

EXAMPLE 4. There exists a torsion-free polycyclic group that has no subgroup of finite index which is the split extension of one nilpotent group by another.

This despite the well-known fact due to Mal'cev that polycyclic groups are nilpotent by abelian finite ([5], 3.25 Corollary).

2. Groups with Hirsch length at most 2.

PROOF OF THEOREM A. Let G be a soluble group of finite total rank whose Hirsch length does not exceed 1. It follows from Lemma 9.34 and Theorem 9.39.3 of [5] that G has a normal series $1 \leqslant T \leqslant M \leqslant G$ wherein T is a divisible abelian group with the minimal condition, M/T is torsion-free abelian of rank $\leqslant 1$ and G/M is finite.

Now $[T, {}_rM] = [T, {}_{r+1}M] = A$, say, for some $r \geqslant 0$; also M/A is nilpotent. It follows by Theorem C of [7] together with Lemma 10 of [6] that G has a subgroup X with the properties $G = XA$ and $|X \cap A| < \infty$. It is easy to see that X is nilpotent by finite, so we may write $X = YL$ where Y is finitely generated and L is nilpotent and normal in X. Clearly Y is polycyclic, while $G = YLA$. Thus we may assume G to be an infinite polycyclic group.

There exists a normal infinite cyclic subgroup A of G with finite index. Thus G/A possesses a Sylow basis $(P_1/A, P_2/A, \ldots, P_k/A)$; we shall assume that P_1/A is a 2-group. Naturally $G = P_1 \ldots P_k$ and $P_iP_j = P_jP_i$.

Let C denote $C_G(A)$; then $|G:C| \leqslant 2$, so that $P_i \leqslant C$ if $i > 1$. Hence $C = (P_1 \cap C) P_2 \ldots P_k$. Clearly $P_1 \cap C$ and P_i, $i > 1$, are nilpotent. Moreover if $i > 1$,

$$(P_1 \cap C)P_i = (P_1P_i) \cap C = (P_iP_1) \cap C = (P_i \cap C)P_1 .$$

Hence $(P_1 \cap C/A, P_2/A, \ldots, P_k/A)$ is a Sylow basis of C/A.

Next, since $P_1/P_1 \cap C$ is cyclic, we may write $P_1 = \langle t \rangle (P_1 \cap C)$. Now $\big((P_1 \cap C)^t/A, P_2^t/A, \ldots, P_k^t/A\big)$ is a Sylow basis of C/A, so it must be conjugate to $(P_1 \cap C/A, P_2/A, \ldots, P_k/A)$ in C. Hence there is a c in C such that $(P_1 \cap C)^c = (P_1 \cap C)^t$ and $P_i^c = P_i^t$ if $i > 1$. This means that $u = tc^{-1}$ normalizes $P_1 \cap C$ and also P_i if $i > 1$. Therefore $G = \langle u \rangle (P_1 \cap C) P_2 \ldots P_k$ is a product of pairwise permutable nilpotent subgroups.

The proof of Theorem B is expedited by a simple lemma.

Lemma. Let H be a supersoluble group which has a free abelian normal subgroup A of rank 2 such that $A = C_H(A)$ and H/A is finite but not cyclic. Then there exist elements x and y such that $H = \langle x, y, A \rangle$ and $H' \leqslant C_A(x)\, C_A(y)$.

Proof. Let $\bar{H} = H/A$. This is effectively a finite group of integral triangular 2×2 matrices, so it is a 4-group. Let $H = \langle x, y, A \rangle$. If $\bar{H}$ is diagonal, we may choose x and y to operate on A according to the matrices

$$\bar{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and } \bar{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

in which event the result is obvious.

Supposing $\bar{H}$ not to be diagonal, we may assume, after an appropriate choice of basis, that x operates like

$$\bar{x} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},$$

let us say the former. Now the centralizer of $\bar{x}$ in the group of triangular matrices is $\langle \bar{x} \rangle \times \langle -1 \rangle$, so this must be $\bar{H}$. Therefore we may choose y to operate like

$$\bar{y} = -\bar{x} = \begin{pmatrix} -1 & -1 \\ 0 & +1 \end{pmatrix}$$

Thus $C_A(x)C_A(y) = \langle (2,0),(0,1) \rangle = K$ say. Clearly $[A, H] \leqslant K$. Suppose that $[x, y] \notin K$. Then $[x, y]$ must have the form $(2r+1, s)$ for some integers r, s. Hence $[x^2, y] = [x, y]^{x+1} = (4r+2, 2r+1)$. On the other hand $x^2 \in C_A(x)$, so $x^2 = (2t, t)$ for some t: hence $[x^2, y] = (-4t, -2t)$, a contradiction. Thus $H' \leqslant K$. The second possibility for $\bar{x}$ is handled in a similar way.

Proof of Theorem B. Let G be a supersoluble group with Hirsch length $h(G) = 2$. It is obvious that we may assume the centre of G be trivial. Since G is supersoluble, it follows that $O_2(G) = 1$ and G' contains no elements of order 2.

There exists a normal free abelian subgroup A with rank 2 and finite index—this is a consequence of a familiar theorem of Hirsch (see [5], 9.39.3). Put $C = C_G(A)$. Then G/C, being isomorphic with a finite group of triangular 2×2 matrices, is elementary abelian of order 2 or 4. We shall assume that $|G:C| = 4$, the other case being simpler and amenable to treatment in the manner of Theorem A.

Let $(P_1/A, P_2/A, \dots, P_k/A)$ be a Sylow basis of G/A with P_1/A a 2-group. Then $P_i \leqslant C$ if $i > 1$. Since $G = P_1 P_2 \dots P_k$, we have

$$C = TP_2 \dots P_k$$

where $T = P_1 \cap C$; these factors are permutable nilpotent subgroups.

Next A is contained in $Z(T)$, the centre of T, and T/A is a 2-group. Therefore T' is a finite 2-group, which means that $T' = 1$ and T is abelian. Denote the torsion subgroup of T by T_0; of course this is a finite 2-group.

Write $\bar{P}_1 = P_1/T_0$ etc. We claim that $C_{\bar{P}_1}(\bar{T}) = \bar{T}$. For if $\bar{z} \in C_{\bar{P}_1}(\bar{T})$, then $[A, z] \leqslant T_0 \cap A = 1$ and $z \in P_1 \cap C = T$. Note also that $\bar{T}$ is free abelian of rank 2 and $\bar{P}_1/\bar{T} \simeq G/C$. Applying the Lemma we conclude that P_1 contains elements x and y such that

$$P_1 = \langle x, y, T \rangle \quad \text{and} \quad \bar{P}_1' \leqslant C_{\bar{T}}(\bar{x})\, C_{\bar{T}}(\bar{y}) .$$

Define L/T_0 and M/T_0 to be the preimages of $C_{\bar{T}}(\bar{x})$ and $C_{\bar{T}}(\bar{y})$ under the natural homomorphism $T \to \bar{T}$, and write

$$X = \langle x, L \rangle \quad \text{and } Y = \langle y, M \rangle .$$

Then $X' \leqslant T_0$; since x acts on the finite 2-group T_0 as an element of order 2, we conclude that X is nilpotent. Similarly Y is nilpotent.

Observe that $L \lhd P_1$ and $M \lhd P_1$ since T is abelian. In addition $[x, y] \in LM$. Therefore

$$XY = \langle x \rangle L \langle y \rangle M = \langle x \rangle \langle y \rangle LM = \langle y \rangle \langle x \rangle LM = YX .$$

Obviously $P_1 = XYT$, so that

$$G = XYTP_2 \dots P_k .$$

Now the $T, P_2, \dots, P_k$ permute among themselves. Also $T \lhd P_1$, while X and Y are subgroups of P_1. Therefore it remains only to prove that X and Y permute with the P_i, $i > 2$.

Since a finite supersoluble group is 2-nilpotent, $D = P_2 \dots P_k$ is a normal subgroup of G. Consequently if $i > 1$,

$$XP_i \leqslant (P_i P_1) \cap (DX) = P_i(P_1 \cap (DX)) = P_i AX = P_i X .$$

Therefore $XP_i = P_i X$, and the proof is complete.

3. Generalizations of Zaičev's theorem.

THEOREM C. Let N be a normal subgroup of a group G. If N is a soluble group with finite total rank and G/N is nilpotent-by-finite, there exists a nilpotent subgroup X such that $|G:XN|$ is finite.

PROOF. We can form a characteristic series

$$1 = N_0 \leqslant N_1 \leqslant \dots \leqslant N_k = N$$

such that each factor is abelian and is either torsion-free or satisfies the minimal condition (see [5], 9.34). We may assume that $k > 0$ and use induction on k. Thus there is a nilpotent subgroup Y/N_1 such that $|G:YN| < \infty$.

There is an integer r such that $A = [N_1, {}_rY]$ has the following property: if N_1 is torsion-free, $A/[A, Y]$ is a torsion group, while if N_1 satisfies the minimal condition, $A = [A, Y]$. Applying Theorem 4 of [6] or Theorems C and D of [7], we conclude that there is a subgroup X_1 of Y such that $|Y:X_1A| < \infty$ and $|X_1 \cap A| < \infty$. Now X_1 is finite-by-nilpotent, so it has a normal nilpotent subgroup X of finite index. Then $|X_1A:XA| < \infty$, so that $|Y:XA| < \infty$ and $|YN:XN| < \infty$. Finally $|G:XN| < \infty$ as required.

An example of Zaičev [8] shows that it is insufficient to assume that N (or even G) has finite Prüfer rank. Also it is not in general possible to choose X so that $X \cap N$ is finite—as Example 4 shows. However, this can be done if G acts sufficiently non-trivially on abelian factors.

THEOREM D. *Let N be a normal nilpotent subgroup of a group G. Assume that N has finite total rank and that G/N is nilpotent by finite. Suppose further that if F is an infinite abelian G-factor of N, then $G/C_G(F)$ is infinite. Then G has a nilpotent subgroup X such that $|G:XN|$ and $|X \cap N|$ are finite.*

Since a soluble group with finite total rank is nilpotent by abelian by finite, Theorem D gives a condition for such a group to have a subgroup of finite index which is a split extension of nilpotent groups.

PROOF OF THEOREM D. We begin by forming a G-admissible series in N whose factors are N-central and either are torsion-free or satisfy the minimal condition. After refinement we can suppose the factors of the series to be « G-irreducible » in the sense that non-trivial G-admissible subgroups of torsion-free factors have torsion quotients while proper G-admissible subgroups of torsion factors are finite. Let the resulting series be $1 = N_0 \leqslant N_1 \leqslant \ldots \leqslant N_k = N$. We can assume that $k > 0$ and proceed by induction on k. Thus there is a nilpotent subgroup Y/N_1 such that $|G:YN|$ and $|Y \cap N:N_1|$ are finite.

Consider the case where N_1 is torsion-free. Suppose that $N_1/[N_1, Y]$ is infinite. Then it follows via [7], Lemma 5.12, that $N_1/[N_1, Y]$ is not a torsion group. Now $[N_1, Y] = [N_1, YN]$ since N_1 is central in N. Consequently $N_1/[N_1, C]$ is not a torsion group where C is the core of YN in G. Hence $[N_1, C] = 1$ by irreducibility. But G/C is finite, so we have a contradiction. When N_1 is torsion, $C_{N_1}(Y)$ is finite, as an analogous argument shows.

The near splitting theorems now give a subgroup X_1 such that $|Y:X_1N_1|$ and $|X_1 \cap N_1|$ are finite. But X_1 has a nilpotent subgroup X of finite index. Clearly $|G:XN|$ and $|X \cap N|$ are finite.

4. Embedding theorems.

PROOF OF THEOREM E. Let G be an arbitrary soluble group of finite total rank. Then G has a normal nilpotent subgroup N such that G/N is abelian by finite. According to Theorem C there is a nilpotent subgroup X such that $|G:XN|$ is finite. Since the core of XN has finite index in G and contains N, we may assume that $XN \lhd G$. Write $Q = G/XN$.

By a well-known principal we may embed G in the standard wreath product $W = (XN) \wr Q$ (see [4]). Notice that W is a soluble group with finite total rank.

Let B denote the base group of the wreath product. Since Q is a finite soluble group, it has a Sylow basis $(P_1, P_2, \ldots, P_k)$ and so $W = P_1P_2 \ldots P_kB$. Now we may write $B = B_1B_2$ where $B_1 = X^Q$ and $B_2 = N^Q$. Note that $B_2 \lhd W$ since $N \lhd G$; also $B_1 \lhd B_1Q$. Therefore $W = P_1P_2 \ldots P_kB_1B_2$ is a product of pairwise permutable nilpotent subgroups.

The same argument applies to polycyclic groups.

In view of Theorem C it is reasonable to ask whether a soluble group of finite total rank can always be embedded as a subgroup of finite index in a group of the same type that is a product of pairwise permutable nilpotent subgroups. We do not know if this can always be done. However in the special case of abelian by finite by nilpotent groups such an embedding always exists, as we shall establish in Theorem F. The proof of this second embedding theorem must be deferred until the next section where the necessary module techniques are developed.

5. Modules over finite by nilpotent groups.

We consider a finite by nilpotent group Q and a Q-module A which has finite total rank (as an abelian group). We shall be interested in the connection between Q-trivial submodules and Q-trivial images of the module A.

Recall that A^Q is the largest Q-trivial submodule of A, that is the set of Q-fixed points of A, and that A_Q is the largest Q-trivial image of A, that is, $A/[A, Q]$, where $[A, Q]$ is the subgroup generated by all $a(x-1)$, $a \in A$, $x \in Q$. Thus $A^Q = H^0(Q, A)$ and $A_Q = H_0(Q, A)$.

THEOREM G. Suppose that Q is a finite by nilpotent group and A is a Q-module which has finite total rank as an abelian group. Then the following conditions are equivalent:

I) A_Q is finite

II) A^Q is finite.

PROOF. 1) *The equivalence of* I) *and* II) *when Q is finite* (of order q say).

I) $\Rightarrow$ II). Set $s = \sum_{x \in Q} x \in \mathbb{Z}Q$. Then we have

$$q([A, Q]^Q) = ([A, Q]^Q)s \leqslant ([A, Q])s = 0$$

since $ys = s$ for all $y \in Q$. It follows that $[A, Q]^Q = [A, Q] \cap A^Q$ is finite. But $A^Q/[A, Q] \cap A^Q$ is also finite since A_Q is finite. Therefore A^Q is finite.

II) $\Rightarrow$ I). Let $a \in A$. Then

$$as = \sum_{x \in Q} a(x-1) + qa ,$$

which shows that

$$qA \leqslant [A, Q] + A^Q .$$

But A^Q is finite, so there is an $n > 0$ such that $nA \leqslant [A, Q]$. However this implies that A_Q is finite.

Since the finite case has been dealt with, we may assume that Q is infinite. By a theorem of P. Hall ([5], § 4.2) the centre of Q is non-trivial. It is also clear that we can always suppose Q to act faithfully on A.

In what follows $r_p(H)$ denotes the p-rank of an abelian group H.

2) I) $\Rightarrow$ II) *when A is a torsion-free group.*

Choose a non-trivial element x from the centre of Q and let θ be the non-zero Q-endomorphism $a \mapsto a(x-1)$. If $\operatorname{Ker}\theta = 0$, then $A^Q = 0$

and we are done. Assume therefore that $\operatorname{Ker}\theta \neq 0$. Then $r_0(A\theta) < r_0(A)$. Since $A\theta$ inherits the hypotheses on A, we have by induction that $(A\theta)^Q$ is finite and therefore zero. Let $S/A\theta$ be the torsion-subgroup of $A/A\theta$. Then S is a Q-module. Now $S^Q + A\theta/A\theta$ is torsion, so that S^Q is torsion since $S^Q \cap A\theta = 0$. Hence $S^Q = 0$. Moreover $r_0(A/S) < r_0(A)$, so by induction $A^Q \leqslant S$ and hence $A^Q = S^Q = 0$, as required.

3) I) $\Rightarrow$ II) *when A is a torsion group.*

We can assume that A is divisible by factoring out a finite Q-invariant subgroup. Choose x and θ as in 2). If $\operatorname{Ker}\theta$ is finite, we are done, so let it be infinite. Hence $r_p(A\theta) < r_p(A)$ for some prime p. It follows by an obvious induction that $(A\theta)^Q$ is finite and $(A/A\theta)^Q$ is finite. Therefore A^Q is finite.

4) II) $\Rightarrow$ I) *when A is a torsion-free group.*

Choose x and θ as in 2). If $A/A\theta$ is finite, we are finished. Suppose that $A/A\theta$ is infinite: then by a result of Fuchs (see [6], Lemma 9) $\operatorname{Ker}\theta \neq 0$ and $r_0(A\theta) < r_0(A)$; hence by induction $(A\theta)_Q$ is finite. Moreover $A\theta \overset{Q}{\simeq} A/\operatorname{Ker}\theta$ and $(\operatorname{Ker}\theta)_Q$ is finite, again by induction. It follows that A_Q is finite.

5) II) $\Rightarrow$ I) *when A is a torsion group.*

Replacing A by a multiple if necessary, we may assume that A is divisible. As in 4) we are justified in supposing $A/A\theta$ to be infinite. But in that case $r_p(A\theta) < r_p(A)$ for some prime p and so by induction $(A\theta)_Q$ is finite. Since $(\operatorname{Ker}\theta)_Q$ is finite, we conclude that A_Q is finite.

The mired case.

6) I) $\Rightarrow$ II). Supposing this to be false, we choose for A a counterexample with minimal torsion-free rank. Let T be the torsion subgroup of A. Then it follows from 2) that $A^Q \leqslant T$. If T_Q were finite, we could apply 3) to obtain $T^Q = A^Q$ finite, as required. Thus we may assume that T is infinite and $[T, Q] = 0$.

We claim next that A/T is irreducible in the sense that any non-zero submodule has a torsion quotient in A. If this is not true, there exists a submodule B such that $T < B < A$ and A/B is torsion-free. Now $(A/T)^Q = 0$, so $(B/T)^Q = 0$. Thus by 4) we have that $(B/T)_Q$ is finite, so B_Q, and hence $A/[B, Q]$, has finite total rank.

Suppose that $r_0(A/[B, Q]) < r_0(A)$. Then by minimality we conclude that $(A/[B, Q])^Q$ is finite, which implies that B_Q is finite. Furthermore $(A/B)^Q = 0$ by 2). It follows that $A^Q = B^Q$ is finite, a contradiction.

Hence $r_0(A/[B, Q]) = r_0(A)$, so that $r_0\big(A/([B, Q] + T)\big) = r_0(A/T)$, which can only mean that $[B, Q] \leqslant T$. However $(A/T)^Q = 0$, whence $B = T$, another contradiction. The irreducibility of A/T has therefore been established.

Let now C be the centre of Q. We consider first of all the case where $[A, C] \leqslant T$, so that $[A, C, C] = 0$, since $[T, Q] = 0$. Because A_Q is finite, $[A, C]/[A, Q, C]$ is finite. But $[A, Q, C] = [A, C, Q]$ since C is central; therefore $[A, C]_Q$ is finite.

Now $[A, C]$ is a Q/C-module. Thus by induction on the upper central height of Q we obtain that $[A, C]^Q$ is finite. In addition $A/[A, C]$ has finite total rank since $[A, C] \leqslant T$. Thus, again by induction, $(A/[A, C])^Q$ is finite, Hence A^Q is finite, a contradiction.

It follows that $[A, C] \nleqslant T$ and therefore there exists an $x \in C$ such that $[A, x] \nleqslant T$. Let θ be the Q-endomorphism $a \mapsto a(x-1)$. Then since A/T is irreducible, θ induces a monomorphism in A/T. Consequently $\operatorname{Ker}\theta = \operatorname{Ker}\theta^2 = T$, from which it follows that $A\theta \cap T = 0$. Also, by the result of Fuchs referred to above, the group $A/(A\theta + T)$ has finite order, m say. Thus $mA \leqslant A\theta + T$, which implies that $[mA, Q] \leqslant A\theta$. Furthermore $A/[mA, Q]$ is finite, so $A/A\theta$ is finite. However this forces T to be finite, a final contradiction in this case.

7) II) $\Rightarrow$ I). Supposing this to be false, we choose A to be a counterexample with $0 \neq r(T) = \sum_p r_p(T)$ minimal—here T is the torsion-subgroup of A. Replacing A by some integer multiple if necessary, it is easy to see that we may assume T to be divisible. If $(A/T)^Q = 0$, then 4) implies the finiteness of $(A/T)_Q$, whereas T_Q is finite by 5); thus A_Q is finite, a contradiction. Hence $(A/T)^Q \neq 0$. Now we may clearly replace A by D where $D/T = (A/T)^Q$. Assume therefore that $(A/T)^Q = A/T \neq 0$.

We claim that T is irreducible in the sense that every proper submodule is finite. If not, there exists an infinite proper submodule S of T. Let $B/S = (A/S)^Q$. Suppose that $r\big(m(B \cap T)\big) < r(T)$ for some $m > 0$. Then $(mB)_Q$, and hence B_Q, is finite by minimality. But $[B, Q] \leqslant S$, so B/S is finite: by minimality $(A/S)_Q$ is finite. Moreover it follows from 5) that S_Q is finite. Hence A_Q is finite, a contradiction.

Thus $r\big(m(B \cap T)\big) = r(T)$ for all $m > 0$. This means that $B \cap T = T$

and $T \leqslant B$ since T is divisible. But T^Q is finite, so T_Q is finite by 5). Hence $(T/S)_Q$ is finite. Therefore $T = [T, Q] + S = S$ since $[T, Q] \leqslant \leqslant [B, Q] \leqslant S$. This is a contradiction. Thus T is irreducible, as claimed.

Let C be the centre of Q and suppose first that $[T, C] = 0$. We claim that $(A/A^C)^Q = D/A^C$, say, is finite. For $[D, Q, C] = 0$, so that $[D, C, Q] = 0$ and $[D, C] \leqslant A^Q$, which is finite of order m, say. It follows that $[mD, C] = 0$ and $mD \leqslant A^C$. Hence D/A^C is finite, as required.

We note that A/A^C has finite total rank. For if A/A^C contains an element of prime order p, so also does A. Moreover $(A/A^C)^C = A/A^C$, so that A/A^C is a Q/C-module. By induction on the upper central height of Q we conclude that $(A/A^C)_Q$ is finite. Induction also shows that $(A^C)_Q$ is finite. Consequently A_Q is finite, a contradiction.

It follows that $[T, C] \neq 0$, so there exists $x \in C$ such that $[T, x] \neq 0$. Let θ be the Q-endomorphism $a \mapsto a(x-1)$ of A. By the irreducibility of T we have $T \cap \operatorname{Ker} \theta$ finite, so that $T = T\theta$. Also $A\theta \leqslant T$, because $(A/T)^Q = A/T$. Hence $A\theta = T\theta = T$ and $A\theta = A\theta^2$. We deduce that $A = A\theta + \operatorname{Ker}\theta = T + \operatorname{Ker}\theta$; now $[\operatorname{Ker}\theta, Q] \leqslant T \cap \operatorname{Ker}\theta$, which is finite. Therefore $m(\operatorname{Ker}\theta) \leqslant A^Q$ for some m, so that $\operatorname{Ker}\theta$ is finite. Hence $A = T + \operatorname{Ker}\theta$ is a torsion group, a final contradiction.

Remark. It is evident from the proofs that Theorem G holds if the hypercentre of Q has finite index provided that A is not mixed. However in the mixed case the theorem fails even if Q is a hypercentral group, as the following examples show.

Let $A = B \oplus C$ where $B = \langle b_1, b_2, \ldots\rangle$ is a quasicyclic 2-group and $C = \langle c\rangle$ is infinite cyclic. Let Q be the locally dihedral 2-group

$$Q = \langle t, x_i : x_{i+1}^2 = x_i,\, x_1^2 = 1 = t^2,\, x_i^t = x_i^{-1}\rangle .$$

Two actions of Q on A are defined in the following manner:

(i) $$b_i^t = b_i \qquad b_i^{x_j} = b_i$$
$$c^t = c^{-1} \qquad c^{x_j} = b_j c$$

and

(ii) $$b_i^t = b_i^{-1} \qquad b_i^{x_j} = b_i$$
$$c^t = c \qquad c^{x_j} = b_j c .$$

With action (i) one has

$$A^Q = B \quad \text{and} \quad A_Q \simeq \mathbb{Z}_2 ,$$

as Hartley and Tomkinson [2] have observed.

In the case of action (ii)

$$A^Q = \langle b_1 \rangle \quad \text{and } A_Q \cong \mathbb{Z} .$$

Thus Theorem G fails in both directions. Notice that Q is abelian by finite here, so Theorem G does not hold for nilpotent by finite groups. Similar examples demonstrate that there is no theorem for Q nilpotent by finite even in the torsion and torsion-free cases.

We now apply Theorem G to give information about (co)homology.

THEOREM H. *Let Q be a finite by nilpotent group and let A be a Q-module which has finite total rank as an abelian group. Assume that A_Q (or equivalently that A^Q) is finite. Then $H^n(Q, A)$ and $H_n(Q, A)$ have finite exponent for all n.*

PROOF. There exists a finite normal subgroup F such that $\bar{Q} = Q/F$ is nilpotent. Consider the Lyndon-Hochschild-Serre spectral sequence for cohomology associated with the extension $F \rightarrowtail Q \twoheadrightarrow \bar{Q}$ and the module A:

$$E^{ij} = H^i(\bar{Q}, H^j(F, A)) \underset{i+j=n}{\Rightarrow} H^n(Q, A) .$$

If $j > 0$, it is clear that E^{ij} has finite exponent. We claim that $E^{i0} = H^i(\bar{Q}, A^F)$ also has finite exponent. Set $M = A^F$: then $M^{\bar{Q}} = A^Q$ is finite.

Denote by T the torsion subgroup of M. Then there exists a finite submodule L such that T/L is divisible. Now $M_{\bar{Q}}$ is finite, so that $(M/T)_{\bar{Q}}$ is finite and $(M/T)^{\bar{Q}} = 0$, by Theorem G. For similar reasons $(T/L)_{\bar{Q}} = 0$. We now apply Theorems C and D of [7] to the modules T/L and M/T respectively to show that

$$H^i(\bar{Q}, T/L) \quad \text{and} \quad H^i(\bar{Q}, M/T)$$

have finite exponent for all i: of course $H^i(\bar{Q}, L)$ also has finite exponent. We now invoke the cohomology sequence to show that $H^i(\bar{Q}, M)$ has finite exponent for all i.

It follows from the convergence of the spectral sequence that $H^n(Q, A)$ has finite exponent. The proof for homology is analogous.

The following result may be compared with Theorems C and D.

COROLLARY. *Suppose that G is a group with a normal abelian subgroup B of finite total rank such that G/B is finite by nilpotent. Then* there exist a normal subgroup A of G contained in B and a nilpotent subgroup X such that $|G:XA|$ and $|X \cap A|$ are finite.

PROOF. Since B has finite total rank, there is an integer r such that $[B, {}_rG]/[B, {}_{r+1}G]$ is finite. We set $A = [B, {}_rG]$. It follows from theorems of Baer and Hall ([5], § 4.2) that $Q = G/A$ is finite by nilpotent. Regarding A as a Q-module in the obvious way, we have A_Q finite. Theorem H now shows that $H^2(Q, A)$ has finite exponent. We conclude that there exists a subgroup Y of G with the properties $|G:YA| < \infty$ and $|Y \cap A| < \infty$. Here Y is finite by nilpotent, so it contains a nilpotent subgroup X of finite index. Clearly $|G:XA|$ and $X \cap A$ are finite.

As a consequence of these results we are now in a position to prove Theorem F.

PROOF OF THEOREM F. Just as in the above Corollary we can find a normal abelian subgroup A such that $H^2(Q, A)$ has finite exponent, where $Q = G/A$. Let Δ be the cohomology class of the extension

$$A \rightarrowtail G \twoheadrightarrow Q .$$

Then there is a positive integer m such that $m\Delta = 0$. Consequently we may form the push-out diagram

$$\begin{array}{ccccc} A & \rightarrowtail & G & \twoheadrightarrow & Q \\ \downarrow & & \downarrow{\scriptstyle\gamma} & & \| \\ A & \rightarrowtail & \bar{G} & \twoheadrightarrow & Q \end{array}$$

where the left hand map is $a \mapsto a^m$. The index $|\bar{G}:G^\gamma| = |A:A^m|$ is finite. Now $\bar{G}$ splits over A, while Q has some term of its upper central series of finite index. Hence $\bar{G}$ is a product of pairwise permutable nilpotent subgroups. Let $F = \operatorname{Ker} \gamma$; then F is clearly finite. If G has no quasicyclic subgroups, we could assume in the first place that A is torsion free, so that $F = 1$.

6. Counterexamples.

EXAMPLE 1. There exists a polycyclic group of Hirsch length 2 that is not a product of pairwise permutable nilpotent subgroups.

Let $A = \langle a \rangle \oplus \langle b \rangle$ be a free abelian group of rank 2. Let $Q = \langle x, y \rangle$

be a dihedral group of order 8, where $x^4 = 1 = y^2$ and $x^y = x^{-1}$. If we allow Q to act on A by means of the assignments

$$x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and } y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then A becomes a Q-module. Now there is a non-split extension of A by Q realizing the given module structure which, with a slight abuse of notation, we may write as

$$G = \langle x, y, A : x^4 = 1 = y^2, x^y = x^{-1}b \rangle .$$

(In fact this is the only non-split extension because $H^2(Q, A) \simeq \mathbb{Z}_2$) Clearly G is polycyclic and $h(G) = 2$.

Suppose that N is a nilpotent subgroup of G such that $N \nleq A$ and $N \cap A$ is non-trivial. Now the group $\langle x^2 d, c \rangle$ is not nilpotent if $d \in A$ and $1 \neq c \in A$; so $x^2 \notin NA$. It follows that N must be a subgroup of one of the following types

$$\langle yr, R \rangle , \quad \langle yxs, S \rangle , \quad \langle yx^2 t, T \rangle , \quad \langle yx^3 u, U \rangle$$

where r, s, t, u are elements and R, S, T, U are subgroups of A. In particular it is not hard to see that N is abelian.

It is easy to prove that if the dihedral group of order 8 is expressed as a product of permutable subgroups, then it is a product of two of them. Assuming that G is a product of permutable nilpotent subgroups, we deduce that there is such a factorization of the form

$$G = ABC .$$

If $G = AB$, then $A \cap B$ would be contained in the hypercentre of G, which is obviously trivial; thus G would be a split extension, which we know to be false. Hence $C \nleq A$ and for similar reasons $B \nleq A$. If neither B nor C is finite, they must have the forms specified above. A check of the permutability of the four subgroups modulo A, that is in the dihedral group, reveals just two possibilities:

$$B = \langle yr, R \rangle \quad \text{and} \quad C = \langle yx^2 t, T \rangle$$

or

$$B = \langle yxs, S \rangle \quad \text{and} \quad C = \langle yx^3 u, U \rangle .$$

However in neither case does ABC equal G.

We may therefore assume that B is finite. If $h(BC) < 2$, then $D = (BC) \cap A$ is infinite cyclic since G does not split over A. But D is normal in G, whereas A contains no infinite cyclic normal subgroups of G, a contradiction.

Hence $h(BC) = 2$ and, since B is finite, C is therefore of finite index in G. Since C is nilpotent, there exist integers m, r such that $[A^m, {}_rC] = 1$. Hence $[A, C] = 1$ and so $C \leqslant A$, a final contradiction.

REMARK. It is interesting to note that every proper homomorphic image of G is a finite abelian by nilpotent group and as such splits over some term of its lower central series by a well known theorem of Gaschütz. Thus, every proper image of G is a product of two nilpotent subgroups, one of them normal.

EXAMPLE 2. There exists a supersoluble group of Hirsch length 3 that is not a product of pairwise permutable nilpotent subgroups.

Let $A = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$ be a free abelian group of rank 3 and let $Q = \langle x, y \rangle$ be a 4-group. An action of Q on A is specified by

$$x \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

so A becomes a Q-module.

Let G be the non-split extension of A by Q given by

$$G = \langle x, y, A : x^2 = a, y^2 = b, [x, y] = a^{-1}bc \rangle .$$

We omit the verification that this is in fact a non-split extension. Clearly G is supersoluble: its Hirsch length is 3.

Suppose that G is a product of pairwise permutable nilpotent subgroups. By looking at factorizations of the 4-group we conclude that G has a factorization

$$G = ABC$$

where B and C are permutable nilpotent subgroups. If G were to equal AC or AB, one can show that the extension would split. Hence $B \nleqslant A$ and $C \nleqslant A$.

A check of the nilpotent subgroups of G reveals the following possibilities for B and C:

$$R = \langle xr, R_0\rangle\ , \qquad S = \langle ys, S_0\rangle\ , \qquad T = \langle xyt, T_0\rangle$$

where $r, s, t \in A$ and $R_0 \leqslant C_A(x)$, $S_0 \leqslant C_A(y)$, $T_0 \leqslant C_A(xy)$.

Suppose, for example, that $RS = SR$. Then

$$(xr)(ys) = (ys)^j(xr)^i a_x a_y$$

where $a_x \in C_A(x)$ and $a_y \in C_A(y)$. Here i and j must be odd. It follows that

$$[x, y] \in C_A(x)\,C_A(y)[A, G]A^2\ .$$

But the latter equals $\langle a, b, c^2\rangle$, in contradiction to $[x, y] = a^{-1}bc$. By similar arguments $ST = TS$ and $TR = RT$ are untenable. This gives our final contradiction.

EXAMPLE 3. The exists a soluble group with finite total rank and Hirsch length 2, having a normal series whose factors are locally cyclic, which is not a product of pairwise permutable nilpotent subgroups.

Denote by A the additive group of rational numbers of the form $m3^n$, $(m, n \in \mathbf{Z})$. Let $Q = \langle x\rangle \times \langle y\rangle$ where x has infinite order and y has order 2. Make A into a Q-module via

$$ax = 3a \qquad \text{and} \qquad ay = -a$$

for all $a \in A$. Note that $[A, Q] = 2A$. Choosing u from $A \setminus 2A$ we obtain a non-split extension

$$G = \langle x, y, A\colon [x, y] = u\rangle\ ,$$

which is in fact unique.

Suppose that N is a nilpotent subgroup of G not contained in A. Then it is easily seen that $N \cap A = 1$, so that in fact N is a finitely generated abelian group. If G had a product decomposition of the forbidden type, we could write $G = XA$, where X is a product of pairwise permutable finitely generated abelian subgroups. It follows from a theorem of Amberg [1], or from a more general theorem of

Lennox and Roseblade [3], that X must be polycyclic. Now $X \cap A$ cannot be trivial since G does not split over A. Hence $X \cap A$ is an infinite cyclic normal subgroup of G contained in A. But this is clearly impossible.

Notice that G is a finitely generated minimax group.

EXAMPLE 4. There exists a torsion-free polycyclic group having no subgroup of finite index which is the split extension of one nilpotent group by another.

We begin with a torsion-free «extra-special» group with four generators. This is the group N with generators a_1, a_2, b_1, b_2 which is nilpotent of class 2 and satisfies in addition

$$[a_1, a_2] = c = [b_1, b_2]$$

and

$$[a_i, b_j] = 1 .$$

Thus $N' = \langle c \rangle$ is the centre of N, and N/N' is free abelian of rank 4.

Two automorphisms x and y of N are defined by the following rules:

$$a_1^x = a_2^{-1}, \quad a_2^x = a_1 a_2^3, \quad b_i^x = b_i$$

and

$$b_1^y = b_2^{-1}, \quad b_2^y = b_1 b_2^3, \quad a_i^y = a_i .$$

It is routine to check that x and y are indeed automorphisms. Moreover they generate a free abelian group $Q = \langle x \rangle \times \langle y \rangle$. Thus K is a Q-module. Define

$$A = \langle a_1, a_2 \rangle \quad \text{and} \quad B = \langle b_1, b_2 \rangle ;$$

then A/A' is $\langle x \rangle$-irreducible and B/B' is $\langle y \rangle$-irreducible.

We now form the non-split extension of N by Q

$$G = \langle x, y, N : [x, y] = c \rangle .$$

This is a torsion-free polycyclic group of Hirsch length 7.

Suppose that there exist nilpotent subgroups M and H such that $H \cap M = 1$, $M \lhd HM$ and $|G : HM| < \infty$. We shall obtain a contradiction.

We claim first that $M \leqslant N$. If this is not true, M contains some

$u = x^r y^s d$ where $(r, s) \neq (0, 0)$ and $d \in N$. Since $|G{:}N_G(M)| < \infty$, some N^t normalizes M, where $t > 0$. One concludes that $[N^t, {}_i u] = 1$ for a suitable i; this is because M is nilpotent. It follows that $[A^t, {}_i x^r] \leq A'$. But A/A' is rationally irreducible with respect to $\langle x \rangle$; thus $[A, x^r] \leq A'$ and $r = 0$. In the same way $s = 0$, so we have a contradiction.

The next point to establish is that M has finite index in N, and hence $h(M) = 5$. Since $|G{:}HM|$ is finite, there is a $t > 0$ such that $\langle x^t, N^t \rangle \leq HM$. Since H is nilpotent, we have $[N^t, {}_i x^t] \leq M$ for some i. Now modulo N' some powers a_1^j, a_2^j, $(j > 0)$, are contained in $[N^t, {}_i x^t]$ and hence in M. Forming commutators we conclude that M contains non-trivial powers of a_1, a_2 and c. Clearly the same holds for b_1 and b_2. Consequently $|N{:}M|$ is finite.

Comparing Hirsch lengths we see that

$$7 = h(HM) = h(H) + h(M) = h(H) + 5\,,$$

so that $h(H) = 2$. Moreover there is a positive integer t such that $H \cap N^t \leq H \cap M = 1$. Since G is torsion-free, this can only mean that $H \cap N = 1$ and H is abelian.

Since $|G{:}HM|$ is finite, there exist r, a positive integer, and d, e, elements of N, such that $x^r d \in H$ and $y^r e \in H$. It follows that $[x^r d, y^r e] = 1$, which in turn implies that

$$[x^r, y^r] = [x^r, e]^{-e^{-1}} [e, d] [y^r, d]^{d^{-1}}\,.$$

Hence $[x^r, e] \in N'$, which must mean that $e \in B$ and $[x^r, e] = 1$. Similarly $d \in A$ and $[y^r, d] = 1$. But $[A, B] = 1$, so $[x^r, y^r] = 1$. This however is impossible because $[x^r, y^r] = c^{r^2} \neq 1$.

Remark. It is worth observing that the group G cannot be isomorphic with a subgroup of finite index in a split extension of one nilpotent group by another.

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