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Abelian Groups in which Every Γ-Isotype Subgroup is a Pure Subgroup, Resp. an Isotype Subgroup.

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All groups considered in this paper are abelian. Concerning the terminology and notation, we refer to [3]. In addition, if G is a group then G_t and G_p are the torsion part of G and the p-component of G_t respectively. Let G be a group and p a prime. Following Rangaswamy [10] we say that a subgroup H of G is p-absorbing, resp. absorbing in G if $(G/H)_p = 0$, resp. $(G/H)_t = 0$. A subgroup H of G is said to be isotype in G if $p^\alpha H = H \cap p^\alpha G$ for all primes p and all ordinals α. Recall that if H is p-absorbing in G then $p^\alpha H = H \cap p^\alpha G$ for every ordinal α (see lemma 103.1 [3]).

Let $\mathbb{N}$ be the set of all positive integers, $p_1, p_2, \ldots$ be the sequence of all primes in the natural order and $\mathcal{K}$ the class of all sequences $(\alpha_1, \alpha_2, \ldots)$, where each α_i is either an ordinal or the symbol ∞ which is considered to be larger than any ordinal. Let G be a group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{K}$. A subgroup H of G is said to be Γ-isotype in G if $p_i^\beta H = H \cap p_i^\beta G$ for every $i \in \mathbb{N}$ and for every ordinal $\beta \leqslant \alpha_i$. If $\Gamma = (0, 0, \ldots)$, $\Gamma = (1, 1, \ldots)$, $\Gamma = (\omega, \omega, \ldots)$, $\Gamma = (\infty, \infty, \ldots)$ then Γ-isotype subgroups of G are precisely subgroups, neat subgroups, pure subgroups, isotype subgroups respectively. Note that if $\Gamma = (\alpha_1, \alpha_2, \ldots)$, $\Gamma' = (\alpha_1', \alpha_2', \ldots) \in \mathcal{K}$ and $\Gamma \leqslant \Gamma'$ (i.e. $\alpha_i \leqslant \alpha_i'$ for each $i \in \mathbb{N}$) then every Γ'-isotype subgroup of G is Γ-isotype in G. Let G be a p-group, γ be an ordinal or the symbol ∞. A subgroup H of G is said to be γ-isotype in G if $p^\beta H = H \cap p^\beta G$ for every ordinal $\beta \leqslant \gamma$.

A direct sum of cyclic groups of the same order p^e is denoted by B_e.

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The purpose of this paper is to describe the classes of all groups in which every Γ-isotype subgroup is a neat, a pure, an isotype subgroup, a direct summand, an absolute direct summand, an absorbing subgroup respectively. Here are so generalized the results of this type from [1], [2], [4], [6]-[9], [11], [13] (see [1]-introduction).

LEMMA 1. Let G be a torsion group and $\Gamma = (\alpha_1, \alpha_2, \dots) \in \mathcal{K}$. A subgroup H of G is Γ-isotype in G iff H_{p_i} is α_i-isotype in G_{p_i} for every $i \in \mathbf{N}$.

PROOF. Obvious.

LEMMA 2. Let G be a p-group, $g \in G$ an element of order p^i, $k \in \mathbf{N}$, $k \leqslant i$. The subgroup $\langle g \rangle$ is k-isotype in G iff $h^G(p^{k-1}g) = k-1$. Moreover, the subgroup $\langle g \rangle$ is pure (isotype) in G iff $h^G(p^{i-1}g) = i-1$.

PROOF. Easy.

LEMMA 3. Let H be a subgroup of a group G and p a prime. If G_p is divisible and $pH = H \cap pG$ then $p^\alpha H = H \cap p^\alpha G$ for every ordinal α.

PROOF. Obviously, H_p is neat in G_p and hence H_p is divisible. Write $H = H_p \oplus H'$ and $G = G_p \oplus G'$, where $H' \subset G'$. Since H' is p-absorbing in G', the result follows.

LEMMA 4. Let G be a p-group and $k \in \mathbf{N}$. If every k-isotype subgroup of G is a pure subgroup of G then either $G = D \oplus B$, where D is divisible and $p^{k-1}B = 0$, or $p^{k-1}G = B_e \oplus B_{e+1}$ for some $e \in \mathbf{N}$.

PROOF. Let $G = D \oplus B$, where D is nonzero divisible and B is reduced. Suppose $B = \langle a \rangle \oplus B'$, where $o(a) = p^j$ and $j \geqslant k$; let $d \in D$ be an element of order p^{j+1}. By lemma 2, $\langle a + d \rangle$ is k-isotype in G but is not pure in G—a contradiction. Hence $p^{k-1}B = 0$.

Let G be reduced. Suppose $G = \langle a \rangle \oplus \langle b \rangle \oplus G'$, where $o(a) = p^j$, $o(b) = p^m$ and $m - 2 \geqslant j \geqslant k$. By lemma 2, the subgroup $\langle a + pb \rangle$ is k-isotype in G but is not pure in G—a contradiction. If B is a basic subgroup of G then obviously $G = B = B_1 \oplus \dots \oplus B_{k-1} \oplus B_m \oplus B_{m+1}$, where $m \geqslant k$, and hence $p^{k-1}G = B_e \oplus B_{e+1}$ $(e = m - k + 1)$.

LEMMA 5. Let G be a group, p a prime and $\alpha < \beta$ ordinals. If $p^\beta G_p$ is not essential in $p^\alpha G_p$ and either $p^{\beta+1}G_p$ is nonzero or $p^\beta G$ is not torsion then there is a sugroup H of G with following properties: H is q-absorbing in G for every prime $q \neq p$, $p^\gamma H = H \cap p^\gamma G$ for every ordinal $\gamma \leqslant \alpha + 1$ and $p^{\beta+1}H \neq H \cap p^{\beta+1}G$.

PROOF (see lemma 3 [1]). There is a nonzero element $n \in p^{\alpha}G_p[p]$ such that $\langle n\rangle \cap p^{\beta}G_p = 0$. Let $g \in p^{\beta}G$ such that either $0 \neq pg \in G_p$ or $o(g) = \infty$. Write $X = \langle p^{\beta}G[p], pg, n + g\rangle$. It is easy to see that $\langle n\rangle \cap X = 0$. Let H be an $\langle n\rangle$-high subgroup of G containing X. By [5], $p^{\gamma}H = H \cap p^{\gamma}G$ for every ordinal $\gamma \leqslant \alpha + 1$. Since $p^{\beta}G[p] \subset H$, $p^{\beta+1}H \neq H \cap p^{\beta+1}G$. By lemma 6 [1], H is q-absorbing in G for every prime $q \neq p$.

THEOREM 1. Let G be a group and $\Gamma = (\alpha_1, \alpha_2, \dots) \in \mathcal{K}$. The following are equivalent:

(i) Every Γ-isotype subgroup of G is a pure subgroup of G.

(ii) For every $i \in \mathrm{N}$, if $\alpha_i < \omega$ then either $G_{p_i} = D \oplus B$, where D is divisible and $p_i^{\alpha_i - 1}B = 0$, or G is torsion and G_{p_i} is elementary or G is torsion and $p_i^{\alpha_i-1}G_{p_i} = B_e \oplus B_{e+1}$ for some $e \in \mathrm{N}$.

PROOF. Assume (i). For each $i \in \mathrm{N}$, every α_i-isotype subgroup of G_{p_i} is Γ-isotype in G and hence pure in G_{p_i}. By lemma 4 and by [4], G_{p_i} is as claimed. Suppose G is not torsion and $\alpha_i = 0$ for some $i \in \mathrm{N}$. If $g \in G$ is an element of infinite order then $g \notin \langle p_i g, G_{p_i}\rangle$ and a subgroup H maximal with respect to the properties $g \notin H$, $\langle p_i g, G_{p_i}\rangle \subset H$ is Γ-isotype in G by lemma 6 [1]. Since H is pure in G, there is an element $h \in H$ such that $p_i g = p_i h$, hence $g - h \in G_{p_i} \subset H$—a contradiction. Finally, if G is not torsion, $\alpha_i < \omega$ for some $i \in \mathrm{N}$ and $p_i^{\alpha_i-1}G_{p_i} = B_e \oplus B_{e+1} \neq 0$ then $p_i^{\alpha_i+e}G_{p_i}$ is not essential in $p_i^{\alpha_i-1}G_{p_i}$, $p_i^{\alpha_i+e}G$ is not torsion and lemma 5 implies a contradiction.

Assume (ii). Let H be an Γ-isotype subgroup of G and $i \in \mathrm{N}$. Write $\beta = \alpha_i$ and $p = p_i$. If $\beta \geqslant \omega$ then H is p-pure in G. If $\beta = 0$ then by assumption G is torsion and G_p is elementary; write $G = G_p \oplus G'$ and $H = H_p \oplus H'$. For every $k \in \mathrm{N}$, $p^k H = H' = H \cap G' = H \cap p^k G$, i.e. H is p-pure in G. Let $0 < \beta < \omega$. Suppose that $p^{\beta-1}G_p = B_e \oplus B_{e+1}$ and G is torsion. By lemma 1,

$$p(p^{\beta-1}H_p) = H_p \cap p(p^{\beta-1}G_p) = p^{\beta-1}H_p \cap p(p^{\beta-1}G_p)\,,$$

i.e. $p^{\beta-1}H_p$ is neat in $p^{\beta-1}G_p$. By [9], $p^{\beta-1}H_p$ is pure in $p^{\beta-1}G_p$ and hence H_p is pure in G_p. Consequently, H is p-pure in G. Suppose that $G_p = D \oplus B$, where D is divisible and $p^{\beta-1}B = 0$. Now,

$$p(p^{\beta-1}H) = H \cap p(p^{\beta-1}G) = p^{\beta-1}H \cap p(p^{\beta-1}G)\,.$$

Since $p^{\beta-1}G_p$ is divisible, $p^{\beta-1}H$ is p-pure in $p^{\beta-1}G$ by lemma 3. Therefore H is p-pure in G.

THEOREM 2. Let G be a group and $\Gamma = (\alpha_1, \alpha_2, \dots) \in \mathcal{K}$. Every Γ-isotype subgroup of G is a direct summand of G iff the following conditions hold:

(i) $G = T \oplus D \oplus N$, where T is torsion reduced, D is divisible and N is a direct sum of a finite number mutually isomorphic torsion-free rank one groups;

(ii) if $\alpha_i < \omega$ then either $p_i^{\alpha_i - 1}T_{p_i} = 0$ or G is torsion and G_{p_i} is elementary or G is torsion and $p_i^{\alpha_i - 1}G_{p_i} = B_e \oplus B_{e+1}$ for some $e \in \mathbf{N}$;

(iii) if $\omega \leqslant \alpha_i$ then T_{p_i} is bounded.

PROOF. If every Γ-isotype subgroup of G is a direct summand of G then every isotype subgroup of G is a direct summand of G and every Γ-isotype subgroup of G is pure in G. Now, theorem 2 [1] and theorem 1 imply (ii). Conversely, by theorem 1, every Γ-isotype subgroup of G is pure in G and by [2], every pure subgroup of G is a direct summand of G.

For the similar result see [12].

THEOREM 3. Let G be a group and $\Gamma = (\alpha_1, \alpha_2, \dots) \in \mathcal{K}$. Every Γ-isotype subgroup of G is an absolute direct summand of G iff G satisfies one of the following two conditions:

(i) G is torsion and for every $i \in \mathbf{N}$,
if $\alpha_i = 0$ then G_{p_i} is elementary,
if $0 < \alpha_i$ then either G_{p_i} is divisible or $G_{p_i} = B_e$ for some $e \in \mathbf{N}$.

(ii) $\alpha_i \neq 0$ for every $i \in \mathbf{N}$ and either G is divisible or $G = G_t \oplus R$, where G_t is divisible and R is of rank one.

PROOF. Every Γ-isotype subgroup of G is an absolute direct summand of G iff every Γ-isotype subgroup of G is a direct summand of G and every direct summand of G is an absolute direct summand of G. Now, theorem 2 and [11] imply the desired result.

THEOREM 4. Let G be a group and $\Gamma = (\alpha_1, \alpha_2, \dots) \in \mathcal{K}$. The following are equivalent:

(i) Every Γ-isotype subgroup of G is a neat subgroup of G.

(ii) If $\alpha_i = 0$ for some $i \in \mathbf{N}$ then G_{p_i} is elementary and G is torsion.

PROOF. Assume (i). Every α_i-isotype subgroup of G_{p_i} is obviously Γ-isotype in G and hence neat in G_{p_i}. By [11], if $\alpha_i = 0$ then G_{p_i} is elementary. Suppose that G is not torsion and $\alpha_i = 0$ for some $i \in \mathrm{N}$. If $g \in G$ is an element of infinite order then a subgroup H of G maximal with respect to the properties $g \notin H$, $\langle p_i g, G_{p_i}\rangle \subset H$ is Γ-isotype in G by lemma 6 [1] but obviously it is not a neat subgroup of G.

Assume (ii). If $\alpha_i > 0$ for each $i \in \mathrm{N}$ then every Γ-isotype subgroup of G is neat in G. Suppose G is torsion and if $\alpha_i = 0$ then G_{p_i} is elementary. If H is an Γ-isotype subgroup of G then H_{p_i} is α_i-isotype in G_{p_i} for each $i \in \mathrm{N}$ and hence neat in G_{p_i} by [11]. Consequently, H is neat in G.

LEMMA 6. Let G be a group, p a prime and β an ordinal. Let H be a $p^\beta G$-high subgroup of G and $a \in p^\beta G$. If $p^\alpha H_p \neq 0$ for each ordinal $\alpha < \beta$ then there is a subgroup X of G such that $p^\alpha X = X \cap p^\alpha G$ for each ordinal $\alpha \leqslant \beta$ and $p^\beta X = \langle a \rangle$.

PROOF. If $(o(a), p) = 1$ then write $X = \langle a \rangle$ and all is well. Hence suppose that $p | o(a)$ or $o(a) = \infty$.

If β is not a limit ordinal then there is an element $b \in p^{\beta-1}G$ such that $pb = a$. If $b \notin p^\beta G$ then write $a_\alpha = b$ for every ordinal $\alpha < \beta$ and $X_\beta = \langle b \rangle$. If $b \in p^\beta G$ and $0 \neq c \in p^{\beta-1}H[p]$ then $b' = b + c \in p^{\beta-1}G \setminus p^\beta G$, $pb' = a$; in this case write $a_\alpha = b'$ for every ordinal $\alpha < \beta$ and $X_\beta = \langle b' \rangle$. Obviously $X_\beta \cap p^\beta G = \langle a \rangle$.

Let β be a limit ordinal. For each ordinal $\alpha < \beta$ there is an element $x \in p^\alpha G \setminus p^\beta G$ such that $a = px$. We use the transfinite induction to define the sets $X_\alpha, \alpha \leqslant \beta$: $X_0 = \langle a \rangle$; obviously $X_0 \cap p^\beta G = \langle a \rangle$ and $(G \cap X_0)[p] \subset \langle a \rangle$. Further, $X_1 = \langle X_0, a_1 \rangle$, where $a_1 \in pG \setminus p^\beta G$ and $pa_1 = a$; obviously $X_1 \cap p^\beta G = \langle a \rangle$ and $(pG \cap X_1)[p] \subset \langle a \rangle$. Suppose that $X_{\alpha-1}$ has been defined such that $X_{\alpha-1} \cap p^\beta G = \langle a \rangle$ and $(p^{\alpha-1}G \cap X_{\alpha-1})[p] \subset \langle a \rangle$, define X_α. If there is an element $x \in X_{\alpha-1} \cap p^\alpha G$ such that $px = a$ then let $a_\alpha = x$ and $X_\alpha = X_{\alpha-1}$. Otherwise let $X_\alpha = \langle X_{\alpha-1}, a_\alpha \rangle$, where $a_\alpha \in p^\alpha G \setminus p^\beta G$ and $pa_\alpha = a$. We show that $X_\alpha \cap p^\beta G = \langle a \rangle$. Let $y + za_\alpha \in p^\beta G$, where $y \in X_{\alpha-1}$ and z is an integer. Obviously $py \in X_{\alpha-1} \cap p^\beta G = \langle a \rangle$; write $py = ma$, where m is an integer. If $(p, m) = 1$ then there are integers u, v such that $upa + vma = a$ and hence $a = p(ua + vy)$, $ua + vy \in X_{\alpha-1} \cap p^\alpha G$ —a contradiction. Hence $m = pm'$, $p(y - m'a) = 0$ and $y - m'a \in (p^{\alpha-1}G \cap X_{\alpha-1})[p] \subset \langle a \rangle$. Therefore $y \in \langle a \rangle$, $y + za_\alpha \in \langle a_\alpha \rangle \cap p^\beta G = \langle a \rangle$. Further we show that $(p^\alpha G \cap X_\alpha)[p] \subset \langle a \rangle$. Let $y + za_\alpha \in (p^\alpha G \cap X_\alpha)[p]$, where $y \in X_{\alpha-1}$ and z is an integer; hence $py = -za$.

If $(p, z) = 1$ then $a = p(ua - vy)$, where u, v are integers, $ua - vy \in \in X_{\alpha-1} \cap p^\alpha G$—a contradiction. Hence

$$z = pz', \quad y + z'a \in (p^{\alpha-1}G \cap X_{\alpha-1})[p] \subset \langle a \rangle$$

and therefore $y \in \langle a \rangle$. Now, $y + za_\alpha \in \langle a \rangle$. Finally, if α is a limit ordinal then let $X_\alpha = \bigcup_{\gamma<\alpha} X_\gamma$.

Let X be a subgroup of G maximal with respect to the properties: $X \cap p^\beta G = \langle a \rangle$, $X^\beta \subset X$. We prove that $p^\alpha X = X \cap p^\alpha G$ for every $\alpha \leqslant \beta$. It is sufficient to show that if this equality holds for $\alpha - 1$ then it holds for α. Let $x \in X \cap p^\alpha G$, i.e. $x = pg$, where $g \in p^{\alpha-1}G$. If $g \in X$ then $g \in X \cap p^{\alpha-1}G = p^{\alpha-1} X$ and $x \in p^\alpha X$. If $g \notin X$ then there is an element $y \in X$ and an integer z such that $zg + y \in p^\beta G \setminus \langle a \rangle$. Obviously $y \in p^{\alpha-1}G$ and $(z, p) = 1$. Since $pzg + py \in X \cap p^\beta G = \langle a \rangle$, $zx + py = ra = rpa_{\alpha-1}$ and $zx = p(ra_{\alpha-1} - y)$. Now, $ra_{\alpha-1} - y \in X \cap \cap p^{\alpha-1}G = p^{\alpha-1}X$, $zx \in p^\alpha X$ and $x \in p^\alpha X$.

LEMMA 7. Let G be a p-group and β an ordinal. The following are equivalent:

(i) Every β-isotype subgroup of G is isotype in G.

(ii) Either $G = D \oplus B$, where D is divisible and $p^\gamma B = 0$ for some ordinal $\gamma < \beta$, or $p^\beta G$ is elementary or $p^{\beta-1}G = B_e \oplus B_{e+1}$ for some $e \in \mathbf{N}$.

PROOF. Assume (i). If $\beta = 0$ then G is elementary by [4]. If β is a limit ordinal then write $\alpha = \beta$, otherwise write $\alpha = \beta - 1$. Let $p^\alpha G = D \oplus R$, where D is divisible and R is reduced. If both D and R are nonzero, write $R = \langle a \rangle \oplus R'$, where $o(a) = p^k$, $k \in \mathbf{N}$. The subgroup $p^{\alpha+k}G$ is not essential in $p^\alpha G$, $p^{\alpha+k+1}G \neq 0$ and lemma 5 implies a contradiction. If $p^\alpha G$ is reduced and $p^\alpha G = \langle a \rangle \oplus \langle b \rangle \oplus R'$, where $o(a) = p^k$, $o(b) = p^j$ and $j - k \geqslant 2$, then $p^{\alpha+k}G$ is not essential in $p^\alpha G$, $p^{\alpha+k+1}G \neq 0$ and lemma 5 implies a contradiction. Consequently, either $p^\alpha G$ is nonzero divisible or $p^\alpha G = B_e \oplus B_{e+1}$ for some $e \in \mathbf{N}$. If $\alpha = \beta - 1$ then we are through, since if $p^\alpha G$ is divisible then $G = p^\alpha G \oplus B$ and obviously $p^\alpha B = 0$. Hence suppose $\alpha = \beta$. Let $p^\beta G$ be nonzero divisible; write $G = p^\beta G \oplus B$. If $p^\gamma B \neq 0$ for every ordinal $\gamma < \beta$ and $0 \neq a \in p^\beta G[p]$ then there is a β-isotype subgroup X of G such that $p^\beta X = \langle a \rangle$ by lemma 6. Now, $p^{\beta+1}X = 0 \neq \langle a \rangle = = X \cap p^{\beta+1}G$—a contradiction. Hence $p^\gamma B = 0$ for some ordinal $\gamma < \beta$. Let $p^\beta G = B_e \oplus B_{e+1}$ and suppose that $p^\beta G$ is not elementary.

If H is $p^\beta G$-high subgroup of G then $p^\gamma H \neq 0$ for every ordinal $\gamma < \beta$, since β is a limit ordinal. Let $a \in p^{\beta+1}G[p]$ be a nonzero element. By lemma 6, there is a β-isotype subgroup X of G such that $p^\beta X = \langle a \rangle$. Now, $p^{\beta+1}X \neq X \cap p^{\beta+1}G$—a contradiction. Hence $p^\beta G$ is elementary.

Assume (ii). Let H be a β-isotype subgroup of G. If $p^{\beta-1}G = B_e \oplus B_{e+1}$ then

$$p(p^{\beta-1}H) = p^\beta H = H \cap p^\beta G = p^{\beta-1}H \cap p(p^{\beta-1}G),$$

hence $p^{\beta-1}H$ is neat in $p^{\beta-1}G$ and therefore $p^{\beta-1}H$ is pure in $p^{\beta-1}G$ by [9]. Consequently,

$$p^n(p^{\beta-1}H) = p^{\beta-1}H \cap p^n(p^{\beta-1}G) = H \cap p^n(p^{\beta-1}G)$$

for every natural number n and moreover, if $n \geqslant e+1$ then $p^n(p^{\beta-1}H) = 0$. If $G = D \oplus B$, where D is divisible and $p^\gamma B = 0$ for some $\gamma < \beta$ then

$$p^\gamma H = H \cap p^\gamma G = H \cap p^\beta G = p^\beta H.$$

If $p^\beta G$ is elementary then

$$p^{\beta+1}H = H \cap p^{\beta+1}G = 0.$$

In all cases, H is isotype in G.

THEOREM 5. Let G be a group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{K}$. The following statements are equivalent:

(i) Every Γ-isotype subgroup of G is isotype in G.

(ii) For every $i \in \mathrm{N}$, either $G_{p_i} = D \oplus B$, where D is divisible and $p_i^\gamma B = 0$ for some ordinal $\gamma < \alpha_i$, or $p_i^{\alpha_i} G$ is torsion and $p_i^{\alpha_i} G_{p_i}$ is elementary or $p_i^{\alpha_i} G$ is torsion and $p_i^{\alpha_i - 1} G_{p_i} = B_e \oplus B_{e+1}$ for some $e \in \mathrm{N}$.

PROOF. Assume (i). Every α_i-isotype subgroup of G_{p_i} is isotype in G_{p_i} and hence G_{p_i} is as claimed in (ii) by lemma 7. Let $i \in \mathrm{N}$; write $\beta = \alpha_i$ and $p = p_i$. Suppose that $p^\beta G$ is not torsion. If $p^{\beta-1}G_p = B_e \oplus B_{e+1} \neq 0$ then $p^{\beta+e}G_p$ is not essential in $p^{\beta-1}G_p$ and $p^{\beta+e}G$ is not torsion. If $p^\beta G_p$ is nonzero elementary then $p^{\beta+1}G_p$ is not essential in $p^\beta G_p$ and $p^{\beta+1}G$ is not torsion. In these both cases, lemma 5 implies a contradiction.

Suppose that $p^\beta G$ is not torsion, $p^\beta G_p = 0$ and $p^\gamma G_p \neq 0$ for each ordinal $\gamma < \beta$. Let $a \in p^\beta G$, $o(a) = \infty$ and A be a $p^\beta G$-high subgroup of G containing G_p. Hence $p^\gamma A_p \neq 0$ for each ordinal $\gamma < \beta$. By lemma 6, there is a subgroup X of G such that $p^\gamma X = X \cap p^\gamma G$ for every ordinal $\gamma \leqslant \beta$ and $p^\beta X = \langle pa \rangle$. Let H be a subgroup of G maximal with respect to the properties: $X \subset H$, $a \notin H$. By lemma 6 [1], H is q-absorbing in G for every $q \neq p$. We prove that $p^\gamma H = H \cap p^\gamma G$ for each ordinal $\gamma \leqslant \beta$. It is sufficient to show that if this equality holds for $\gamma - 1 < \beta$ then it holds also for γ. Let $h \in H \cap p^\gamma G$; there is $g \in p^{\gamma-1} G$ such that $h = pg$. Obviously $h \in p^{\gamma-1} H$. If $g \in H$ then $h \in p^\gamma H$. If $g \notin H$ then $a \in \langle g, H \rangle$, i.e. $a = zg + h'$, where $h' \in H$ and z is an integer. Now, $(z, p) = 1$ and $h' \in H \cap p^{\gamma-1} G = p^{\gamma-1} H$. Further, $pa = zh + ph' \in p^\beta X \subset p^\gamma X$, there is $x' \in p^{\gamma-1} X$ such that $zh + ph' = px'$. Hence $zh = p(x' - h')$, where $x' - h' \in p^{\gamma-1} H$, and therefore $zh \in p^\gamma H$. Now, $ph \in p^\gamma H$, $zh \in p^\gamma H$ and $(p, z) = 1$ imply $h \in p^\gamma H$. Hence H is Γ-isotype in G. Finally, $pa \in H \cap p^{\beta+1} G \setminus p^{\beta+1} H$. For, if $pa = py$, where $y \in p^\beta H$, then $a - y \in G_p \cap p^\beta G = 0$, $a \in H$ —a contradiction. Consequently, H is not isotype in G.

Assume (ii). If H is a Γ-isotype subgroup of G then H_t is Γ-isotype in G_t and by lemma 1, each H_{p_i} is α_i-isotype in G_{p_i}. By lemma 7, each H_{p_i} is isotype in G_{p_i} and by lemma 1, H_t is isotype in G_t.

Let $i \in \mathrm{N}$, write $\beta = \alpha_i$ and $p = p_i$. If $p^\beta G$ is torsion then

$$p^\gamma H = p^\gamma H_t = H_t \cap p^\gamma G_t = H \cap p^\gamma G_t = H \cap p^\gamma G$$

for every $\gamma \geqslant \beta$. Suppose that $G_p = D \oplus B$, where D is divisible and $p^\gamma B = 0$ for some ordinal $\gamma < \beta$. Hence $p^\gamma G_p$ and $p^\gamma H_p$ are divisible. Write $p^\gamma H = p^\gamma H_p \oplus Y$. Since $p^\gamma G_p \cap Y = 0$, $p^\gamma G = p^\gamma G_p \oplus X$, where $Y \subset X$. We show that $p^\varepsilon Y = Y \cap p^\varepsilon X$ for each ordinal ε. It is sufficient to show that if this equality holds for ε then it holds also for $\varepsilon + 1$. Let $y \in Y \cap p^{\varepsilon+1} X$; there is $x \in p^\varepsilon X$ such that $y = px$. Now, $y \in p^{\gamma+1} G \cap H = p^{\gamma+1} H$, there is $h \in p^\gamma H$ such that $y = ph$. Write $h = h' + y'$, where $h' \in p^\gamma H_p$ and $y' \in Y$. Since $y = ph' + py'$, $ph' \in Y \cap p^\gamma H_p = 0$. Hence $y = py'$, $x - y' \in X_p = 0$, $x \in Y \cap p^\varepsilon X = p^\varepsilon Y$ and therefore $y \in p^{\varepsilon+1} Y$. Finally,

$$p^\varepsilon(p^\gamma H) = p^\gamma H_p \oplus p^\varepsilon Y = p^\gamma H_p \oplus (Y \cap p^\varepsilon X) =$$
$$= p^\gamma H \cap (p^\gamma G_p \oplus p^\varepsilon X) = p^\gamma H \cap p^\varepsilon(p^\gamma G) = H \cap p^\varepsilon(p^\gamma G)$$

for each ordinal ε.

THEOREM 6. Let G be a nonzero group and $\Gamma = (\alpha_1, \alpha_2, \ldots) \in \mathcal{K}$. The following are equivalent:

(i) Every Γ-isotype subgroup of G is an absorbing subgroup of G.

(ii) Either G is torsion-free and $\alpha_i > 0$ for each $i \in \mathrm{N}$ or G is cocyclic and if $\alpha_i = 0$ for some $i \in \mathrm{N}$ then either $G_{p_i} = 0$ or $G = Z(p_i)$.

PROOF. Every Γ-isotype subgroup of G is absorbing in G iff every Γ-isotype subgroup of G is isotype in G and every isotype subgroup of G is absorbing in G. Now, theorem 5 and theorem 6 [1] imply the desired result.

REFERENCES

[1] J. BEČVÁŘ, *Abelian groups in which every pure subgroup is an isotype subgroup*, Rend. Sem. Math. Univ. Padova, **62** (1980), pp. 129-136.

[2] S. N. ČERNIKOV, *Gruppy s sistemami dopolnjaemych podgrupp*, Mat. Sb., **35** (1954), pp. 93-128.

[3] L. FUCHS, *Infinite abelian groups I, II*, Academic Press, 1970, 1973.

[4] L. FUCHS - A. KERTÉSZ - T. SZELE, *Abelian groups in which every serving subgroup is a direct summand*, Publ. Math. Debrecen, **3** (1953), pp. 95-105. Errata ibidem.

[5] J. M. IRWIN - E. A. WALKER, *On isotype subgroups of abelian groups*, Bull. Soc. Math. France, **89** (1961), pp. 451-460.

[6] K. KATÔ, *On abelian groups every subgroup of which is a neat subgroup*, Comment. Math. Univ. St. Pauli, **15** (1967), pp. 117-118.

[7] A. KERTÉSZ, *On groups every subgroup of which is a direct summand*, Publ. Math. Debrecen, **2** (1951), pp. 74-75.

[8] R. C. LINTON, *Abelian groups in which every neat subgroup is a direct summand*, Publ. Math. Debrecen, **20** (1973), pp. 157-160.

[9] C. MEGIBBEN, *Kernels of purity in abelian groups*, Publ. Math. Debrecen, **11** (1964), pp. 160-164.

[10] K. M. RANGASWAMY, *Full subgroups of abelian groups*, Indian J. Math., **6** (1964), pp. 21-27.

[11] K. M. RANGASWAMY, *Groups with special properties*, Proc. Nat. Inst. Sci. India, *A* **31** (1965), pp. 513-526.

[12] V. S. ROCHLINA, *Ob ε-čistote v abelevych gruppach*, Sib. Mat. Ž., **11** (1970), pp. 161-167.

[13] K. SIMAUTI, *On abelian groups in which every neat subgroup is a pure subgroup*, Comment. Math. Univ. St. Pauli, **17** (1969), pp. 105-110.

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