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G. DA PRATO
M. IANNELLI

Linear integro-differential equations in Banach spaces

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Linear Integro-Differential Equations in Banach Spaces.

G. DA PRATO - M. IANNELLI (*)

1. Introduction.

Let X be a Banach space (by $|\cdot|$ we denote the norm in X). Let $A; D_A \subset X \to X$ a linear operator in X, $K; \overline{\mathbb{R}}_+ \to \mathbb{R}$ a real function. Throughout the paper, the following properties are supposed to be verified:

(1) A is a closed operator with domain D_A dense in X

(2) $K \in L^1_{\text{loc}}([0, +\infty); \mathbb{R})$ is absolutely Laplace transformable in the half plane $\operatorname{Re}\lambda > \alpha$

From (2) we put:

$$\hat{K}(\lambda) = \int_0^\infty e^{-\lambda t} K(t)\, dt .$$

Consider the following integro-differential problem:

$$(P) \qquad \begin{cases} u'(t) = \displaystyle\int_0^t K(t-s) A u(s)\, ds & t \geqslant 0 \\ u(0) = x & x \in X \end{cases}$$

(*) Indirizzo degli AA.: Dipartimento di Matematica, Università di Trento - Povo.

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Definition 1. *Let* $\{U(t)\}_{t\geq 0}$ *be a family of linear operators* $U(t)\in \mathcal{L}(X)$ (1). *It is said to be an* (M,ω)*-resolvent family for* (P) *if the following properties are verified:*

$$(3) \qquad x\in X \quad u(t)=U(t)x\in C([0,+\infty);X)$$

$$(4) \quad \forall x\in D_A, \quad u(t)=U(t)x\in C'([0,+\infty);X)\cap C([0,+\infty);D_A)$$

and problem (P) *is verified*

$$(5) \qquad \exists M>0, \quad \omega\in\mathbf{R}$$

such that

$$\|U(t)\|\leqslant Me^{\omega t} \qquad \forall t\geqslant 0$$

The existence of a resolvent family for (P) allows to solve, in a strong sense the inhomogeneous problem:

$$(Q) \qquad \begin{cases} u'(t)=\int_0^t K(t-s)Au(s)\,ds+f(t) & t\in[0,T] \\ u(0)=x \end{cases}$$

for any $x\in X$, $f\in C([0,T];X)$. In fact, if $U(t)$ is a resolvent family for (P) then:

$$(6) \qquad u(t)=U(t)x+\int_0^t U(t-s)f(s)\,ds$$

is a strong solution of (Q) in the sense of the following definition:

Definition 2. $u\in C([0,T];X)$ *is a strong solution of* Q *if there exist a sequence* $u_n\in C^1([0,T];X)\cap C([0,T];D_A)$ *such that:*

i) $u_n\to u$ *in* $C([0,T];X)$; $u(0)=x$

ii) $u_n'-\int_0^t K(t-s)Au_n(s)\,ds\to f(t)$ *in* $C([0,T];X)$

(1) $\mathcal{L}(X)$ is the space of linear bounded operators $T\colon X\to X$ endowed with the norm $\|\cdot\|$.

In fact we can choose $x_n \in D_A$, $f_n \in C([0, T]; D_A)$ such that $x_n \to x$, $f_n \to f$ in $C([0, T]; X)$. Then defining the sequence u_n in the following way:

$$u_n(t) = U(t)x_n + \int_0^t U(t-s) f_n(s)\, ds$$

it is easy to check that i) and ii) are verified with respect to u defined in (6).

In the present paper we mainly consider the following hypothesis:

$$(H) \begin{cases} \textit{There exist } M > 0,\ \omega \geqslant \alpha \textit{ such that if } \operatorname{Re}\lambda > w \textit{ then:} \\ \text{i) } \textit{the mapping } (\lambda - K(\lambda)A) : D_A \to X \textit{ is bijective} \\ \text{ii) } \left\| \dfrac{d^k}{d\lambda^k} (\lambda - \hat{K}(\lambda)A)^{-1} \right\| \leqslant \dfrac{Mk!}{(\operatorname{Re}\lambda - \omega)^{k+1}} \qquad k = 0, 1, . \end{cases}$$

or the following one (for simplicity we suppose $\alpha = 0$)

$$(K) \begin{cases} \textit{There exist } \theta \in (\pi/2, \pi) \textit{ and } M > 0 \textit{ such that:} \\ \text{i) } \hat{K}(\lambda) \textit{ has a bounded extension to } \Sigma_\theta \\ \text{ii) } \textit{the mapping } (\lambda - \hat{K}(\lambda)A) : D_A \to X \textit{ is bijective} \qquad \forall \lambda \in \Sigma_\theta \\ \text{iii) } \|(\lambda - \hat{K}(\lambda)A)^{-1}\| \leqslant \dfrac{M}{|\lambda|}, \qquad \lambda \in \Sigma_\theta \end{cases}$$

where $\Sigma_\theta \equiv \{\lambda \in \mathbf{C} \mid |\arg\lambda| < \theta\}$.

We remark that (H, ii) makes a sense because the mapping $\lambda \to (\lambda - \hat{K}(\lambda)A)^{-1}$ is analytic in the half plane $\operatorname{Re}\lambda > \omega$. Moreover we remark that (H, i) (resp. (K, ii)) implies $\hat{K}(\lambda) \neq 0$ if $\operatorname{Re}\lambda > \omega$ (resp. $\lambda \in \Sigma_\theta$).

Our main results are

THEOREM 3. *Let* (H) *be verified, then problem* (P) *has one and only one* (M, ω)*-resolvent family.*

THEOREM 4. *Let* (K) *be verified, then problem* (P) *has one and only one* $(M, 0)$*-resolvent family such that:*

(7) $U(t)$ *has an analytic extension to* $\Sigma_{\theta - \pi/2}$

$$(8) \qquad \left\| \frac{d^k}{dt^k} U(t) \right\| \leqslant \frac{M}{t^k}, \quad k = 0, 1, \ldots, t > 0$$

(9) *if* $t > 0$ *and* $x \in X$ *then* $\int_0^t K(t-s)\,U(s)\,ds \in D_A$ *and*;

$$\frac{d}{dt}\,U(t) = A\int_0^t K(t-s)\,U(s)x\,ds\,.$$

The proofs of theorem 3 and theorem 4 are given in section 2 and section 4 respectively; the other sections are devoted to the necessity of condition (H) and to some remarks on the Hilbert space case.

The same results hold for the following problem:

$$(P_1)\qquad \begin{cases} u'(t) = \int_0^t Au(t-s)\,d\eta(s) \\ u(0) = x \end{cases}$$

where η; $[0, +\infty) \to \mathbb{R}$ has bounded variation and is Laplace trasformable. Moreover we remark that the same method can be used to study the problem:

$$(P_2)\qquad \begin{cases} u'(t) = \int_0^t A(t-s)u(s)\,ds \\ u(0) = x\,, \end{cases}$$

Actually suitable conditions on $A(t)$ are to be imposed in order that the various steps in the proofs can be repeated with slight changes.

If η is the Heaviside function, then problem (P_1) is the well known abstract Cauchy problem and the semi-group theory is available. Actually our results look like a generalization of the methods of this theory, in fact our conditions (H) and (K) generalize the well known conditions for A to be the infinitesimal generator of a strongly continuous or respectively analytical semigroup.

Moreover if $K \equiv 1$, then problem (P) reduce to the abstract wave equation;

$$(10)\qquad u''(t) = Au(t); \quad u(0) = x; \quad u'(0) = 0$$

and $U(t)$ is the abstract cosine function generated by A (see [4]).

Also in this case, condition (H) generalizes the necessary and sufficient condition for A to be the generator of an abstract cosine function.

Integral and integro differential equations of Volterra type have been studied by several authors, by other methods (see [1]-[3], [6]-[11]). We remark that, as in the Hille Yosida theorem, our condition (H) is somewhat theoretical and difficult to use in the applications. On the contrary condition (K) is easy to handle, moreover condition (H) applies well in the Hilbert space case, when dealing with positive operators and positive kernels (see section 5); we have studied this latter case, by other methods in a forthcoming paper ([5]).

2. The proof of theorem 3.

Let us suppose (H) verified and in order to simplify the formulas assume $\omega = 0$; no essential change occurs in the general case.

Put for $\operatorname{Re}\lambda > 0$

$$F(\lambda) = (\lambda - \hat{K}(\lambda)A)^{-1}: X \to D_A \tag{11}$$

we soon have:

LEMMA 5. *For any natural number k and any $x \in X$. it is*

$$\lim_{n\to\infty} n^k F^k(n)x = x \tag{12}$$

consequently D_{A^k} is dense in X.

PGOOF. Let $x \in D_A$ then it is

$$nF(n)x = x + \hat{K}(n)F(n)Ax$$

so that, thanks to $(H$, ii) and to the fact that $\lim \hat{K}(n)/n = 0$, we have:

$$\lim_{n\to\infty} nF(n)x = x \qquad \forall x \in D_A \tag{13}$$

As it is

$$\|nF(n)\| \leqslant M \tag{14}$$

and D_A is dense, (10) is true for any $x \in X$. Finally (12) is true still owing to (14).

Our goal is to define $U(t)$ as the strong limit of an approximate sequence. To this aim let us define:

$$U_n(t) = e^{-nt}\left(I + \sum_{k=0}^{\infty}(-1)^k \frac{(n^2 t)^{k+1}}{k!(k+1)!}\frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}F(n)\right). \tag{15}$$

In fact from (H, ii) it follows that the series is convergent in $\mathcal{L}(X)$, uniformly for t in any bounded interval. It also follows:

$$\|U_n(t)\| \leqslant M \qquad \forall t \geqslant 0 \tag{16}$$

so that $U_n(t)$ is absolutely Laplace transformable in $\mathcal{L}(X)$. Moreover $U_n(t)$ is derivable in $\mathcal{L}(X)$, so that putting:

$$F_n(\lambda) = \int_0^{\infty} e^{-\lambda t}\, U_n(t)\, dt; \qquad \operatorname{Re}\lambda > 0 \tag{17}$$

it is

$$U_n(t) = \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda t} F_n(\lambda)\, d\lambda\,, \qquad \varepsilon > 0 \tag{18}$$

then we have

PROPOSITION 6. *$\forall x \in X$ it there exists the limit*:

$$\lim_{n\to\infty} U_n(t)x$$

PROOF. Owing to (16) it is sufficient to prove the thesis for any x in the dense set D_{A^2}. Now if $x \in D_{A^2}$ the following identity is true:

$$F(\lambda)x = \frac{1}{\lambda}x + \frac{1}{\lambda^2}\hat{K}(\lambda)Ax + \frac{1}{\lambda^2}\hat{K}^2(\lambda)F(\lambda)A^2x \tag{19}$$

so that the integral:

$$\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda t} F(\lambda) x\, d\lambda \qquad \varepsilon > 0 \tag{20}$$

makes a sense. In fact the first two terms in (19) are standard Laplace transforms while the last one is absolutely integrable.

We want to show that:

$$U_n(t)x \xrightarrow[n\to+\infty]{} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\lambda t} F(\lambda) x \, d\lambda \tag{21}$$

Now from (17) and (15) it easily follows:

$$F_n(\lambda) = \frac{1}{n+\lambda} I + \frac{n^2}{(\lambda+n)^2} F\left(\frac{\lambda n}{\lambda+n}\right) \tag{22}$$

so that for $\operatorname{Re}\lambda > 0$ and $x \in D_{A^2}$:

$$F_n(\lambda)x - \frac{1}{\lambda}x = \frac{1}{\lambda^2}\left[\hat{K}^2\left(\frac{n\lambda}{n+\lambda}\right) F\left(\frac{n\lambda}{n+\lambda}\right) A^2 x + \hat{K}\left(\frac{n\lambda}{n+\lambda}\right) Ax\right] \tag{23}$$

which yelds (see (19))

$$\begin{cases} \left(F_n(\lambda)x - \frac{1}{\lambda}x\right) \xrightarrow[n\to\infty]{} F(\lambda)x - \frac{1}{\lambda}x \\ \left|F_n((\lambda)x - \frac{1}{\lambda}x\right| \leqslant \dfrac{\text{const.}}{|\lambda|^2}\,. \end{cases} \tag{24}$$

Finally (24) allows going to the limit in (18) to get (21). The thesis is proved.

By means of Proposition 6 we are able to define the family $\{U(t)\}_{t\geqslant 0}$ $U(t) \in \mathfrak{L}(X)$ putting

$$U(t)x = \lim_{n\to\infty} U_n(t)x; \qquad x \in X \tag{25}$$

From (16) it follows:

$$\| U(t) \| \leqslant M \tag{26}$$

Thus $U(t)$ is absolutely transformable in $\mathfrak{L}(X)$, moreover from (17) by (16), (22), (25) it follows:

$$\hat{U}(\lambda) = \int_0^\infty e^{-\lambda t} U(t)\, dt = F(\lambda); \qquad \operatorname{Re}\lambda > 0 \tag{27}$$

REMARK 7. From the proof of proposition 4 it follows that if $x \in D_{A^2}$ is is

$$U(t)x = \int_{\varepsilon - i\infty}^{2+i\infty} e^{\lambda t} F(\lambda) x \, d\lambda \,, \qquad t \geqslant 0 \tag{28}$$

i.e. (27) can be inverted on D_{A^2}. Actually we can claim that (28) is true for any $x \in X$ such that the integral on the right of (28) is convergent. In fact if the integral converges, putting $x_n = n^2 F^2(n) x$ we have:

$$U(t) x_n = \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda t} F(\lambda) x_n \, d\lambda = n^2 G^2(n) \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda t} F(\lambda) x \, d\lambda$$

and (28) follows going to the limit.

PROPOSITION 8. *The family* $\{U(t)\}_{t \geqslant 0}$ *defined in* (25) *is an* $(M, 0)$-*resolvent family for problem* (P).

PROOF. (3) follows from (25) and (5) has been stated in (26). Let $x \in D_A$, from (15) it follows;

$$U_n(t) x \in D_A \quad \text{ and } A U_n(t) x = U_n(t) A x$$

in fact it can be easily shown by induction that:

$$\frac{d^k}{d\lambda^k} F(n) x \in D_A \quad \text{ and } \quad A \frac{d^k}{d\lambda^k} F(n) x = \frac{d^k}{d\lambda^k} F(n) A x \,.$$

Then we have:

$$U(t) x \in D_A; \qquad A U(t) x = U(t) A x$$

Finally:

$$\int_0^\infty e^{-\lambda t} \int_0^t K(t-s) U(s) A x \, ds \, dt = \hat{K}(\lambda) F(\lambda) A x = -x + \lambda F(\lambda) x$$

so that the integral $\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} e^{\lambda t}F(\lambda)x\,d\lambda$ converges, and by Remark 7:

$$\int_0^t\int_0^s K(s-r)\,U(r)Ax\,dr\,ds = \int_{\varepsilon+i\infty}^{\varepsilon+i\infty} e^{\lambda t}\left[F(\lambda)x-\frac{1}{\lambda}x\right]d\lambda = U(t)x-x$$

and the thesis is proved.

We remark that in the proof of Proposition 8 we have also proved that (28) is true for $x\in D_A$.

We conclude the proof of theorem 3 by uniqueness.

PROPOSITION 9. *Let* $V(t)$ *an* (M',ω')*-resolvent family for problem* (P) *then it is* $V(t)=U(t)$ $\forall t\geqslant 0$.

PROOF. $V(t)$ is Laplace transformable in $\mathfrak{L}(X)$. Put:

$$\hat{V}(\lambda)=\int_0^\infty e^{-\lambda t}V(t)\,dt\,,\qquad \operatorname{Re}\lambda>\omega'\,.$$

For $x\in D_A$ it is:

$$\frac{d}{dt}\hat{K}(1)F(1)V(t)x=\int_0^t K(t-s)[F(1)V(s)x-V(s)x]\,ds$$

so that for $\operatorname{Re}\lambda>\omega'$, $x\in D_A$:

$$\lambda\hat{K}(1)F(1)\,\hat{V}(\lambda)x=\hat{K}(1)F(1)x+\hat{K}(\lambda)F(1)\,\hat{V}(\lambda)x-\hat{K}(\lambda)\,\hat{V}(\lambda)x$$

that is

$$\hat{V}(\lambda)x=F(\lambda)x$$

which implies:

$$\hat{V}(\lambda)=F(\lambda)\,,\qquad \operatorname{Re}\lambda>\omega'$$

and the thesis follows.

3. Necessity of (H).

We want to clear in what sense condition (H) is necessary to have existence of an (M, ω)-resolvent family for problem (P).

PROPOSITION 10. *Condition (H) is necessary for problem (P) to have an (M, ω)-resolvent family such that:*

$$\forall x \in D_A \quad \text{it is } U(t)Ax = AU(t)x \tag{29}$$

PROOF. Put for $\operatorname{Re}\lambda > \omega$

$$F(\lambda) = \int_0^\infty e^{-\lambda t} U(t)\, dt \tag{30}$$

where the integral is absolutely convergent in $\mathcal{L}(X)$ thanks to (6). Then it is:

$$\left\| \frac{d^k}{d\lambda^k} F(\lambda) \right\| \leqslant \frac{Mk!}{(\operatorname{Re}\lambda - \omega)^{k+1}} . \tag{31}$$

If, $x \in D_A$ from (29) we have $F(\lambda)x \in D_A$, $AF(\lambda)x = F(\lambda)Ax$, moreover from (P) it follows:

(32I $\qquad \lambda F(\lambda)x = x + \hat{K}(\lambda)F(\lambda)Ax$

that is:

$$(\lambda - \hat{K}(\lambda)A)F(\lambda)x = F(\lambda)(\lambda - \hat{K}(\lambda)A)x = x \tag{32}$$

This proves the thesis.

REMARK 11. If uniqueness of the solution of (P) occurs and A is the infinitesimal generator of a strongly continuous semi-group, then a possible resolvent family verifies (29). In fact, let $x \in D_A$ (suppose to semplify $\omega < 0$) and put:

$$v(t) = A^{-1}U(t)x$$

It is:

$$v'(t) = \int_0^t K(t-s)Av(s)\, ds; \qquad v(0) = A^{-1}x$$

so that;

$$A^{-1}U(t)x = U(t)A^{-1}x$$

which implies (29) for $x \in D_A$, which is dense in X.

4. The Proof of theorem 4.

Let (K) be verified, then we may define:

$$U(t) = \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\theta)} e^{\lambda t}(\lambda - \hat{K}(\lambda)A)^{-1}d\lambda\,, \qquad t \in \Sigma_{\theta-\pi/2} \tag{34}$$

where $\gamma(\varepsilon, \theta) = \gamma^0_{(\varepsilon,\theta)} \cup \gamma^+_{(\varepsilon,\theta)_U} \cap \gamma^-_{(\varepsilon,\theta)}$ and

$$\begin{cases} \gamma^0_{(\varepsilon,\theta)} = \{z \,|\, z = \varepsilon e^{i\eta} & |\eta| \leqslant \theta\} \\ \gamma^{\pm}_{(\varepsilon,\theta)} = \{z \,|\, z = re^{\pm i\theta} & r \geqslant \varepsilon\} \end{cases} \tag{35}$$

In fact (34) converges thanks to (K, iii). Obviously $U(t)$ is analytic in $\Sigma_{\theta-\pi/2}$. First of all we prove that $U(t)$ verifies (8); to see this put $\lambda t = \xi$ then (we consider only the case $k = 0$):

$$U(t) = \frac{1}{2\pi i}\int_{\gamma(\varepsilon|t|,\theta+\arg t)} \frac{e^{\xi}}{t}(\xi/t - \hat{K}(\xi/t)A)^{-1}d\xi = \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\theta)} \frac{1}{t}e^{\xi}(\xi/t - \hat{K}(\xi/t)A)^{-1}d\xi$$

where the change of contour is possible by the analyticity of $(\lambda - \hat{K}(\lambda)A)^{-1}$ and (H, iii). Thus (8) follows by an easy estimate of the last integral. Let now $x \in D_A$ it is:

$$U(t)x - x = \frac{1}{2\pi i}\int_{\gamma(\varepsilon,\theta)} \frac{1}{\lambda}e^{\lambda t}\hat{K}(\lambda)(\lambda - \hat{K}(\lambda)A)^{-1}Ax\,d\lambda \tag{36}$$

On the other hand (8) implies that $t \to U(t);\ (0, +\infty) \to \mathfrak{L}(X)$ is absolutely transformable, and by (36):

$$\hat{U}(\lambda) = (\lambda - \hat{K}(\lambda)A)^{-1}\,, \qquad \operatorname{Re}\lambda > 0 \tag{37}$$

$$U(t)x - x = \int_0^t\int_0^s K(s-r)\,U(r)Ax\,dr\,ds\,, \qquad t \geqslant 0 \tag{38}$$

so that for $x \in D_A$ we have that (9) holds for $t \geqslant 0$ and:

$$\lim_{t \to 0^+} U(t)x = x \tag{39}$$

Finally (8) allows extending (39) to any $x \in X$, moreover (8) and the closedness of A yeld (9).

In conclusion, $U(t)$, continued in $t = 0$ by putting $U(0) = I$, is an $(M, 0)$-resolvent family for problem (P), verifying (7), (8), (9). Uniqueness follows as in Theorem 3.

EXEMPLE 12. Let $K(t) = at^{-\alpha}$, $\alpha \in (0, 1)$ and suppose $A: D_A \to X$ to be the infinitesimal generator of an analytic semi-group in $\Sigma_{\varphi - \pi/2}$ with $\varphi > \pi - \alpha\pi/2$. Then condition (K) is fulfilled with $\theta = \varphi/(2 - \alpha)$

5. Verifying condition (H).

When considering applications, condition (K) is more easy to handle than condition (H). This latter is, in fact, very difficult to check directly. However, when A is a self-adjoint operator in a Hilbert space then condition (H) can be easily related to the properties of the solution of the following scalar equation:

$$u_\xi'(t) = -\xi \int_0^t K(t-s) u_\xi(s)\, ds; \qquad u_\xi(0) = 1 \tag{40}$$

where $\xi \geqslant 0$.

Let us suppose that

$$|u_\xi(t)| \leqslant M e^{\omega t} \qquad \forall \xi \geqslant 0, \quad t \geqslant 0 \tag{41}$$

for some $M > 0$, $\omega \in R$; then for $\operatorname{Re} \lambda > \omega$ it is:

$$\begin{cases} F_\xi(\lambda) = 1/(\lambda + \xi \hat{K}(\lambda)) = \displaystyle\int_0^\infty e^{-\lambda t} u_\xi(t)\, dt \\ \left| \dfrac{d^k}{d\lambda^k} F_\xi(\lambda) \right| \leqslant \dfrac{Mk!}{(\operatorname{Re} \lambda - \omega)^{k+1}} . \end{cases} \tag{42}$$

Thus, via the spectral decomposition of A we get:

PROPOSITION 13. *Let X be an Hilbert space and $A: D_A \to X$ a positive, self-adjoint operator. If $K(t)$ is such that* (41) *is verified, then condition* (H) *is fulfilled.*

EXAMPLE 14. If $K(t)$ is such that that the operator:

$$u(t) \to \int_0^t K(t-s)u(s)\,ds: L^2([0, T]; X) \to L^2([0, T]; X)$$

in positive, then condition (41) is verified with $M = 1$, $\omega = 0$.

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