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Linear Abstract Integro-Differential Equations of Hyperbolic Type in Hilbert Spaces.

G. DA PRATO - M. IANNELLI (*)

Introduction.

This paper is concerned with the study of the problem:

$$\begin{cases} u'(t) = A(t)\,u(t) + (K * u)(t) + f(t) \\ u(0) = x \end{cases}$$

where for $t \geqslant 0$, $A(t)$ is the infinitesimal generator of a strongly continuous semi-group in a Hilbert space H and K is of the form

$$K(t) = \int_0^1 \exp[-t\xi]\,B(\xi)\,d\xi$$

where $B(\xi)$ is self-adjoint and semi-bounded; K is then a vectorial generalization of a completely positive kernel.

We study this problem with the same methods of sum of linear operators as in [8]. We are able, under suitable hypotheses, to show existence and uniqueness of a continuous strong solution u for every $f \in L^2(0, T; H)$ and $x \in H$; moreover u is a classical solution if $f \in L^2(0, T; K)$, where K is a Hilbert space densely and continuously embedded in H and $x \in K$.

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Lavoro svolto nell'ambito del G.N.A.F.A.

Similar problems have been studied by several authors by different methods, in the autonomous case. We remark that we do not assume that the domains of $A(t)$ and $B(\xi)$ are constant.

For the proofs we need an « energy equality » (see formula (21)) that we think will be useful to study asymptotic properties of u.

1. Notations.

We note by H a real Hilbert space (1) (inner product $(\ ,\ )$, norm $(\cdot)$ and by $L^2(0, T; H)$ the Hilbert space of the measurables mappings $u\colon [0, T] \to H$ such that $|u|^2$ is integrable in $[0, T]$, endowed with the inner product:

$$(u, v) = \int_0^T (u(t), v(t))\, dt \qquad u, v \in L^2(0, T; H)$$

We put:

$$W^1([0, T]; H) = \left\{u \in L^2(0, T; H);\ \frac{du}{dt} \in L^2(0, T; H)\right\}.$$

It is well known that every $u \in W^1([0, T]; H)$ can be identified with a continuous function (2); in the following we always make such an identification.

We note also by $C([0, T]; H)$ (resp. $C^1([0, T]; H)$) the set of mappings $[0, T] \to H$ continuous (resp. continuously derivable); it is $W^1([0, T]; H) \subset C([0, T]; H)$.

Finally we put:

$$W^1_0([0, T]; H) = \{u \in W^1([0, T]; H);\ u(0) = 0\}.$$

We write, for brevity, $L^2(H)$, $W^1(H)$, $W^1_0(H)$, $C(H)$, $C^1(H)$. We study the problem:

$$\text{(P)} \qquad \begin{cases} u'(t) = Au(t) + (Bk * u)(t) + f(t) & t \in [0, T],\ T > 0 \\ u(0) = x \end{cases}$$

(1) We suppose H real for simplicity.

(2) See for exemple [3].

We assume:

$$
(\mathrm{H})\quad \begin{cases} a)\ \exists \omega_A \in \mathbb{R} \text{ such that } \varrho(A) \supset]\omega_A, +\infty[\ (^3) \text{ and} \\ \qquad (Ax, x) \leqslant \omega_A |x|^2 \qquad \forall x \in D_A \ (^4)\,, \\ b)\ B \text{ is self-adjoint and } \exists \omega_B \in \mathbb{R} \text{ such that } B - \omega_B \leqslant 0, \\ c)\ \exists c\colon [0, 1] \to \mathbb{R}_+ \text{ measurable and bounded such that} \\ \qquad k(t) = \int_0^1 \exp[-t\xi]\, c(\xi)\, d\xi \ (^5)\,. \end{cases}
$$

We write (P) in the following form:

$$
(1) \qquad \gamma_0 \cdot u = \{f, x\}
$$

where $\{f, x\}$ is given in $L^2(H) \oplus H$ and γ_0 is defined by:

$$
\begin{cases} \gamma_0\colon\ W^1(H) \cap L^2(D_A) \cap L^2(D_B) \to L^2(H) \oplus H \ (^6)\,, \\ u \to \gamma_0\,, \qquad u = \{u' - Au - Bk * u,\ u(0)\}\,. \end{cases}
$$

We consider also the approximating problem:

$$
(\mathrm{P}_n) \qquad \begin{cases} u_n' = Au_n + B_n k * u_n + f \\ u_n(0) = x \end{cases}
$$

where $B_n = n^2 R(n, B) - n$. It is well known ([8]) that (P_n) has a unique strong solution $u_n \in C(H)$; moreover if $f \in W^1(H)$ and $x \in D_A$ it is:

$$
(2) \qquad u_n \in W^1(H) \cap L^2(D_A) \qquad \forall f \in W^1(H)
$$

because $k \in C^1([0, T])$.

(3) If $L\colon D_L \subset H \to H$ is a linear operator we note by $\varrho(L)$ (resp. $\sigma(L)$) the resolvent set (resp. the spectrum) of L and by $R(\lambda, L)$ the resolvent operator of L.

(4) It is known that D_A is dense in H.

(5) This hypothesis implies that the operator $Lu = k * u$ is completely positive in $L^2(0, T; H)$; for the existence of a solution of (P) it will be sufficient to assume L positive.

(6) D_A and D_B are endowed with the graph norm.

2. A priori bound.

PROPOSITION 1. *Assume* (H) *and let* u_n *be the solution of* (P_n), *then it is:*

$$(3)\qquad \begin{cases} |u_n(t)|^2 \leqslant \exp[(2(\omega+\varepsilon)]|x|^2 + \dfrac{1}{2\varepsilon}\displaystyle\int_0^t \exp[2(\omega+\varepsilon)(t-s)]|f(s)|^2\,ds \\ \omega = \omega_A + |k|_{L^1(0,T)}|\omega_B| \qquad\qquad \forall\varepsilon>0\,. \end{cases}$$

PROOF. Choose $f\in W^1(H)$, $x\in D_A$; put $\exp[-t\xi]*u_n = v_{n\xi}$, it is $u_n = v'_{n\xi}+\xi v_{n\xi}$; multiply (P_n) by $u_n(t)$ then it is:

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|^2 = \big(Au_n(t), u_n(t)\big) + \int_0^1 c(\xi)\big((B_n-\omega_B)v_{n\xi}(t),\, v'_{n\xi}(t)+\xi v_{n\xi}(t)\big)\,d\xi + $$
$$+\,\omega_B\big((k*u_n)(t), u_n(t)\big) + \big(f(t), u_n(t)\big) \leqslant \omega_A|u_n(t)|^2 + $$
$$+\,\omega_B\big((k*u_n)(t), u_n(t)\big) + \frac{1}{2}\frac{d}{dt}\int_0^1 \big((B_n-\omega_B)v_{n\xi}(t), v_{n\xi}(t)\big)c(\xi)\,d\xi + $$
$$+\int_0^1 \big((B_n-\omega_B)v_{n\xi}(t), v_{n\xi}(t)\big)\xi\, c(\xi)\,d\xi + \big(f(t), u_n(t)\big)$$

integrating from 0 to t we get:

$$(4)\qquad |u_n(t)|^2 \leqslant |x|^2 + 2\big(\omega_A + |k|_{L^1(0,T)}|\omega_B|\big)\int_0^t |u_n(s)|^2\,ds + $$
$$+\,2\int_0^t \big(f(s), u_n(s)\big)\,ds \leqslant |x|^2 + 2\big(\omega_A + |k|_{L^1(0,T)}|\omega_B| + \varepsilon\big)\cdot$$
$$\cdot\int_0^t |u_n(s)|^2\,ds + \frac{1}{2\varepsilon}\int_0^t |f(s)|^2\,ds$$

and the conclusion follows from the Gronwall lemma for $f \in W((H)$; in the general case we use the density of $W^1(H)$ in $L^2(H)$.

COROLLARY 2. *Under the hypotheses of the Proposition* 1, *if* u *is a classical solution of* (P) (7) *it is*:

$$(5) \qquad |u(t)|^2 \leqslant \exp[2(\omega+\varepsilon)t]|x|^2 + \frac{1}{2\varepsilon}\int_0^t \exp[2(\omega+\varepsilon)(t-s)]|f(s)|^2\,ds$$

$$(6) \qquad |u(t)| \leqslant K(|x| + |f|)$$

where K *is a suitable constant.*

PROOF. It is:

$$u' - Au - B_n k * u = (B - B_n)\,k * u + f$$

using (3) we obtain

$$|u(t)|^2 \leqslant \exp[2(\omega+\varepsilon)t]|x|^2 + \frac{1}{2\varepsilon}\int_0^t \exp[2(\omega+\varepsilon)(t-s)]\cdot$$
$$\cdot\{|(B - B_n)(k^* u)(s) + f(s)|^2\}\,ds$$

and the conclusion follows by dominated convergence.

3. Strong solutions.

PROPOSITION 3. *Assume* (H) *and suppose* $D_A \cap D_B$ *dense in* H; *then* γ_0 *is pre-closed.*

PROOF. Let $\{u_i\} \subset D_{\gamma_0}$ such that:

$$\begin{cases} u_i \to 0 & \text{in } L^2(H)\,, \\ \gamma_0 \cdot u_i = \{f_i, u_i\} \to \{f, x\} & \text{in } L^2(H) \oplus H\,, \end{cases}$$

(7) That is $u \in D(\gamma_0)$.

we have to show that $f = 0$, $x = 0$. From (5) it follows

$$|u_k(t) - u_h(t)|^2 \leqslant \exp[2(\omega + \varepsilon)t]|x_k - x_h|^2 + \\ + \frac{1}{2\varepsilon}\int_0^t \exp[2(\omega+\varepsilon)(t-s)]|f_k(s) - f_h(s)|^2 ds$$

therefore $\{u_i\}$ is a Cauchy sequence in $C(H)$ and furthermore $u_i \to 0$ in $C(H)$; then $x = \lim_{i\to\infty} u_i(0) = 0$. We, now, go to show that $f = 0$. Remark that, since $A - \omega_A$, $(B - \omega_B)k*$ are positive operators in $L^2(H)$ it is:

$$(7) \qquad (\lambda - \omega)|h|^2 \leqslant \tfrac{1}{2}|h(0)|^2 + (h, \lambda h + h' - Ah - Bk*h), \quad h \in D(\gamma_0).$$

Choose g in $W_0^1(H) \cap L^2(D_A) \cap L^2(D_B)$, which is dense in $L^2(H)$, from (7) it follows

$$(8) \qquad |g - \lambda u_i|^2 \leqslant \\ \leqslant \frac{1}{\lambda - \omega}\big(\lambda^2|x_i|^2 + (g - \lambda u_i, \lambda g + g' - Ag - Bk*g - \lambda^2 u_i - \lambda f_i)\big).$$

Then for $i \to \infty$

$$(9) \qquad |g|^2 \leqslant \frac{1}{\lambda - \omega}(g, \lambda g + g' - Ag - Bk*g - \lambda f)$$

and

$$(10) \qquad |g| \leqslant \frac{1}{\lambda - \omega}|\lambda(g - f) + g' - Ag - Bk*g|.$$

Finally for $\lambda \to \infty$

$$(11) \qquad |g| \leqslant |g - f|, \qquad \forall g \in W_0^1(H) \cap L^2(D_A) \cap L^2(D_B)$$

which implies $f = 0$.

In the following we denote by γ the closure of γ_0. We call $u \in L^2(H)$ a *strong solution* of (P) if it is:

$$(12) \qquad \gamma u = \{f, x\}$$

i.e. if there exists $\{u_i\} \in W^1(H) \cap L^2(D_A) \cap L^2(D_B)$ such that

$$(13) \qquad \begin{cases} u_i' - Au_i - Bk * u_i \to f & \text{in } L^2(H), \\ u_i(0) \to x & \text{in } H. \end{cases}$$

It is not easy in general to characterize the domain $D(\gamma)$ of γ; however we can show that $D(\gamma) \subset C(H)$ and that if $u \in D(\gamma)$, $\gamma \cdot u = \{f, x\}$ then $u(0) = x$.

The following proposition is straightforward:

PROPOSITION 4. *Under the hypotheses of the Proposition* 3 *if* $u \in D(\gamma)$ *and* $\gamma \cdot u = \{f, x\}$ *then* (5) *and* (6) *hold.*

Moreover γ *is one-to-one and has a closed range; consequently* (P) *has, at most, one strong solution.*

PROPOSITION 5. *Assume that the hypotheses of Proposition* 3 *are fulfilled. Let* $u \in D(\gamma)$, $\gamma \cdot u = \{f, x\}$; *then* $u \in C(H)$ *and it is*:

$$(14) \qquad u_i \to u \qquad \textit{in } C(H).$$

PROOF. Let $\{v_i\} \subset D(\gamma_0)$ such that

$$(15) \qquad \begin{cases} v_i \to u & \text{in } L^2(H) \\ f_i = v_i' - Av_i - Bk * v_i \to f & \text{in } L^2(H) \\ x_i = v_i(0) \to x & \text{in } H \end{cases}$$

from (6) it follows

$$|v_i(t) - v_h(t)| \leqslant K\big(|x_i - x_h| + |f_i - f_h|\big).$$

Thus $\{v_i\}$ is a Cauchy sequence in $C(H)$ which implies $u \in C(H)$, $v_i \to u$ in $C(H)$. Finally put $z = u_n - v_i$; then z is a strong solution of the problem:

$$\begin{cases} z' - Az - B_n k * z = f - f_i + (B_n - B) k * v_i, \\ z(0) = x - x_i, \end{cases}$$

from (5) it follows

$$|u_n(t) - v_i(t)| \leqslant K\big(|x - x_i| + |(B_n - B)k * v_i| + |f - f_i|\big)$$

and $u_n \to u$ in $C(H)$. #

4. Existence.

If $L\colon D_L \subset H \to H$ is a linear mapping and K a sub-space of H we denote by L_K the following mapping in K:

$$\text{(16)} \qquad \begin{cases} D(L_K) = \{x \in D_L \cap K;\ Lx \in K\}\,, \\ L_K x = L\cdot x \qquad \forall x \in D(L_K)\,. \end{cases}$$

It is easy to see that if $\lambda \in \varrho(L) \cap \varrho(L_K)$ then it is $R(\lambda, L)(K) \subset K$ and $R_K(\lambda, L) = R(\lambda, L_K)$ (8).

THEOREM 6. *Assume that the hypothesis* (H) *holds and that there exists a Hilbert space* K *(inner product* $((\ ,\ ))$*, norm* $\|\ \|$*) densely embedded in* H *such that*:

$$(\mathrm{H}_E) \qquad \begin{cases} a)\ K \hookrightarrow D_B,\ K \cap D_A \text{ is dense in } H \text{ (9)}, \\ b)\ \exists \eta_A \in \mathbb{R} \text{ such that } \varrho(A_K) \supset]\eta_A, +\infty[\text{ and } ((A_K y, y)) \leqslant \eta_A \|y\|^2, \\ c)\ B_K \text{ is self-adjoint and } \exists \eta_B \in \mathbb{R} \text{ such that } B_K - \eta_B \leqslant 0. \end{cases}$$

Then $\forall f \in L^2(H)$, $\forall x \in H$, *the problem* (P) *has a unique strong solution* u *such that* $u \in C(H)$, $u(0) = x$.

Moreover $\forall f \in W^1(K)$ *and* $x \in K \cap D_A$ *the solution* u *belongs to*

$$C^1(H) \cap L^\infty(D_B) \cap L^2(D_A)$$

i.e. it is a classical solution.

(8) $R_K(\lambda, L)$ is the restriction of $R(\lambda, L)$ to K.

(9) $K \hookrightarrow D_B$ means that K is continuously and densely embedded in D_B.

PROOF. By virtue of the closed graph theorem it is $B \in \mathcal{L}(K, H)$ put

$$|B|_{\mathcal{L}(K,H)} = \beta \,. \tag{17}$$

It is clear that $D_A \cap D_B$ is dense in H, therefore γ_0 is pre-closed (Proposition 3). Finally, due to the Corollary 4, to show existence it is sufficient to prove that γ has a range dense in $L^2(H) \oplus H$.

Take $f \in W^1(K)$, $x \in K$ and let u_n be the classical solution in H of the problem [10]:

$$\begin{cases} u_n' - Au_n - B_n k * u_n = f\,, \\ u_n(0) = x\,, \end{cases}$$

from Proposition 1 there exists $N > 0$ such that

$$\|u_n(t)\| \leqslant N(\|x\| + \|f\|) \tag{18}$$

it follows

$$|B_n u_n| \leqslant \frac{n}{n-\omega}\,|Bu_n| \leqslant \frac{n\beta}{n-\omega}\,\|u_n\|$$

due to (18) $\exists N' > 0$ such that

$$\begin{cases} |B_n u_n| \quad \leqslant N'(\|x\| + \|f\|)\,, \\ |B_n k * u_n| \leqslant N'(\|x\| + \|f\|)\,. \end{cases} \tag{19}$$

It is

$$\gamma_0 \cdot u_n - \{f, x\} = \{(B_n - B)k * u_n, 0\}$$

then if $\varphi \in L^2(D_B)$ it is

$$\big((B_n - B)k * u_n, \varphi\big) = \big(k * u_n, (B_n - B)\varphi\big) \to 0\,.$$

By virtue of (19) $\{(B_n - B)k * u_n\}$ is bounded in $L^2(H)$, it follows

$$\gamma_0 \cdot u_n \rightharpoonup \{f, x\} \text{ [11]} \qquad \forall \{f, x\} \in W^1(K) \oplus K$$

because $L^2(D_B)$ is dense in $L^2(H)$.

(10) See inclusion (2).

(11) $\rightharpoonup$ means weak convergence.

Consequently the range of γ_0 is weakly dense in $L^2(H)\oplus H$ and γ is onto.

We prove now the regularity result. Recall that $u_n \to u$ in $L^2(H)$ (Proposition 5); moreover due to (19) $\exists$ a sub-sequence $\{u_{n_k}\}$ such that $\{Bu_{n_k}\}$ is weakly Cauchy; consequently $u \in D_B$ and by virtue of (18) $u \in L^\infty(K)$.

Consider now the problem:

$$\begin{cases} v' - Av - Bk * v = k(t)Bx + f' , \\ v(0) = Ax + f(0) , \end{cases}$$

and the approximating one

$$\begin{cases} v_n' - Av_n - B_n k * v_n = K(t)B_n x + f' , \\ v_n(0) = Ax + f(0) . \end{cases}$$

It is $v_n = u_n'$ and $v_n \to v$ (Proposition 5); it follows $v = u'$ and $u \in C^1(H)$ because $v \in C(H)$.

Finally it is easy to see that $u \in D_A$ and $Au = u' - f - Bk * u \in \in L^2(H)$. #

5. Generalizations.

We generalize now the problem (P). We consider two families $\mathcal{A} = \{A(t)\}_{t\in[0,T]}$, $\mathcal{B} = \{B(\xi)\}_{\xi\in[0,1]}$ of linear operators in H. We put:

$$\begin{cases} K(t) = \int_0^1 \exp[-t\xi]B(\xi)\, d\xi , \\ K_n(t) = \int_0^1 \exp[-t\xi]B_n(\xi)\, d\xi , \qquad B_n(\xi) = n^2 R(n, B(\xi)) - n , \end{cases}$$

and the study the problem:

$$\text{(P')} \qquad \begin{cases} u'(t) = A(t)u(t) + (K*u)(t) + f(t) , \\ u(0) = x . \end{cases}$$

We write (P′) in the following form:

$$\gamma_0' = \{f, x\}$$

where γ_0' is defined by:

$$(20)\qquad \begin{cases} D(\gamma_0') = \{u \in L^2(H);\ u', Au, K * u \in L^2(H)\}\,, \\ \gamma_0'(u) = \{u' - Au - K * u,\ u(0)\}\,. \end{cases}$$

We consider also the approximating problem:

$$(\mathrm{P}_n')\qquad \begin{cases} u_n' = A(t)\,u_n(t) + (K_n * u_n)(t) + f(t)\,, \\ u_n(0) = x\,. \end{cases}$$

If $u \in D(\gamma_0')$, multiplying (P′) for $u(t)$ and putting $v_\xi = \exp[-t\xi] * u$ we get the *energy equality*:

$$(21)\qquad \frac{1}{2}\frac{d}{dt}\left(|u(t)|^2 - \int_0^1 \big(B(\xi)\,v_\xi(t),\ v_\xi(t)\big)\,d\xi\right) - \big(A(t)\,u(t),\ u(t)\big) + \\ - \int_0^1 \xi\big(B(\xi)\,v_\xi(t),\ v_\xi(t)\big)\,d\xi = 0\,.$$

We prove the theorem:

THEOREM 7. *Assume that*:

(H′)

a) $\exists$ *a Hilbert space* $K \hookrightarrow D\big(A(t)\big) \cap D\big(B(\xi)\big)$, $\forall t \in [0, T]$, $\xi \in [0, 1]$ *and* $\beta_A, \beta_B \geqslant 0$ *such that*:

$$|A(t)|_{\mathcal{L}(H,K)} \leqslant \beta_A\,, \qquad |B(\xi)|_{\mathcal{L}(H,K)} \leqslant \beta_B\,.$$

b) $\exists \omega_A, \eta_A \in \mathbb{R}$ *such that* $\varrho\big(A(t)\big) \supset]\omega_A, +\infty[$, $\varrho\big(A_K(t)\big) \supset]\eta_A, +\infty[$, $\big(A(t)\,x, x\big) \leqslant \omega_A |x|^2$, $\big(\!\big(A_k(t)\,x, x\big)\!\big) \leqslant \eta_A \|x\|^2$ *and* $R\big(\lambda, A(\cdot)\big)$ *(resp.* $R(\lambda, A_k(\cdot))\big)$ *is strongly measurable in* H *(resp.* K*).*

c) $B(\xi)$ *is self-adjoint and* $\exists \omega_B, \eta_B \in \mathbb{R}$ *such that* $B - \omega_B \leqslant 0$, $B_K - \eta_B \leqslant 0$ *and* $R\big(\lambda, B(\cdot)\big)$ *(resp.* $R(\lambda, B_K(\cdot))\big)$ *is strongly measurable in* H *(resp.* K*).*

Then $\forall u \in L^2(H)$, $x \in H$, $\exists a$ *unique strong solution* $u \in C(H)$. *Moreover if* $u \in L^2(K)$, $x \in K$ *then it is* $u \in W^1(H) \cap L^\infty(K)$.

We can write the problem (P'_n) in the following form:

$$u_n(t) = Z(t, 0)x + \int_0^t Z(t, s)[(K_n * u_n)(s) + f(s)]\,ds \tag{22}$$

where $Z(t, s)$ are the evolutions operators attached to the family $\mathcal{A}$. If $x \in K$, $f \in L^2(K)$ we can solve (22) by the contraction principle ([12]); therefore $K_n * u_n + f \in L^2(K)$ which implies $u_n \in W^1(H) \cap L^\infty(K)$ ([8]). Using the energy equality in the K space we get:

$$\|u_n(t)\|^2 \leqslant \exp[2(\eta + \varepsilon)]\,\|x\|^2 + \int_0^t \exp[2(\eta + \varepsilon)(t - s)]\,\|f(s)\|^2\,ds\,, \tag{23}$$

$\forall f \in L^2(K)$, $x \in K$, $\varepsilon > 0$; where $\eta = \eta_A + \eta_B$.

Proceeding as in previous sections we are able to show:

a) γ'_0 is preclosed and its closure γ' is one-to-one and with a closed range.

b) $\exists N' > 0$ such that

$$|B_n u_n| \leqslant N'(\|x\| + \|f\|) \qquad x \in K\,,\ f \in L^2(K)\,. \tag{24}$$

c) It is

$$\gamma'_0 \cdot u_n \rightharpoonup \{f, x\}\,.$$

It follows that γ' is onto and $\exists$ a unique strong solution u of (P') for every $x \in H$, $f \in L^2(H)$; moreover $u_n \to u$ in $L^2(H)$.

Concerning regularity we remark that if $x \in K$, $f \in L^2(K)$, then by virtue of (24) Bu belongs to $L^2(H)$ and recalling (23) $u \in L^\infty(K)$, consequently $Au \in L^2(H)$ and $u' \in L^2(H)$.

REMARK 8. For sake of simplicity we have assumed $K \subset D(A(t)) \cap \cap D(B(\xi))$ instead of $K \subset D(B(\xi))$ as in previous section; actually similar results can be proved with this latter assumption.

([12]) In fact Z is strongly measurable and bounded in K [8].

6. Resolvent family.

Assume that the hypotheses either (H′) or (H) and (H_E) hold. Consider the problem:

$$(P'_s) \qquad \begin{cases} u'(t) = Au(t) + K * u(t) & t \in [s, T] \\ u(s) = x \end{cases}$$

where $K = Bk$ if (H) and (H_E) hold.

Due to Theorem 7 (P'_s) has a unique strong solution. Moreover if (H′) (resp. (H) and (H_E)) hold and $x \in K$ (resp. $K \cap D_A$) then u is classical. Put:

$$u(t) = G(t, s)x \tag{25}$$

it is $G(t, s) \in \mathcal{L}(H)$. The mapping:

$$G\colon \Delta_T = \{(t, s) \in [0, T]^2;\ t \geqslant s\} \to \mathcal{L}(H)\,, \qquad (t, s) \to G(t, s) \tag{26}$$

is called the *evolution operator* for the problem (P'_s) and the family $\{G(t, s)\}_{(t,s)\in\Delta_T}$ the *resolvent family.*

Consider also the approximating problem:

$$(P'_{s,n}) \qquad u'_n = Au + K_n * u_n\,, \qquad u_n(s) = x\,,\ t \in [s, T]$$

and put $u_n(t) = G_n(t, s)x$.

Proposition 9. *For every* $x \in H$ *it is*:

$$\lim_{n\to\infty} G_n(t, s)x = G(t, s)x \quad \textit{uniformly in } \Delta_T\,. \tag{27}$$

Therefore $G(\cdot, \cdot)x \in C(\Delta_T; H)$.

Proof. It is sufficient to prove (27) for $x \in K$ (resp. $D_A \cap K$). Put $v = u - u_n$, then it is:

$$\begin{cases} v' - Av - K * v = (K - K_n) * u\,, \\ v(s) = 0\,, \end{cases}$$

from which

$$|G(t,s)x - G_n(t,s)x|^2 =$$

$$= |v(t)|^2 \leqslant \frac{1}{2\varepsilon}\int_s^t \exp\,[2(\omega+\varepsilon)(t-z)]|(K-K_n)*u|^2\,dz$$

and the thesis follows from dominated convergence. $\#$

PROPOSITION 10. *Assume that the hypotheses* (H′) (*resp.* (H) *and* (H_E)) *hold. If* $x\in H$, $f\in L^2(H)$ *and* u *is the strong solution of* (P′) (*resp.* (P)) *it is*:

$$u(t) = G(t,0)x + \int_0^t G(t,s)f(s)\,ds\,. \tag{28}$$

PROOF. Go to the limit in the equality:

$$u_n(t) = G_n(t,0)x + \int_0^t G_n(t,s)f(s)\,ds\,.$$

REMARK 11. By similar arguments we can study the problem:

$$(\mathrm{P}'')\qquad \begin{cases} u'(t) = A(t)u(t) + \displaystyle\int_0^t K(t,s)u(s)\,ds + f(t) \\ u(0) = x \end{cases}$$

where

$$K(t,s) = \int_0^1 \exp\left[-\int_s^t p(z,\xi)\,dz\right]B(\xi)\,d\xi \tag{29}$$

and $p\colon [0,T]\times[0,1]\to\mathbb{R}_+$ is continuous.

The energy equality is in this case:

$$\frac{1}{2}\frac{d}{dt}\left(|u(t)|^2 - \int_0^1 \big(B(\xi)v_\xi(t), v_\xi(t)\big)\,d\xi\right) - \big(A(t)u(t), u(t)\big) +$$
$$- \int_0^1 p(t,\xi)\big(B(\xi)v_\xi(t), v_\xi(t)\big)\,d\xi + \big(f(t), u(t)\big) \tag{30}$$

where

$$v_\xi(t) = \int_0^1 \exp\left[-\int_s^t p(z, \xi)\, d\xi\right] u(s)\, ds \tag{31}$$

is the solution of the problem

$$v'_\xi + p(t, \xi) v_\xi = u\,, \qquad v(0) = 0\,. \tag{32}$$

EXAMPLE 12. Let $H = L^2(R)$, $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ and

$$\begin{cases} D(A(t)) = \{u \in L^2(\mathbb{R});\ \varphi(t, \cdot)\, u \in L^2(\mathbb{R})\}\,, \\ A(t)\, u = \varphi(t, x)\, u_x\,, \end{cases} \tag{33}$$

$$\begin{cases} B(\xi)\, u = \dfrac{\partial}{\partial x}\big(a(x, \xi)\big)\dfrac{\partial u}{\partial x} & a \text{ continue, } a(x, \xi) \geqslant a > 0\,, \\ D(B(\xi)) = W^2(\mathbb{R})\,. \end{cases} \tag{34}$$

Then (P′) is equivalent to

$$\begin{cases} u = \varphi(t, x)\, u_x + \displaystyle\int_0^1 \frac{\partial}{\partial x}\big(a(x, \xi) \exp[-t\xi] * u_x\big)\, d\xi + f(t, x)\,, \\ u(0, x) = u_0(x)\,. \end{cases} \tag{35}$$

If $u_0 \in L^2(\mathbb{R})$ and $f \in L^2([0, T] \times \mathbb{R})$ then (35) has a unique strong solution $u \in C([0, T]; L^2(\mathbb{R}))$; if moreover $f_{xx} \in L^2([0, T] \times \mathbb{R})$ and $u_0 \in W^2(\mathbb{R})$ then u is classical and it is

$$u \in L^\infty([0, T];\ W^2(\mathbb{R})) \cap W^1([0, T];\ L^2(\mathbb{R}))\,.$$

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