RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

GAETANO ZAMPIERI

μ-minkowskianity of the Schwarzschild universe

Rendiconti del Seminario Matematico della Università di Padova, tome 62 (1980), p. 17-21

<http://www.numdam.org/item?id=RSMUP_1980__62__17_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 62 (1980)

μ-Minkowskianity of the Schwarzschild Universe.

GAETANO ZAMPIERI (*)

1. Brief introduction (1).

In [1] the author defines the μ-Minkowskian space-time in order to describe precisely insular gravitating systems, by means of certain properties having a physical meaning. Moreover there it was possible to associate in a natural way each μ-Minkowskian space-time with an asymptotic Minkowski space and (the right number of) asymptotic inertial spaces.

In this note the validity of our hypothesis in the case of every space-time admitting the Schwarzschild exterior metric form is proved.

2. Existence of the Minkowski asymptotic space for the Schwarzschild universe.

Let S_4 be a space-time admitting the Schwarzschild exterior metric form (2)

$$(1) \qquad \Phi = (1-2m/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - (1-2m/r)\,dt^2\,, \qquad r \geqslant a > 2m\,,$$

(*) Indirizzo dell'A.: Seminario Matematico, Università di Padova Via Belzoni 7 - 35100 Padova.

Lavoro eseguito nell'ambito dei Gruppi di Ricerca di Fisica Matematica del C.N.R.

(1) Originally this note constituted an appendix of [1], and it has been printed separately by typografical reasons.

(2) See e.g. [2] formula (184) in chapter VII. We refer to this chapter also for possible explanations on the coordinates in (1).

where a is a positive constant and m has the well known meaning. Let us denote by S_4^e the region of S_4 where Φ_e is defined, and let (x') be a frame such that

$$(2) \qquad x'^0 = t\,, \quad x'^1 = r\,, \quad x'^2 = \theta\,, \quad x'^3 = \varphi \quad \text{in } \mathsf{S}_4^e\,.$$

By (1) the metric tensor satisfies

$$(3) \qquad g'_{i0} = 0\,, \quad g'_{\alpha\beta,0} = 0\,, \quad g'_{\alpha\beta,3} = 0\,,$$
$$\alpha, \beta \in \{0, 1, 2, 3\}\,, \quad i \in \{1, 2, 3\}\,, \quad \text{in } \mathsf{S}_4^e\,.$$

By $(3)_{1,2}$ S_4^e is static. Moreover, from $(3)_{2,3}$ we have at once that the vectors whose components in (x') are (δ_0^α) and (δ_3^α) respectively, are Killing vectors in S_4^e, i.e. they satisfy the condition $v_{\alpha/\beta} + v_{\beta/\alpha} = 0$. By these equations, along any geodesic in S_4^e, the scalar product of the tangent vector with each Killing vector is constant.

We are interested in space-like geodesics of S_4^e. As is well known there is no loss of generality if we consider only those belonging to the hypersurface $\theta = \pi/2$. By the considerations above each of them satisfies the following equations involving the unit tangent vector $\boldsymbol{dx}/ds$ and the constants E and Λ joined to the aforementioned Killing vectors.

$$(4) \qquad (1 - 2m/r)(dt/ds) = E\,,$$

$$(5) \qquad r^2(d\varphi/ds) = \Lambda\,,$$

$$(6) \qquad (dr/ds)^2 = E^2 + (1 - 2m/r)(1 - \Lambda^2/r^2)\,.$$

Let us prove the μ-Minkowskianity—see [1], Def 5.1—of the frame (x) in which the spatial coordinates x^i are obtained from those x'^i in (x') by the usual transformations relating Cartesian to polar coordinates, and the temporal coordinate x^0 coincides with x'^0.

Let b be a positive constant with $b \geqslant a$—see (1)—, and let T_b be the set of all events x such that $r \leqslant b$. Observe that T_b is a world-bundle —see (d) in [1], N. 4. The clause (ii) in [1], Def 5.1, is immediately proved for the frame (x) by using world-bundles of the above kind. In fact we see at once that, for some constant k_1, $|g_{\alpha\beta}(x) - m_{\alpha\beta}| \leqslant k_1 r^{-1}$ at any $x \in \mathsf{S}_4^e$— where $(m_{\alpha\beta})$ is the diagonal matrix $(-1, 1, 1, 1)$.

We shall prove that the clause (i) in [1], Def 5.1, holds for (x) with respect to $\mathfrak{W}_0 = T_b$, where b satisfies

$$b > 3m \quad \text{and} \quad b \geqslant a \,. \tag{7}$$

We see at once that, for some positive constant k_2,

$$\left| \begin{Bmatrix} \alpha \\ \beta \ \sigma \end{Bmatrix} (x) \right| \leqslant k_2 r^{-2}\,, \quad x \in (\mathbf{S}_4^e)^c (= \mathbf{S}_4 - \mathbf{S}_4^e)\,, \quad \alpha, \beta, \sigma \in \{0, 1, 2, 3\}. \tag{8}$$

Thus, to show that the subclause (i″) holds for (x), it suffices to prove that for some positive function g, continuous in $\mathfrak{W}_0^c$, along every space-like geodesic in $\mathfrak{W}_0^c$,

$$g(\overline{\mathscr{E}})s \leqslant r(s)\,, \tag{9}$$

where s is the arc parameter and $\overline{\mathscr{E}}$ is the origin of the geodesic.

Let us fix one of the aforementioned geodesics l. Along it, (4) to (6) hold for some constant E and Λ. By (7), $(d^2r/ds^2) = mr^{-2} + (1 - 3m/r)\Lambda^2 r^{-3} > 0$, which shows that dr/ds is a strictly increasing function on l. If $(dr/ds)_{s=0} \geqslant 0$

$$s \leqslant [b/b - 3m] r(s)\,. \tag{10}$$

This immediately follows from (6) in the case where $\Lambda = 0$. In the contrary case let us set

$$\xi = r(E^2 + 1 - 2m/r)^{\frac{1}{2}}[\Lambda^2(1 - 2m/r)]^{-\frac{1}{2}}\,. \tag{11}$$

By (7) we see at once that $\xi(r)$ is invertible and

$$0 < \frac{dr}{d\xi} \leqslant \frac{r}{\xi} \frac{r - 2m}{r - 3m}\,. \tag{12}$$

By (6), (11), and (12)

$$s \leqslant \frac{b - 2m}{b - 3m} \frac{|\Lambda|}{E^2 + 1 - 2m/b} \int\limits_{\xi(0)}^{\xi(s)} \xi(\xi^2 - 1)^{-\frac{1}{2}} d\xi \leqslant \tag{13}$$
$$\leqslant \frac{b - 2m}{b - 3m} \frac{r(s)}{(E^2 + 1 - 2m/b)^{\frac{1}{2}}(1 - 2m/b)^{\frac{1}{2}}}\,.$$

This prove (10). Now let us consider the case where $(dr/ds)_{s=0} < 0$. In it, $r(s)$ strictly decreases in some interval $[0, \hat{s}]$, and, if the length of l is larger than $\hat{s}$, it is strictly increasing for $s > \hat{s}$. By reasoning as we did in the preceding case, we arrive at the inequality $\hat{s} \leqslant \leqslant [b/b - 3m]r(0)$, which yields immediately $s \leqslant [2r(0)/b - 3m]r(s)$. By setting $g(x) = [b - 3m/2r]$, the preceding inequality and (10) prove (9).

Now let us prove that our frame satisfies the subclause (i′) in [1], Def 5.1, with respect to the aforementioned choice of $\mathcal{W}_0$.

By [1]—Def 3.3—we must show that the following condition, on the typical space-like geodesic $l \subset \mathcal{W}_0^c$ having endpoints, holds. Let $\mathscr{E}_1$ be the origin of l, let $\mathscr{E}_2$ be the other endpoint, let $\boldsymbol{\gamma}$ be the 4-velocity of the ideal fluid joined to the frame (x)—and hence to (x')—, and let $\boldsymbol{u}(s)$ be the field obtained from $\boldsymbol{\gamma}(\mathscr{E}_1)$ by parallel transport along l; then for some constant Γ independent of l,

$$\gamma(l) \equiv -(\boldsymbol{u}\cdot\boldsymbol{\gamma})(\mathscr{E}_2) \leqslant \Gamma . \tag{14}$$

It is easily shown that in our case we have

$$d(-\boldsymbol{u}\cdot\boldsymbol{\gamma}) = mr^{-2}(1-2m/r)^{-1}u'^1\gamma'_0\,dt .$$

Since $|u'^1/u'^0| \leqslant 1$, and $r > b$, we have

$$\ln\gamma(l) \leqslant \left|\int_l (1-2m/b)^{-1} mr^{-2}\,dt\right| . \tag{15}$$

By the preceding discussion of clause (i″), there is only one point, $\overline{\mathscr{E}} \in l$, such that $r(\overline{\mathscr{E}}) = \inf_l r$. Let $l_1[l_2]$ be the arc of l with origin $\overline{\mathscr{E}}$ and endpoint $\mathscr{E}_1[\mathscr{E}_2]$ (3). Then, by (4) and (15), for some constant E (dependent on the choice of l)

$$\ln\gamma(l) \leqslant (1-2m/b)^{-2} m|E| \sum_{i=1}^{2} \int_{l_i} r^{-2}\,ds . \tag{16}$$

If $\Lambda \neq 0$, (13) holds for each l_i because $(dr/ds)_{s=0} \geqslant 0$ for them. But the first term in (13) is less than or equal to the third also in the case

(3) Obviously one of these arcs can consist of a single point.

where $\Lambda = 0$. Therefore (16) yields

$$\ln \gamma(l) \leqslant 2(1 - 2m/b)^{-2} m |E| \cdot \left[\int_0^{b/|E|} b^{-2}\, ds + \int_{b/|E|}^{\infty} \left(\frac{b - 2m}{b - 3m} \right)^2 \frac{s^{-2}}{(1 - 2m/b)(E^2 + 1 - 2m/b)}\, ds \right],$$

for $E \neq 0$. Thus

$$\ln \gamma(l) \leqslant 2(1 - 2m/b)^{-2} \frac{m}{b} \left[1 + \frac{b(b - 2m)}{(b - 3m)^2} \right],$$

which prove the assertion including (14).

We have thus proved that (x) satisfies all the clauses in [1], Def 5.1. Then, observing that Condition 5.1 in [1] is obviously fulfilled by it, we have that our frame is μ-Minkowskian. This implies the μ-Minkowskianity of every space-time admitting the Schwarzschild exterior metric form, and hence the existence of the Minkowski asymptotic space for such space-times—see [1], N. 8.

REFERENCES

[1] G. Zampieri, *Minkowski asymptotic spaces in general relativity*, Rend. Sem. Mat. Univ. Padova, **61** (1979).

[2] J. L. Synge, *Relativity: The General Theory*, North-Holland, 1960.

Manoscritto pervenuto in redazione il 26 gennaio 1979.