RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

GIUSEPPE ZAMPIERI

On some conjectures by E. De Giorgi relative to the global resolvability of overdetermined systems of differential equations

Rendiconti del Seminario Matematico della Università di Padova, tome 62 (1980), p. 115-128
<http://www.numdam.org/item?id=RSMUP_1980__62__115_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 62 (1980)

On Some Conjectures by E. De Giorgi Relative to the Global Resolvability of Overdetermined Systems of Differential Equations.

GIUSEPPE ZAMPIERI (*)

SUMMARY - In the following we prove that semiglobal resolvability in an open set of R^n for overdetermined systems ($Pu = f$, $Qu = 0$) with constant coefficients and Q elliptic comports global resolvability. This proves De Giorgi's first conjecture of [2] in the case $A = B$, thus generalizing L. Modica's result relative to open bounded sets in which the solutions of the equation $Qu = 0$ are approximable by solutions of the same equation in R^n. Then, for Q elliptic, we prove conjecture 2, arising from 1, and conjecture 5 for which we give the resolving map of the system in question by means of closed graph theorems.

1. In this paragraph we compile the necessary classical results needed to affront the De Giorgi's conjectures; these theorems will be indicated with capital letters while our original theorems by numbers. The essential information are theorems on resolvability in convex regions for systems of differential equations with constant coefficients, the theory of real analytic functionals, and theorems on extensions and approximations of solutions to homogeneous elliptic systems and equations.

(*) Indirizzo dell'A.: Seminario Matematico - Via Belzoni 7 - I-35100 Padova.

Let P_{ij} $(i = 1, \ldots, I,\ j = 1, \ldots, J)$ be polynomials in n variables and consider the system of differential equations

$$(1) \qquad \sum_{j=1}^{J} P_{ij}(D)\, u_j = f_i\,, \qquad i = 1, \ldots, I\,,$$

where $D = -i(\partial/\partial x)$. Consider the module over the polynomial ring of the polynomial relations among the matrix's rows and let $(Q_{11}, \ldots, Q_{1I})$, $\ldots$, $(Q_{k1}, \ldots, Q_{kI})$ be a set of generators (it will be useful in the following to know that this one also generates the relations with coefficients in the local ring of germs of holomorphic functions). If we write (1) in the form $Pu = f$ and interpret Q similarly, then obviously a necessary condition for the existence of solutions to (1) is that $Qf = 0$. Conversely we have

THEOREM A (Lojasiewicz-Malgrange). If Ω is a convex open set then the system $Pu = f$ has a solution $u \in \mathcal{E}^J(\Omega)$ $\big(= C^{\infty^J}(\Omega)\big)$ for every $f \in \mathcal{E}^I(\Omega)$ s.t. $Qf = 0$.

In the following we'll deal only with overdetermined systems in the form $(Pu = f,\, Qu = g)$ with P and Q relatively prime differential polinomials and we give a sketch of proof for these particular systems. First of all the condition of compatibility over the data is that $Qf - Pg = 0$. Thus the function: $P \times Q\colon \mathcal{E}(\Omega) \to \{(f, g) \in \mathcal{E}(\Omega) \times \mathcal{E}(\Omega)\colon Qf = Pg\}$ has dense range because let $(u, v) \in \mathcal{E}'(\Omega) \times \mathcal{E}'(\Omega)$ and ${}^tPu + {}^tQv = 0$ $\big({}^tP(D) = P(-D)\big)$ then ${}^tP\hat{u} + {}^tQ\hat{v} = 0$ ($\hat{u}$ denotes the Fourier-Laplace transform of u); so, as we noted above, ${}^tQ$ and $-{}^tP$ devide $\hat{u}$ and $\hat{v}$ respectively from which there exists $w \in \mathcal{E}'(R^n)$ s.t. $u = {}^tQw$ and $v = -{}^tPw$ (see [11]); at last this w is forced to belong to $\mathcal{E}'(\Omega)$ because of the Ω's convexity. To conclude it is enough to prove that ${}^tP\mathcal{E}'(\Omega) + {}^tQ\mathcal{E}'(\Omega)$ is closed. Now let $u \in \big({}^tP\mathcal{E}'(\Omega) + {}^tQ\mathcal{E}'(\Omega)\big)^-$ (closure) thus in particular u is orthogonal to $\mathcal{E}_Q(R^n) \cap \mathcal{E}_P(R^n)$ $\big(\mathcal{E}_Q(R^n) = \{f \in \mathcal{E}(R^n)\colon Qf = 0\}\big)$; then it is easy to see that $\forall \zeta \in \mathbf{C}^n$ we can resolve $\hat{u}_\zeta = {}^tPG_{1_\zeta} + {}^tQG_{2_\zeta}$ in the ring of formal power series (where $\hat{u}_\zeta$ denotes the germ of u at ζ); this equation can also be resolved in the ring of germs at ζ (A_ζ), because every submodule of A_ζ is closed in A_ζ (see Theorem 6.3.5 of [4]). By means of the Cartan theorem we can « globalize » such a solution and so the equation $\hat{u} = {}^tPG_1 + {}^tQG_2$ is solvable in the ring of holomorphic functions. By the Paley-Wiener theorem, which relates the supports of the distributions with the growth of their Fourier-Laplace transforms we conclude that there exists F

holomorphic s.t. $G_1 + {}^tQF$ and $G_2 - {}^tPF$ are the Fourier-Laplace transforms of distributions of $\mathcal{E}'(\Omega)$. So also we proved the following

THEOREM B. Let Ω be a convex open set. If $(u, v) \in \mathcal{E}'(R^n)\times\mathcal{E}'(R^n)$ and ${}^tPu + {}^tQv \in \mathcal{E}'(\Omega)$ then there exists $w \in \mathcal{E}'(R^n)$ s.t. $u + {}^tQw$ and $v - {}^tPw$ both belong to $\mathcal{E}'(\Omega)$.

Let A be a generic open and suppose Q elliptic. The space $\mathcal{E}_Q(A)$ endowed with the topology induced by $\mathcal{E}(A)$, is a Fréchet reflexive space; its dual $\mathcal{E}'_Q(A)$ is canonically isomorphic to $\mathcal{E}'(A)/{}^tQ\mathcal{E}'(A)$ by Hahn-Banach theorem.

DEFINITION. We say that $L \in \mathcal{E}'_Q(A)$ is carried by a compact subset K of A if there is $c > 0$ and $m \in N$ s.t.

$$|\langle L, f\rangle| \leqslant c \sup_{x\in K} \sum_{|\alpha|\geqslant 0}^{m} |D^\alpha f(x)| \qquad \forall f \in \mathcal{E}_Q(A)\,. \tag{2}$$

Obviously we say that L is carried by an open B of A if L is carried by some compact K of B. First of all observe that $\forall L \in \mathcal{E}'_Q(A)$ there exists K s.t. the relation (2) is satisfied with $m = 0$ because in $\mathcal{E}_Q(A)$ the topology induced by $\mathcal{E}(A)$ and by $\mathcal{D}'(A)$ coincide. Thus we can't talk about the order for an element of $\mathcal{E}'_Q(A)$. Besides it is certainly true that every L has some compact carrier but, differently from the case of the supports of distributions, the intersection of two carriers of L is not a carrier of L, generally. For instance if $\delta|_{\mathcal{E}_\Delta(R^n)}$ is the restriction to harmonic functions of the Dirac measure then, in view of the maximum principle, we have:

$$|\langle\delta, f\rangle| = |f(0)| \leqslant \sup_{|x|=1} |f(x)|\,, \qquad \forall f \in \mathcal{E}_\Delta(R^n)\,;$$

but of course $\delta|_{\mathcal{E}_\Delta(R^n)}$ can't be carried by $\{0\}\cap\{x\colon |x| = 1\} = \emptyset$.

However by means of the following results on the representation of $\mathcal{E}'_Q(A)$ we'll see that, under certain hypotheses, carriers behave like supports. Let $L \in \mathcal{E}'_Q(A)$ and let B be a relatively compact open of A that carries L, s.t. $A \sim B$ has no component which is relatively compact in A. Define ΨL by the formula: $\langle \Psi L, \varphi\rangle = \langle L, E * \varphi\rangle$, $\forall \varphi \in \mathcal{D}(R^n \sim \bar{B})$ (where E is a fundamental solution of Q). Obviously $\Psi L \in \mathcal{E}_{{}^tQ}(R^n \sim \bar{B})$. Via Ψ we can give a characterization of $\mathcal{E}'_Q(A)$; i.e. one proves that $\mathcal{E}'_Q(A)$ and $(\mathcal{E}_{{}^tQ}(\tilde{R}^n \sim A))/(\mathcal{E}_{{}^tQ}(R^n))$ are algebraically and topologically isomorphic (see [7]). Here $\tilde{R}^n$ denotes the Alexandroff compactification of R^n and $\mathcal{E}_{t_Q}(\tilde{R}^n \sim A)$ the inductive limit of $\mathcal{E}_{t_Q}(W \cap R^n)$ when W runs over the family of the open neighbourhoods of $\tilde{R}^n \sim A$.

THEOREM C. A functional $L \in \mathcal{E}'_Q(A)$ is carried by an open subset B of A if and only if ΨL has an analytic continuation on $R^n \sim K$ for some $K \Subset B$ (see [8]). So we can state:

THEOREM D. Let $\{A_n\}$ be a family of open subsets of A s.t. $\forall n, m$ $A \sim (A_n \cup A_m)$ has no relatively compact component; if $L \in \mathcal{E}'_Q(A)$ is carried by A_n $\forall n$, then L is carried by $\bigcap_n A_n$ also.

Besides from the representation of $\mathcal{E}'_Q(A)$ it is clear that if $K \Subset A$ then in $\mathcal{E}_Q(A)$ the topology induced by $\mathcal{E}(A)$ and by $\mathcal{E}(A \sim K)$ coincide; infact obviously we can give an analytic (non unique) extension of ΨL to a neighbourhood of $\tilde{R}^n \sim (A \sim K)$. Thus if $A_1 \supseteq A_2$ and $A_1 \sim A_2$ has some component which is relatively compact in A_1, then if $f \in \mathcal{E}_Q(A_2)$ can be approximated by functions of $\mathcal{E}_Q(A_1)$, it follows that f has an extension on such a component $\tilde{f}$ which also satisfy $Qf = 0$. Therefore if $f \in \mathcal{E}_Q(A_2)$ verifies $Qf = \delta_x$ (translation of δ by x belonging to some relatively compact component) then f can't be approximated because it can't be extended. Moreover note that if there is no such component thus let $u \in \mathcal{E}'(A_1)$ s.t. ${}^tQu \in \mathcal{E}'(A_2)$ then u belongs to $\mathcal{E}'(A_2)$; therefore it follows:

THEOREM E (Prop. 8, pg. 336 of [6]). Let $A_1 \supseteq A_2$. Every $f \in \mathcal{E}_Q(A_2)$ is approximable by functions belonging to $\mathcal{E}_Q(A_1)$ if and only if $A_1 \sim A_2$ has no relatively compact component.

At all differently behave the solutions of the overdetermined system $(Pu = 0,\ Qu = 0)$.

THEOREM F. If P has no elliptic factor and K is a compact subset of A s.t. $R^n \sim K$ is connected then every $u \in \mathcal{E}_Q(A \sim K) \cap \mathcal{E}_P(A \sim K)$ can be extended as an element of $\mathcal{E}_Q(A) \cap \mathcal{E}_P(A)$.

We make the above hypothesis on P because thus the theorem banally arises from [5] and because this hypothesis is non restrictive in problems of analytic convexity. It is clear that we could have analogous results supposing P and Q relatively prime; indeed if W is a neighbouhood of K s.t. $Z_W = \emptyset$ (Z_W is the union of all components of $R^n \sim W$ that are compact) then ${}^tP\colon \mathcal{E}'_Q(W) \to \mathcal{E}'_Q(W)$ is injective. Thus we conclude using the representation of $\mathcal{E}'_Q(W)$ and observing that P and Ψ commute.

2. Some conjectures by E. De Giorgi.

In [2] E. De Giorgi proposed five conjectures arising from his works (e.g. [3]) on the resolvability of partial differential equations in the space of real analytic functions. The problem of analytic convexity for an open of R^n can be translated into the compatibility, respect to an overdetermined system, between opens of the space R^m $(m > n)$ to which we « suspend » the differential equation by means of the Cauchy-Kovalevsky theorem. We express the above compatibility as follows:

Definition. Given two open sets $B \subseteq A$ of R^n and two operators P, Q defined in $\mathcal{E}(A)$ then (A, B, P, Q) is ($\mathcal{E}$-) compatible if $\forall f \in \mathcal{E}(A)$ for which the system $Pu = f$, $Qu = 0$ is ($\mathcal{E}$-) locally resolvable in A (i.e. $\forall y \in A$ there exists a neighbourhood V_y and an $u \in \mathcal{E}_Q(V_y)$ s.t. $Pu = f$ in V_y) for every such f the system is ($\mathcal{E}$-) resolvable in B (i.e. there exists an $u \in \mathcal{E}_Q(B)$ s.t. $Pu = f$ in B).

In the following we suppose that P and Q have constant coefficients and that they are relatively prime; so the condition of local resolvability of the system is that f belongs to $\mathcal{E}_Q(A)$ (see Lojasiewicz-Malgrange theorem); furthermore we suppose that Q is elliptic because there exists a class of opens and of hypoelliptic operators for which the conjectures are false (see [12] in which we developed a counterexample of [9]). These hypotheses are sufficient for the following

Lemma. If A is Q-convex (in particular if Q is elliptic), then (A, A, P, Q) is compatible if and only if A is (P, Q)-convex (i.e. $\forall (f, g) \in \mathcal{E}(A) \times \mathcal{E}(A)$ with $Qf = Pg$ the system $Pu = f$, $Qu = g$ is resolvable in A). For the proof see [12].

In conjecture 1 De Giorgi presumes that if $\{B_n\}$ is a rising sequence of opens of R^n s.t. $A \supseteq \bigcup_n B_n = B$ and if $\forall n$ (A, B_n, P, Q) is compatible, then (A, B, P, Q) is also compatible. We will not prove this conjecture in all its generality (i.e. with $B \subsetneq A$) because it is arduous, if not impossible, to give, for a generic open, a density theorem of $\mathcal{E}_Q(A) \cap \mathcal{E}_P(A)$ in $\mathcal{E}_Q(A_\theta) \cap \mathcal{E}_P(A_\theta)$ (where $A_\theta = \{x \in A : d(x, R^n \sim A) > \theta\}$).

This will impede us to answer conjecture 4 (substantially reconducible to 1 with $B \subsetneq A$) and conjecture 3, which is quite a bit stronger than 1. In any case if $B = A$, then from the compatibility of (A, B_n, P, Q) $\forall n$, we obtain a density theorem of the kind mentioned. That is the resolvability of an (elliptic) overdetermined system in relatively-compact opens of A is restricting enough (for an equation it is trivially verified by means of the existence theorem of the fundamental solution) to imply global resolvability (which, in the case of equations, requires additional hypotheses on A's geometry).

Conjecture 2 is resolved combining the results obtained in the course of the proof of conjectures 1 and 5 together with the theorems in [1] and [11]. In conjecture 5, which we here resolve in almost full generality, De Giorgi proposes the following problem: if $\forall y \in R^n \sim A$ $(R^n \sim \{y\}, B, P, Q)$ is compatible, is (A, B, P, Q) also compatible? An elementary example of this situation, is the following:

$$B = A \subseteq R^2, \quad P = \frac{\partial}{\partial x_1}, \quad Q = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}$$

($z = x_1 + ix_2$ is the variable in C).

Given $z_0 \in R^2 \sim A$, if $(R^2 \sim \{z_0\}, A, P, Q)$ is compatible, then in particular the form $(1/(z - z_0))\, dz$ is exact in A; repeating $\forall z_0 \in R^2 \sim A$ one concludes that A is simply-connected from which every form $f\,dz$ which is closed in A (i.e. such that f is holomorphic in A) is exact (i.e. there exists g holomorphic in A s.t. $(d/dz)g = (\partial/\partial x_1)g = f$). Before giving the proofs we need the following

THEOREM G. Given a rising sequence of open sets $\{B_n\}$ s.t. $B = \bigcup_n B_n$, if $\forall n$ (A, B_n, P, Q) is compatible and if the space $\mathcal{E}_Q(B_{n+2}) \cap \mathcal{E}_P(B_{n+2})$ is dense in the space $\mathcal{E}_Q(B_{n+1}) \cap \mathcal{E}_P(B_{n+1})$ endowed with the topology induced by $\mathcal{E}(B_n)$, then (A, B, P, Q) is compatible (1).

This theorem is well known in the case that the topology in question is that induced by $\mathcal{E}(B_{n+1})$ (see Prop. 2, pg. 296 of [10]) and makes use of the usual device of the telescopic series.

When the opens B_{n+2}, B_{n+1}, B_n are, with respect to (P, Q), in the previous relation we say (following L. Modica) that they form a Runge triple for (P, Q).

3. THEOREM 1. *Given a rising sequence $\{A_n\}$ of open sets of R^n with $A = \bigcup_n A_n$, if $\forall n$ (A, A_n, P, Q) is compatible then (A, A, P, Q) is also compatible* (2).

(1) Here it is not necessary to assume Q elliptic.

(2) Here, as in the following, we don't specify that P and Q have constant coefficients and that they are relatively prime with Q elliptic.

PROOF. Let $Z_A = \emptyset$. We want to show that, for every fixed $m \in N$, $(A, A \cap S(m), P, Q)$ is compatible (where $S(m) = \{x \in R^n: |x| < m\}$). Let $B_n^m = A_{1/2^n} \cap S(m - 1/2^n)$ (where $A_{1/2^n} = \{x \in A: d(x, R^n \sim A) > 1/2^n\}$) then obviously (A, B_n^m, P, Q) is compatible $\forall n$ and $A \cap S(m) = \bigcup_n B_n^m$. Besides

$$A_{1/2^n} \cap S\left(m - \frac{1}{2^n}\right) \subseteq \left(A_{1/2^{n+2}} \cap S\left(m - \frac{1}{2^{n+2}}\right)\right)_{1/2^{n+1}} \subseteq$$

$$\subseteq A_{1/2^{n+1}} \cap S\left(m - \frac{1}{2^{n+1}}\right) \subseteq A_{1/2^{n+2}} \cap S\left(m - \frac{1}{2^{n+2}}\right)$$

as it is easy to prove considering that $1/2^{n+2} + 1/2^{n+1} < 1/2^n$.

Let $L \in \mathcal{E}'_Q((B_{n+2}^m)_{1/2^{n+1}})$ be orthogonal to the space $\mathcal{E}_Q(B_{n+2}^m) \cap \mathcal{E}_P \cdot (B_{n+2}^m)$; then in view of the theorem of [11] there exists $T \in \mathcal{E}'_Q(A)$ s.t. $L = {}^tPT$. If we show that $T \in \mathcal{E}'_Q(B_{n+1}^m)$ it will follow that B_{n+2}^m, B_{n+1}^m, $(B_{n+2}^m)_{1/2^{n+1}}$ (and *a fortiori* $B_{n+2}^m, B_{n+1}^m, B_n^m$) is a Runge triple for (P, Q), which enables us to conclude in view of Theorem G. If η is small enough then $\forall y \in R^n$ s.t. $|y| < 1/2^{n+1} + \eta$ the functional $\tau_y L$ (3) is carried by B_{n+2}^m and belongs to $(\mathcal{E}_Q(B_{n+2}^m) \cap \mathcal{E}_P(B_{n+2}^m))^\perp$ (4); indeed if $f \in \mathcal{E}_Q(B_{n+2}^m) \cap \mathcal{E}_P(B_{n+2}^m)$ then the application:

$$y \mapsto \langle \tau_y L, f \rangle \qquad \forall |y| < \frac{1}{2^{n+1}} + \eta$$

is analytic and vanishing together with all its derivatives at $y = 0$, it is identically zero. Therefore $\forall |y| < 1/2^{n+1} + \eta$, L belongs to $(\mathcal{E}_Q \cdot (\tau_y B_{n+2}^m) \cap \mathcal{E}_P(\tau_y B_{n+2}^m))^\perp$ from which, $(\tau_y A, \tau_y B_{n+2}^m, P, Q)$ being compatible, there exists $T_y \in \mathcal{E}'_Q(\tau_y A)$ s.t. $L = {}^tPT_y$. Obviously $\forall y\ T = T_y$. Indeed $P: \mathcal{E}_Q(R^n) \to \mathcal{E}_Q(R^n)$ is surjective and, since ${}^tPT_y = {}^tPT$, then $T_y|_{\mathcal{E}_Q(R^n)} = T|_{\mathcal{E}_Q(R^n)}$; so we conclude considering the density of $\mathcal{E}_Q(R^n)$ in $\mathcal{E}_Q(A)$ and $\mathcal{E}_Q(\tau_y A)$ which arises from the hypothesis $Z_A = \emptyset$. Therefore $T \in \bigcap_{|y| < 1/2^{n+1} + \eta} \mathcal{E}'_Q(\tau_y A)$ and so $T \in \mathcal{E}'_Q(A_{1/2^{n+1}})$ in view of Theorem D.

We now show that T belongs to $\mathcal{E}'_Q(S(m - 1/2^{n+1}))$. Let $\tilde{T}$ be a distribution of $\mathcal{E}'(R^n)$ which induces T on $\mathcal{E}_Q(R^n)$. ${}^tP\tilde{T}$ induces ${}^tPT$ and so there exists $\tilde{W} \in \mathcal{E}'(R^n)$ s.t. ${}^tP\tilde{T} + {}^tQ\tilde{W} \in \mathcal{E}'(S(m - 1/2^{n+1}))$.

(3) If $L \in \mathcal{E}_Q(A)\, \tau_y L$ is the translation of L by y;
$\forall f \in \mathcal{E}_Q(\tau_{-y} A) (= \mathcal{E}_Q(-y + A)) \quad \langle \tau_y L, f \rangle = \langle L, f(x - y) \rangle$.

(4) $= \{L \in \mathcal{E}'_Q(B_{n+2}^m): L$ vanishes in $\mathcal{E}_Q(B_{n+2}^m) \cap \mathcal{E}_P(B_{n+2}^m)\}$.

Therefore, by Theorem B, there exists $\tilde{U} \in \mathcal{E}'(R^n)$ s.t. $\tilde{T} + {}^tQ\tilde{U}$, $\tilde{W} - {}^tP\tilde{U}$ belong to $\mathcal{E}'(S(m - 1/2^{n+1}))$. Finally T is carried by $S(m - - 1/2^{n+1})$ and being carried by $A_{1/2^{n+1}}$ also, it belongs to

$$\mathcal{E}'_Q\left(A_{1/2^{n+1}} \cap S\left(m - \frac{1}{2^{n+1}}\right)\right) = \mathcal{E}'_Q(B^m_{n+1}) .$$

Thus we proved that, $\forall m \in N$, $(A, A \cap S(m), P, Q)$ is compatible from which (A, A, P, Q) is also compatible considering that, again by Theorem B, $A \cap S(m+1)$, $A \cap S(m)$ is a Runge pair for (P, Q). We'll now give a sketch of another proof of the theorem similar to the previous one; it will, however, be useful in the following because it shows how to utilize the theorems on surjections between F-spaces ([11]). It amounts to show that the $T \in \mathcal{E}'_Q(A)$ which realizes $L = {}^tPT$ in view of the compatibility of (A, B^m_{n+2}, P, Q), in fact belongs to $\mathcal{E}'_Q(B^m_{n+1})$. In primis $T \in \mathcal{E}'_Q(A_{1/2^{n+1}})$. Otherwise, let

$$\alpha = \sup\{t\colon\ \tau_y T \in \mathcal{E}'_Q(A),\ \forall |y| < t\} ,$$

then $\{\tau_y{}^tPT\}_{|y|<\alpha}$ is a family of functionals carried by one and the same compact of B^m_{n+2}, while there isn't a compact of A which carries every functional of the family $\{\tau_y T\}_{|y|<\alpha}$ (see Theorem 1.9 of [8]) (5).

Now, by what we have already proved,

$$\tau_y{}^tPT \in \left(\mathcal{E}_Q(B^m_{n+2}) \cap \mathcal{E}_P(B^m_{n+2})\right)^\perp \qquad \forall |y| < \alpha .$$

Furthermore $\tau_y{}^tPT = {}^tP\tau_y T$ (the equality is obvious in $\mathcal{E}'_{Q(R^n)}$, and hence in $\mathcal{E}'_Q(A)$ by density); this contradicts the corollary of [11]. To finish the proof one now proceedes as above.

We show now that, in the hypotheses of the theorem, Z_A must be empty. Indeed $\forall n$ we must have $Z_{A_n} \subseteq A$, from which $Z_A = \emptyset$, according to the following:

LEMMA. Given two open sets $B \subseteq A$, then $P\mathcal{E}_Q(B)$ is $\mathcal{E}_Q(B)$-dense in $\mathcal{E}_Q(A)$ (if and) only if $Z_B \subseteq A$.

(5) If Q is a hypoelliptic operator and A an open, then in $\mathcal{E}_Q(A)$ the topology induced by $\mathcal{E}(A)$ and by $C^0(A)$ (even by $\mathcal{D}'(A)$) coincide. Therefore we can canonically identify the continuous seminorms on $\mathcal{E}_Q(A)$ with the compacts of A.

The « if » is proved in [11]. Conversely if we suppose $Z_B \not\subseteq A$ we can take a compact K of Z_B open in $R^n \sim B$ with $K \cap (R^n \sim A) \neq \emptyset$ (Lemma 2, pg. 333 of [6]). So if $\chi \in \mathcal{D}(B \cup K)$, $\chi \equiv 1$ in a neighbourhood of K, and $f \in \mathcal{E}_Q(R^n) \cap \mathcal{E}_P(R^n)$ with $f \neq 0$ in some point of $K \cap (R^n \sim A)$ (such f exists if and only if P and Q have common complex zeros), then $\chi f \in \mathcal{E}'(R^n)$, ${}^tP\chi f$ and ${}^tQ\chi f$ belong to $\mathcal{E}'(B)$, but $\chi f \notin \mathcal{E}'(A)$.

THEOREM 2. *Suppose (A, B, P, Q) compatible; then if $\{A_n\}$ is a rising sequence of open sets s.t. $A = \bigcup_n A_n$ then there exists a rising sequence $\{B_n\}$ with $B = \bigcup_n B_n$ s.t. $\forall n$ (A_n, B_n, P, Q) is compatible.*

PROOF. Let $Z_A = \emptyset$. W.l.g. we can suppose that $A_{1/n} \cap S(n) \subseteq A_n$ (otherwise we can take a subsequence of $\{A_n\}$). Since, if $L \in \big(\mathcal{E}_Q \cdot (B_{1/n} \cap S(n)) \cap \mathcal{E}_P(B_{1/n} \cap S(n))\big)^\perp$ then $L \in (\mathcal{E}_Q(B) \cap \mathcal{E}_P(B))^\perp \cap \mathcal{E}'_Q(B_{1/n} \cap S(n))$, it follows by Theorem 1 that $L \in {}^tP\mathcal{E}'_Q(A_{1/n} \cap S(n))$. This assures the compatibility of $(A_n, B_{1/n} \cap S(n), P, Q)$, $\forall n$; since furthermore $B = \bigcup_n B_{1/n} \cap S(n)$ we conclude. Let $Z_A \neq \emptyset$. If (A, B, P, Q) is compatible so is $(A_{1/n}, B_{1/n}, P, Q)$ as we'll see in the course of the proof of Theorem 3. So, observing that

$$\mathcal{E}_Q(A_{1/n} \cap S(n)) = \frac{\mathcal{E}_Q(A_{1/n}) \oplus \mathcal{E}_Q(S(n))}{\mathcal{E}_Q(A_{1/n} \cup S(n))},$$

we conclude that $(A_n, B_{1/n} \cap S(n), P, Q)$ is also compatible.

THEOREM 3. *Given two open sets $B \subseteq A$ of R^n suppose either B bounded or $B = A$. Then if $\forall y \in R^n \sim A$ $(R^n \sim \{y\}, B, P, Q)$ is compatible so is (A, B, P, Q).*

PROOF. If $B = A$ is unbounded, supposing the case of B bounded being proved, we obtain that $(A, A \cap S(n), P, Q)$ is compatible and hence (A, A, P, Q) is compatible by Theorem 1. Therefore suppose B bounded with $S(n) \supsetneq B$. Let, for the time being, $Z_A = \emptyset$. Fixed $L \in (\mathcal{E}_Q(B) \cap \mathcal{E}_P(B))^\perp \cap \mathcal{E}'_Q(B_\varepsilon)$ $(L = \lim_j {}^tPT_j$ with $T_j \in \mathcal{E}'_Q(B))$ (6) define T as follows: $\forall f \in \mathcal{E}_Q(S(n))$ $\langle T, f\rangle = \langle L, h\rangle$ if h resolves the system $(Ph = f,\ Qh = 0)$ in $S(n)$; obviously $T \in \mathcal{E}'_Q(S(n))$. It is clear

(6) Obviously $(\mathcal{E}_Q(B) \cap \mathcal{E}_P(B))^\perp = {}^tP\mathcal{E}'_Q(B)^-$ (closure).

that $L = {}^tPT$ and that

$$\langle T, f\rangle = \langle L, h\rangle = \lim_j \langle {}^tPT_j, h\rangle = \lim_j \langle T_j, f\rangle .$$

Fixed $y_0 \in R^n \sim A$ let $\{y_i\}_{i=1,\dots,N}$ be a set of points belonging to $R^n \sim A$ with $y_N \in R^n \sim S(n)$ s.t. $\forall i\ d(y_i, y_{i+1}) < \varepsilon$. Since $(R^n \sim \{y_i\}, B, P, Q)$ is compatible $\forall i$ then $(R^n \sim \bigcup_i y_i, B, P, Q)$ is also. Indeed the application:

$$\bigoplus_i \mathcal{E}_Q(R^n \sim \{y_i\}) \to \mathcal{E}_Q(R^n \sim \bigcup_i y_i) ,$$

$$\bigoplus_i u_i \to u_0|_{R^n \sim \bigcup_i y_i} - \sum_{i=1} u_i|_{R^n \sim \bigcup_i y_i} ,$$

is a surjective homomorphism (this is well-known in the case of Laplace's operator; see Prop. 1, pg. 499 of [7]). Therefore if $f \in \mathcal{E}_Q \cdot (R^n \sim \bigcup_i y_i)$, splitting f in $f_0 - \sum_{i=1}^N f_i$ with $f_i \in \mathcal{E}_Q(R^n \sim y_i)$, we have $f = P\big(g_0 - \sum_{i=1}^N g_i\big)$ in B if $g_i \in \mathcal{E}_Q(B)$ resolves $Pg_i = f_i$ in B. Now consider the map: L: $\mathcal{E}_Q(R^n \sim \bigcup_i y_i) \to ({}^tP\mathcal{E}'_Q(B))'_s$ defined by the position $L(g)({}^tPT) = \langle T, g\rangle$ (here s indicates the weak topology).

$L(g)$ is well defined since if ${}^tPT = 0$ then T vanishes on $P\mathcal{E}_Q(B) \supseteq \mathcal{E}_Q(R^n \sim \bigcup_i y_i)$. $L(g)$ is linear (obvious), continuous for if ${}^tPT_j \to 0$ thus if $f \in \mathcal{E}_Q(B)$ resolves $Pf = g$ in B we have:

$$\lim_j L(g)({}^tPT_j) = \lim_j \langle T_j, g\rangle = \lim_j \langle T_j, Pf\rangle = \lim_j \langle {}^tPT_j, f\rangle = 0 .$$

L is obviously linear and continuous. Let

$$M\colon \mathcal{E}_Q(R^n \sim \bigcup_i y_i) \to ({}^tP\mathcal{E}'_Q(B)^-)' = \frac{\mathcal{E}_Q(B)}{\mathcal{E}_Q(B) \cap \mathcal{E}_P(B)}$$

where $M(g)$ is the extension by continuity of $L(g)$ to ${}^tP\mathcal{E}'_Q(B)^-$. The map M is continuous by the closed-graph theorem; therefore for every continuous seminorm p on $\mathcal{E}_Q(B)$ there exists a continuous seminorm h on $\mathcal{E}_Q(R^n \sim \bigcup_i y_i)$ s.t.

$$M\colon \mathcal{E}_Q\big(R^n \sim \bigcup_i y_i\big)_h \to \frac{\mathcal{E}_Q(B)_p}{\mathcal{E}_Q(B) \cap \mathcal{E}_P(B)} \quad \text{is continuous } (^7).$$

(7) If X is a V.S. by X_p we indicate X topologized by means of the seminorm p.

Transposing we obtain:

$$ {}^tM\colon \left(\mathcal{E}_Q(B)\cap\mathcal{E}_P(B)\right)^\perp \cap \left(\mathcal{E}_Q(B)_p\right)' \to \Big(\mathcal{E}_Q\Big(R^n\sim\bigcup_i y_i\Big)_h\Big)'\,. $$

It is obvious that $L = {}^tP\,{}^tML$; we now show that ${}^tML\in\mathcal{E}'_Q\Big(\Big(R^n\sim\bigcup_i y_i\Big)_\varepsilon\Big)$. If not, letting $\alpha=\sup\{t\colon {}^tML \text{ is carried by } \big(R^n\sim\bigcup_i y_i\big)_t\}$ we would have $\alpha\leqslant\varepsilon$. Then observing that the functional τ_yL again belongs to $\left(\mathcal{E}_Q(B)\cap\mathcal{E}_P(B)\right)^\perp$ $\forall|y|<\alpha\leqslant\varepsilon$, as seen in Theorem 1, we would have that there exists a compact of B which carries every functional of the family $\{\tau_yL\}_{|y|<\alpha}$ while there is no compact of $R^n\sim\bigcup_i y_i$ which carries every element of the family $\{\tau_y{}^tML\}_{|y|<\alpha}$.

If we prove the following

LEMMA. $\forall|y|<\alpha,\ \ \tau_y{}^tML={}^tM\tau_yL,$

we obtain a contradiction. Fixed $g\in\mathcal{E}_Q\big(R^n\sim\bigcup_i y_i\big)$, consider the two applications:

$$ F_1\colon y\mapsto\langle\tau_y{}^tML, g\rangle\,,\qquad F_2\colon y\mapsto\langle{}^tM\tau_yL, g\rangle\,. $$

They are (defined and) analytic $\forall|y|<\alpha$; for F_1 we have seen this in Theorem 1. As far as F_2 is concerned, observe that $\langle{}^tM\tau_yL, g\rangle = \langle\tau_yL, Mg\rangle = \langle\tau_yL, h\rangle = \langle L, h(x-y)\rangle$ where h is an arbitrary element of the class Mg of the quotient $\left(\mathcal{E}_Q(B)\right)/\left(\mathcal{E}_Q(B)\cap\mathcal{E}_P(B)\right)$. If we prove that F_1 and F_2 coincide together with their derivatives in $y=0$ we have finished. Indeed

$$ \begin{aligned} \left(D^\alpha F_1(y)\right)_{y=0} &= \left(D^\alpha\langle{}^tML, g(x-y)\rangle\right)_{y=0} = \langle{}^tML, (-1)^{|\alpha|}(D^\alpha g)\rangle = \\ &= (-1)^{|\alpha|}\lim_j\langle T_j, D^\alpha g\rangle = \lim_j\langle D^\alpha T_j, g\rangle = \langle{}^tMD^\alpha L, g\rangle = \langle D^\alpha L, h\rangle = \\ &= (-1)^{|\alpha|}\langle L, D^\alpha h\rangle = \left(D^\alpha\langle L, h(x-y)\rangle\right)_{y=0} = \left(D^\alpha F_2(y)\right)_{y=0}\,. \end{aligned} $$

This implies that $\Psi\,{}^tML$ $\Big($where Ψ is the representing isomorphism of $\mathcal{E}'_Q\Big(\Big(R^n\sim\bigcup_i y_i\Big)\Big)$ of n. 1$\Big)$ is analytic in the connected open set $\bigcup_i S(y_i,\varepsilon)$. Furthermore ΨT and $\Psi\,{}^tML$ coincide together with their derivatives at the point y_N. Indeed:

$$ \begin{aligned} D^\alpha(\Psi\,{}^tML)(y_N) &= (-1)^{|\alpha|}\langle\Psi\,{}^tML, D^\alpha\delta_{y_N}\rangle = \\ &= (-1)^{|\alpha|}\langle{}^tML, E*D^\alpha\delta_{y_N}\rangle \ (\text{if } QE=\delta) = (-1)^{|\alpha|}\lim_j\langle T_j, E*D^\alpha\delta_{y_N}\rangle = \\ &= (-1)^{|\alpha|}\langle\Psi T, D^\alpha\delta_{y_N}\rangle = D^\alpha\Psi T(y_N)\,. \end{aligned} $$

Repeating for every $y_0 \in R^n \sim A$ we conclude that there exists a compact $K \Subset A$ s.t. T has an analytic extension to $R^n \sim K$ from which $T \in \mathcal{E}'_Q(A)$; so in view of [1] and [11] we conclude if $Z_A = \emptyset$.

The case $Z_A \neq \emptyset$; we have seen that $(A \cup Z_A, B, P, Q)$ is compatible from which $\big((A \cup Z_A) \cap S(n), B, P, Q\big)$ is banally also compatible. Fixed ε and a set of points $\{y_i\}_{i=1,\dots,M}$ of $R^n \sim A$ s.t.

$$\bigcup_{i=1}^{M} S(y_i, \varepsilon) \supseteq \bar{Z}_A \cap \overline{S(n)},$$

(where $S(y_i, \varepsilon)$ is the sphere of centre y_i and radius ε), then the map M defined above is the resolving map for the equaton $Pu = f$ with data in $\mathcal{E}_Q\Big(\big(R^n \sim \bigcup_i y_i\big)_\varepsilon\Big)$ (and hence in particular in $\mathcal{E}_Q\Big(R^n \sim (\bar{Z}_A \cap \overline{S(n)})\Big)\Big)$ and solutions in $\mathcal{E}_Q(B_\varepsilon)$. In fact consider:

$${}^tM\colon\ {}^tP\mathcal{E}'_Q(B)^- \cap \mathcal{E}'_Q(B_\varepsilon) \to \mathcal{E}'_Q\Big(\big(R^n \sim \bigcup_i y_i\big)_\varepsilon\Big).$$

The first space is closed in $\mathcal{E}'_Q(B_\varepsilon)_s$ and hence, endowed with the topology induced by $\mathcal{E}'_Q(B_\varepsilon)_s$ is isomorphic to

$$\left(\frac{\mathcal{E}_Q(B_\varepsilon)}{\big({}^tP\mathcal{E}'_Q(B)^- \cap \mathcal{E}'_Q(B_\varepsilon)\big)^\perp}\right)'_s$$

where $\big({}^tP\mathcal{E}'_Q(B)^- \cap \mathcal{E}'_Q(B_\varepsilon)\big)^\perp$ is obviously the closure of $\mathcal{E}_Q(B) \cap \mathcal{E}_P(B)$ in $\mathcal{E}_Q(B_\varepsilon)$. If we also endow the second space with the weak topology, ${}^tM$ is continuous by the closed graph theorem for weak duals of F-spaces (Theorem 1 of [11]). In fact if $(L_j, {}^tML_j) \to (L, T)$ then $\forall f \in \mathcal{E}_Q\big(R^n \sim \bigcup_i y_i\big)$ we have on one hand: $\langle {}^tML_j, f\rangle \to \langle T, f\rangle$ and on the other:

$$\langle {}^tML_j, f\rangle = \langle L_j, Mf\rangle \to \langle L, Mf\rangle = \langle {}^tML, f\rangle;$$

thus ${}^tML|_{\mathcal{E}_Q(R^n \sim \bigcup_i y_i)} = T|_{\mathcal{E}_Q(R^n \sim \bigcup_i y_i)}$ from which ${}^tML = T$ by density. Transposing we obtain:

$$M_\varepsilon\colon\ \mathcal{E}_Q\Big(\big(R^n \sim \bigcup_i y_i\big)_\varepsilon\Big) \to \frac{\mathcal{E}_Q(B_\varepsilon)}{\big({}^tP\mathcal{E}'_Q(B)^- \cap \mathcal{E}'_Q(B_\varepsilon)\big)^\perp}$$

which is the resolving map in question. Indeed if $g \in \mathcal{E}_Q\Big(\big(R^n \sim \bigcup_i y_i\big)_\varepsilon\Big)$

and $L \in \mathcal{E}'_Q(B_\varepsilon)$ we have:

$$\langle L, PM_\varepsilon g\rangle = \langle {}^tPL, M_\varepsilon g\rangle = \langle {}^tM\,{}^tPL, g\rangle =$$
$$= \Big(\text{if } f_n \Big(\in \mathcal{E}_Q\big(R^n \sim \bigcup_i y_i\big)\Big) \to g \text{ in } \mathcal{E}_Q\Big(\big(R^n \sim \bigcup_i y_i\big)_\varepsilon\Big)\Big) \lim_n \langle {}^tM\,{}^tPL, f_n\rangle =$$
$$= \lim_n \langle L, f_n\rangle = \langle L, g\rangle .$$

If next $\varepsilon_1 < \varepsilon$ we must add some points $\{y_i\}_{i=M,\dots,N}$ to the set $\{y_i\}_{i=1,\dots,M}$ in order that $\bigcup_{i=1}^{N} S(y_i, \varepsilon_1) \supseteq \bar{Z}_A \cap \overline{S(n)}$. We obtain by similar construction, another resolving map M_{ε_1}. It is moreover evident that if $f \in \mathcal{E}_Q\big(R^n \sim (\bar{Z}_A \cap \overline{S(n)})\big)$ then $M_{\varepsilon_1} f$ thought of as an element of

$$\frac{\mathcal{E}_Q(B_\varepsilon)}{\big({}^tP\mathcal{E}'_Q(B)^- \cap \mathcal{E}'_Q(B_\varepsilon)\big)^\perp}$$

is equal to $M_\varepsilon f$. This means that if $\widetilde{M_{\varepsilon_1} f}$ and $\widetilde{M_\varepsilon f}$ are two generic representatives of the classes $M_{\varepsilon_1} f$ and $M_\varepsilon f$ respectively then $\widetilde{M_{\varepsilon_1} f}|_{B_\varepsilon} - \widetilde{M_\varepsilon f}$ belongs to the closure of $\mathcal{E}_Q(B) \cap \mathcal{E}_P(B)$ in $\mathcal{E}_Q(B_\varepsilon)$. Therefore if $\{\varepsilon_n\}$ is a decreasing sequence which converges to zero, chosen $h_n \in \mathcal{E}_Q(B) \cap \mathcal{E}_P(B)$ $\forall n$ s.t.:

$$\sup_{x \in B_{\varepsilon_{n-1}} \cap S(n-1)} |\widetilde{M_{\varepsilon_{n+1}} f}(x) - \widetilde{M_{\varepsilon_n} f}(x) - h_n(x)| < \frac{1}{2^n},$$

then the following series: $M_{\varepsilon_0} f + \sum_{n=1}^{\infty} (\widetilde{M_{\varepsilon_{n+1}} f} - \widetilde{M_{\varepsilon_n} f} - h_n)$ converges in $\mathcal{E}_Q(B)$ to a solution of the equation $Pu = f$. Therefore $\big(R^n \sim (\bar{Z}_A \cap \overline{S(n)}), B, P, Q\big)$ is compatible. Observe now that:

$$\mathcal{E}_Q\big(A \cap S(n)\big) = \frac{\mathcal{E}_Q\big((A \cup Z_A) \cap S(n)\big) \oplus \mathcal{E}_Q\big(R^n \sim (\bar{Z}_A \cap \overline{S(n)})\big)}{\mathcal{E}_Q\Big(\big((A \cup Z_A) \cap S(n)\big) \cup \big(R^n \sim (\bar{Z}_A \cap \overline{S(n)})\big)\Big)}$$

from which (A, B, P, Q) is compatible. It is clear that we could have given a proof simultaneously resolving the cases $Z_A = \emptyset$ and $Z_A \neq \emptyset$ by taking a set of points $\{y_i\}_{i=1,\dots,M}$ of $R^n \sim \big(A \cap S(n)\big)$, $\forall \varepsilon$, s.t. $\bigcup_i \big(S(y_i, \varepsilon)\big)$ is a neighbourhood of $\big(\Gamma(A \cap S(n))\big) \cup Z_{A \cap S(n)}$ (where Γ indicates the boundary). We prefered to give a separate proof in the case $Z_A = \emptyset$ since there the complications are not too intolerable.

BIBLIOGRAPHY

[1] G. Bratti, *Un'applicazione del teorema del grafico chiuso alla risolubilità di sistemi del tipo* ($Pu = f$, $Qu = 0$), Rend. Sem. Mat. Univ. Padova, Vol. 61 (1979).

[2] E. De Giorgi, *Sulle soluzioni globali di alcuni sistemi di equazioni differenziali*, Boll. Un. Mat. Ital. (4), **11** (1975), pp. 77-79.

[3] E. De Giorgi - L. Cattabriga, *Una dimostrazione diretta dell'esistenza di soluzioni analitiche nel piano reale di equazioni a derivate parziali a coefficienti costanti*, Boll. Un. Mat. Ital. (4), **4** (1971), pp. 1015-1027.

[4] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, D. Van Nostrand Company, Princeton, New Jersey, 1966.

[5] A. Kaneko, *Prolongement des solutions analytiques réelles d'équations aux dérivées partielles à coefficients constants*, Séminaire Goulaouic-Schwartz (1976/77).

[6] B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier (Grénoble), **6** (1955/56), pp. 271-355.

[7] F. Mantovani - S. Spagnolo, *Funzionali analitici reali e funzioni armoniche*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. Mat. Fis. Natur., Sez. III, **18** (1964), pp. 475-513.

[8] L. Modica, *A Runge theorem for overdetermined systems with constant coefficients*, typescript.

[9] M. Nacinovich, *Un'osservazione su una congettura di De Giorgi*, Boll. Un. Mat. Ital. (4), **12** (1975), pp. 9-14.

[10] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[11] G. Zampieri, *Un'estensione del teorema sulle suriezioni fra spazi di Fréchet. Qualche sua appplicazione*, Rend. Sem. Mat. Univ. Padova, Vol. 61 (1979).

[12] G. Zampieri, *Sistemi del tipo* ($Pu = f$, $Qu = g$) *non globalmente risolubili*, Rend. Sem. Mat. Univ. Padova, Vol. 61 (1979).

Manoscritto pervenuto in redazione il 26 aprile 1979.