RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 61 (1979), p. 365-392

<http://www.numdam.org/item?id=RSMUP_1979__61__365_0>

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REND. SEM. MAT. UNIV. PADOVA, Vol. 61 (1979)

On Wave Functions in Quantum Mechanics.

PART 3

A Theory of Quantum Mechanics where Wave Functions are Defined by Means of Surely Fundamental Observables.

ALDO BRESSAN (*)

SUMMARY - An axiomatic theory, $\mathfrak{T}_1$, of quantum mechanics is constructed in which wave functions are defined, Born's rule need not be postulated, and a fundamental proportionality property of the wave functions of a same state, Theor. 2.1, is proved. This theory has two main aims:

(i) to reduce the primitive notions of ordinary theories on the same subject, in order to avoid using as (primitive) fundamental, observables that are not surely so, and

(ii) to state a rigorous and complete set of quantistic axioms with an explicit list of primitive notions.

In order to carry out (ii), possibility axioms are important in $\mathfrak{T}_1$. Of course they can be understood intuitively; from the logico-mathematical point of view they are based on the theory of modal logic stated in [1]. Let us add that the postulate on the state ψ^+ immediately after a measurement, used in Part I, and involving the ordinary notion of orthogonality of states, has a counterpart in $\mathfrak{T}_1$ which is deeply different, especially because this is tightly connected with the evolution properties of quantal systems. At last it is briefly shown how to define in $\mathfrak{T}_1$ fundamental notions of the theory of measurement. For the sake of simplicity only systems of pairwise distinguishable spinless particles are treated, see footnote (1).

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This paper has been worked out in the sphere of activity of the research group no. 3 of the C.N.R. (Consiglio Nazionale delle Ricerche) for the academic years 1976-'77 and 1977-'78.

(1) The meaning of causal implication and physical possibility can be read e.g. in [8] and [2] to [4]. The logico-mathematical theory on these no-

12. Introduction to Part 3.

The present work aims at introducing quantum mechanics (for pairwise distinguishable spinless particles) without assuming that any observable is fundamental—i.e. can be measured by some apparatus—in case we don't know this fact. The interest of this aim has been emphasized especially in Part 2—see its introductory section, N. 9. In Part 1 the same aim is reached by assuming the notion of quantistic states to be primitive—as is usually done. However such a state s is usually characterized intuitively by means of the expected values $\mathcal{E}_s(\omega)$ in the usual *direct* sense, of all fundamental observables ω; and the avoidance of the assumption above makes the above notion of quantistic states objectionable. Therefore it is important to define quantistic states without adding other notions as primitive. This definition is one of the two main aims of Part 3; more precisely it is embodied in theory $\mathfrak{T}_1$ of quantum mechanics where, in addition Born's rule is proved. The proof in $\mathfrak{T}_1$ of the fundamental proportionality theorem 2.1 is obviously more complex than the one available in Part 1.

When wave functions are defined in $\mathfrak{T}_1$, s can be characterized by means of the expected values $\mathcal{E}_s(\omega)$ in a certain *indirect* sense, of all (possibly non-fundamental) observables ω.

The other main aim of Part 3 is to state a rigorous and complete set of (quantistic) axioms with an explicit list of primitive notions.

Now consider the Hilbert space $\mathcal{H}$, where the states of $\mathfrak{S}$ are represented. For every self-adjoint operator A in $\mathcal{H}$, Theor 2.1 allows us to consider $\langle A\rangle_s = \langle\psi|A\psi\rangle$, where ψ is any wave function of the state s, as the expected value, in an *indirect* sense, of « the observable represented by A » (for $\mathfrak{S}$ in s). When this indirect sense is considered Post 2.1, an analogue of assumption 2.1, is true also from our point of

tions can be found, in part, e.g. in [8] where relations, functions, or descriptions are not dealt with, and especially in [1] where a general theory is developed. Precisely on this theory does the author consider $\mathfrak{T}_1$ to be based.

Theory $\mathfrak{T}_1$ shares with [17] (*a*), (*b*) the property of being based on a theory of modal logic. However it is deeply different from that works in that [17] (*a*), (*b*) concentrate on states (they give a motive for using modal logic in dealing with mixtures) whereas in $\mathfrak{T}_1$ modal logic is used in connection with the whole theory of quantum mechanics for the systems being considered. Furthermore the theory of modal logic on which $\mathfrak{T}_1$ is based is very general.

view. Thus all theorems of ordinary theories of quantum mechanics —cf. e.g. [10], [12], [13], [14], [15] and [18]—conform with this point of view, possibly under a slightly different interpretation.

Let us remark that theory $\mathfrak{T}_1$ has been constructed without being sure of the falsity of assumption 2.1, because even if this assumption is true, $\mathfrak{T}_1$ seems useful in that by it a reduction of the primitive observables has been performed with respect to ordinary quantum theories. More precisely the latter theories consider, as primitive, indefinitely many observables that are defined in classical mechanics in terms of position. In $\mathfrak{T}_1$ these observables are defined in terms of position and momentum in a way different from the classical procedure. Thus $\mathfrak{T}_1$ makes the quantistic relations among (those) observables clearer and contributes to the comparison of classical and quantum mechanics. Incidentally from this point of view it is desirable to improve $\mathfrak{T}_1$ into a theory, $\mathfrak{T}_2$, where all mechanical observables are defined in terms of position.

* * *

In Part 1 Theor 2.1 is proved on the basis of certain rather usual postulates that are reasonably supported by experiments. Among them there is Post 7.2 on the state ψ^+ of the quantal system $\mathfrak{S}$ immediately after a measurement; and there is no easy postulate on ψ^+ such as either Post 2.4, which was used in previous proofs of Theor 2.1 but is disproved by actual experiments, or Post 3.1 (a direct analogue of Post 2.4) which apparently is in agreement with experiments (connected with superselection rules) but is not supported by them satisfactorily.

In Part 1 the notion of wave functions of a pure state s is assumed as a primitive; and in Post 7.2 (on ψ^+) this notion is essential to speak of certain orthogonality relations (substantially in connection with ideal measurements of the first kind). It is also essential in the aforementioned characterization of a connex pure state s by means of the results of a certain system σ of measurements, in that on the one hand the statistical properties of σ can be represented in the usual way by a wave function ψ of s. On the other hand in N. 16 we show by an esample that the same properties can be expressed in the same way by a wave function, ψ^*, of a different state s^*. Hence by the system σ above alone (without taking into account the various possible processes of $\mathfrak{S}$ starting at the state s of $\mathfrak{S}$ being considered) we cannot determine the wave functions of s.

Let us add that when the afore-mentioned usual characterization of (quantistic) states is accepted, it is natural to consider the notion of wave functions as quite clear. This is no longer true if, as is done in the present work, one accepts the afore-mentioned aims which, as was hinted at, induce some changes in some basic notions including the one of wave functions. In spite of this no intuitive characterization of this notion has been given explicitly in Part 1, so that the characterization of pure states given there appears to depend on the assumption that one knows what wave functions are.

Of course, at this point one can speak of implicit definition of the primitive concepts by the axioms. Obviously this does not mean that they are determined by the axioms; however this determination is hoped to be achieved by both the axioms and the intuitive properties explicitly stated to hold for the primitive notions. In Part 1 it is not clear whether or not this determination hold (and granted that it holds, it is not clear in which way).

The best and clearest answer that can be given in similar situations is obtained by replacing the old theory, say $\mathcal{T}_0$, with a new one that includes definitions of the notions that in $\mathcal{T}_0$ are used as primitive but were not characterized intuitively in a satisfactory way. Along these lines, in Part 3 we briefly construct the assiomatic theory $\mathcal{T}_1$, where the same theorems proved in Part 1 hold, but the notions of wave functions and pure states are defined. The notion of (possibly non-pure) states of $\mathfrak{S}$—characterized by preparations of $\mathfrak{S}$—is taken as primitive. Born's correspondence rule need not be postulated in $\mathcal{T}_1$.

* * *

The task of stating the counterpart in $\mathcal{T}_1$ of the postulates on ψ^+ considered in Part 1 is complex. In fact it is fulfilled by taking the evolution properties of $\mathfrak{S}$ strongly into account. Thus our definition of wave functions is essentially based on those properties. Is this natural? The affirmative answer is tenable e.g. on the basis of the following requirement that connects the notion of states with evolutions.

The state of the physical system $\mathfrak{S}$ at the instant τ determines those at all later instants.

It dominates classical physics, which is quite natural in that it complies perfectly with the classical determination principle. It is to remark that the same requirement was kept holding also in passing to quantum mechanics, in spite of the compliance of this theory with the indetermination principle, the corresponding indetermination being reflected by some statistical properties of states.

* * *

Now let us describe the content of Part 3 in more detail.

We exhibit a list of primitive notions of the quantistic theory $\mathcal{T}_1$ including the one of apparatus (or experimental set up) to measure position, and the analogous notion for momentum [N. 13]. We do not write explicitly the axioms of $\mathcal{T}_1$ that also belong to classical physics—such as axioms on space-time and a few (structural) axioms on the electromagnetic field (Maxwell equations will not be used in this paper). First we state some structural axioms and a possibility axiom [N. 14]. Some of these axioms in $\mathcal{T}_1$ and others involve physical possibility and related notions such as casual implication (1).

In N. 15 we introduce a preliminary notion to define wave functions (and hence orthogonality relations): the function ψ *q-p-representing* a state, s, with respect to an apparatus $\mathcal{A}_Q$ to measure positions; furthermore we prove a preliminary proportionality theorem on couples of these functions connected with a same apparatus, which is an analogue of Theor 4.1 on wave functions, i.e. a weak version of the fundamental proportionality theorem 2.1. The analogue for p-q-representation holds.

An example in N. 16 shows that functions q-p- or p-q-representing s with respect to $\mathcal{A}_Q$ are not necessarily wave functions of s. Therefore we define ψ ***represents*** s with respect to $\mathcal{A}_Q$ [Def. 16.1] by combining the notion of q-p-representation with evolution properties of the quantal system $\mathfrak{S}$ being considered. Axiom 16.1 implies that such a function ψ is independent of $\mathcal{A}_Q$, which allows us to call it a wave function of s. The truth of Axiom 16.1 seems acceptable on direct intuitive ground. Incidentally, in a future paper it is planned to be made more directly evident.

The joinability (of any state with a connex one, postulated by) Axiom 17.1 (already stated in Part 1) and Axiom 17.2 which substantially asserts the validity of a regular Schrödinger equation for $\mathfrak{S}$, allow us to prove in $\mathcal{T}_1$ the fundamental proportionality Theor 2.1 for wave functions [N. 17].

In N. 18 the first axiom asserts the existence of suitably many pure states in a way widely compatible with superselection rules, the third substantially says that every state is pure or is a mixture; and the uniqueness axiom 18.2 can be avoided by a suitable redefinition of states (shown explicitly in N. 19). The axioms are briefly discussed from the Born-Bohr point of view and from Einstein's. From the former one the epistemological status of $\mathcal{T}_1$ appears more satisfactory.

In N. 19 general observables are defined on the basis of wave functions and the proportionality theorem 2.1; indirect expected values are defined for them. They appear to equal the direct ones in the case of observables of the form $f(q)$ or $g(q)$. The Born rule is a theorem in $\mathcal{T}_1$.

In N. 20 fundamental notions of the theory of measurement are defined as examples. They are interesting for they involve possibility statement that are implicit in ordinary treatments.

The notion of *the pure state* $s^{\mathfrak{S}}_{t_0 t}$ *of* $\mathfrak{S}$ *at* t, *relative to the recognition instant* t_0 ($\leqslant t$), suggested by the reading of [17(a)], is introduced [n. 18] and is used in Axioms 18.2-3 as an alternative—cf. fn. 2 in n. 13.

13. First notions in the theory $\mathcal{T}_1$ of classical quantum mechanics, in which wave functions are defined and only positions and momenta are primitive fundamental observables.

Since for the sake of simplicity in the present work we consider only any quantal system $\mathfrak{S} = (\mathfrak{S}^{(i)}, H^{(e)})$—cf. N. 5—having a classical analogue $\mathfrak{S}_c$, as is done in the first part of ordinary text books such as [14], it is natural to remark that, generally, the theory developed for these quantal systems is substantially a mixture of a really quantistic theory with the classical theory of gravitation and electromagnetism, e.g. the one for continuous media. Notions such as inertial space, time, mass, charge, gravitation, electric or magnetic field, and ponderomotive force are taken from classical physics without changes, as well as the rules for constructing the classical Hamiltonian H_c of $\mathfrak{S}$, starting out from the masses and charges of the particles M_1 to M_n forming $\mathfrak{S}$ and the external gravitational and electromagnetic fields. In some cases a (repulsive) interaction among those particles has to be added (at least for very little mutual distances). Text-books such as [14] say how to construct the (quantistic) Hamiltonian H of $\mathfrak{S}$ from H_c, which incidentally determines the classical analogue $\mathfrak{S}_c$ of $\mathfrak{S}$.

Briefly, in the axiomatic theory $\mathcal{T}_1$ of quantum mechanics that we now want to construct we take as primitive the notions above except that for the sake of simplicity we fix an inertial space-time frame $\mathcal{F}$ once for all. Hence «instant» can be regarded as a sinonimous of «real number» and the primitive notions in $\mathcal{T}_1$ include *absolute reference space* (joint to $\mathcal{F}$), *particle*, *mass*, *charge*, *conservative interaction force* (expressed in $\mathcal{F}$) and *external gravitational* and *electromagnetic fields* (expressed in $\mathcal{F}$). This allows us to consider an (n-particle) system $\mathfrak{S}$ of the kind above and to define its classical and quantistic Hamiltonians H_c and H (up to their well known indetermination elements).

We consider the following quantistic primitive notions in $\mathcal{C}_1$:

(1) the relation $St_{\mathfrak{S},s,\tau}$: *(the quantual system)* $\mathfrak{S}$ *is in the state* s *at (the instant)* τ, which means that $\mathfrak{S}$ has had a certain preparation $\mathcal{I}$ at some instant $\theta < \tau$.

(2) the relation $App_{\nu,\mathcal{A},\mathfrak{S}}$: $\mathcal{A}$ *is an apparatus to measure* ν *(independent compatible) magnitudes on (the above system)* $\mathfrak{S}$,

(3) the relation $Us_{\nu,\mathcal{A},\mathfrak{S},\tau,B}$: $App_{\nu,\mathcal{A},\mathfrak{S}}$ *(holds) and* $\mathcal{A}$ *is used at (the instant)* τ *on* $\mathfrak{S}$ *and* $B(\in \mathfrak{B}_\nu)$, i.e. the apparatus $\mathcal{A}$ is used to see whether or not the ν-tuple of certain magnitudes—those to be observed by $\mathcal{A}$—have a value in B,

(4) the relation $Y_{\nu,\mathcal{A},\mathfrak{S},\tau,B}$: $Us_{\nu,\mathcal{A},\mathfrak{S},\tau,B}$ *(holds) and the answer to the question in* (3) *is yes*, i.e. the value of the ν-tuple of magnitudes observed on $\mathfrak{S}$ by $\mathcal{A}$ at the instant τ appears to be in B,

(5) [(6)] the relation $A^{(Q)}_{n,\mathcal{A},\mathfrak{S}}$ $[App^{(P)}_{n,\mathcal{A},\mathfrak{S}}]$: $\mathcal{A}$ *is an apparatus to measure the* q_1 *to* q_N *[their conjugate momenta]* where $N = 3n$ and q_{3r-3+s} is the s-th co-ordinate in the inertial frame $\mathcal{F}$, of the r-th particle M_r of $\mathfrak{S}$ $(r = 1, \dots, n;\ s = 1, 2, 3)$.

Let us remark that the preparation mentioned in (1) is, for instance, a preparative measurement—see e.g. [15], p. 7—followed by the action of a macroscopic additional field. External microscopic interactions are excluded, which avoids a troublesome dilemma concerning the traditional interpretation of quantum mixtures—cf. [9].

The notions (5) and (6) are meant to be characterized intuitively as the ideal apparata described in ordinary text-books to measure positions and momenta. Positions and momenta are not to be mentioned in the above characterization which, on the other hand, must include some devices to distinguish the particles forming $\mathfrak{S}$ and, in connection with momenta, to take the external electromagnetic field into account, so that from the result of the use of the apparatus we know mechanical and electromagnetic momenta.

As a primitive notion in $\mathcal{C}_1$ we have also to consider the *probability* $pr(\alpha, \beta)$ *(or* $pr_{\alpha,\beta}$*) that if (the condition)* α *holds, then so does* β. In spite of the « if ..., then ... » used in the usual expression above for $pr_{\alpha,\beta}$, from a rigorous point of view the $pr_{\alpha,\beta}$ is a real number (in $0 \vdash\!\dashv 1$) considered as a (non-extensional) function of the two conditions (or assertions) α and β. More, from a rigorous point of view one has to consider two versions of pr, an objective or quantistic one, say $pr^{(q)}$ and a (partially) subjective or frequentistic one, say $pr^{(f)}$, depending partially on our ignorance. However both can be defined in terms of a single notion of probability, pr (2).

(2) The following point of view is in harmony with the so called hortodox interpretation of quantum mechanics. Let the preparation $\mathcal{T}$ be performed on $\mathfrak{S}$ at the instant $\theta < \tau$. Then the corresponding probability distributions for the results of systems of (not necessarily simultaneous) measurements (starting) at τ, with given apparata, are determined.

If any observation on $\mathcal{T}$ made before τ cannot improve the above probability distributions—i.e. cannot decrease their dispersions—then $\mathcal{T}$ can be (called) a preparation of a pure state of $\mathfrak{S}$. In this case those probabilities are to be regarded as objective or irriducible. On the other hand, for most choices of $\mathcal{T}$, a particular performance of $\mathcal{T}$ occurs in that a preparation $\mathcal{T}_1$ is unconsciously carried out, to which better probability distributions correspond. It is in harmony with the Copenhagen school to accept that 1) $\mathcal{T}_1$ can be identified with the preparation of a pure state; and that 2) given $\mathcal{T}$, the probability that a performance of $\mathcal{T}$ occurs through $\mathcal{T}_1$ can be learned by repeating $\mathcal{T}$ and by observing the corresponding relative frequence of $\mathcal{T}_1$ (like what happens with the probabilities related with a physical roulette). This knowledge is usually regarded as objective in connection with a whole system $\Sigma_{\mathcal{T}}$ of repetitions of $\mathcal{T}$ (therefore we speak of frequentistic probability). On the other hand the assignment of such a probability to the occurrence of $\mathcal{T}_1$ in a particular performance of $\mathcal{T}$—made « by abuse of language » according to Reichenbach, see [16]—is partially subjective and is also called reducible in that it depends on our ignoring only which particular element of the set $\Sigma_{\mathcal{T}}$ (of known global behaviour) we are dealing with.

Let us add that the problem of the resolution of a quantistic mixture, s, into orthogonal pure states $s_1, s_2, \ldots$ of given respective probabilities, is complicated by the fact that, at least a priori, in most cases the orthogonal set of states $s_1, s_2, \ldots$ is not unique (mathematically). For the sake of simplicity we can consider e.g. the

ASSUMPTION. *One can restrict oneself to preparations $\mathcal{T}$ of the arbitrary state for which there is a unique set of preparations $\mathcal{T}_i$ of pure states s_i and a unique sequence $p_1, p_2, \ldots$ $(\in \mathbf{R})$ such that the states $s_1, s_2, \ldots$ are orthogonal and $\mathcal{T}_i$ has the (frequentistic) probability p_i relative to $\mathcal{T}$.*

This assumption is compatible with known experiments but is far from being proved by them. We can accept it in spite of its failing to conform with requirement (*c*) in n. 1 on the basis of the reasonable methodological criterion *to proceed step by step*, taking into account that the assumption above does not affect our treatment of pure states.

By the same criterion we limit ourselves to systems formed with spinless pairwise distinguishable particles and we assume at the outset that the space $\mathcal{H}$

For the sake of brevity $\diamond \alpha$ will mean that (the condition) α can happen or more precisely that it is physically possible. Incidentally sometimes one attempts to define this possibility as the logical com-compatibility of α with the laws of physics. However physical possibility is used within the axioms of $\mathcal{T}_1$—as well as in foundations of classical particle mechanics according to Mach an Painlevé—so that the acceptance of that attempt would be even circular in connection with $\mathcal{T}_1$. We can introduce $\diamond$ directly as a special kind of causal possibility, $\diamond_c$—see [2] to [4]. Now there are very general mathematical theories of modal logic—i.e. (non-extensional) possibility logic—fit as bases of the afore-mentioned physical theories where physical possibility is meant as ideal generalized technical possibility—cf. footnote (1) (3).

We shall shay that *s is a state for* $\mathfrak{S}$ *at* τ if $\diamond\, St_{\mathfrak{S},s,\tau}$, i.e. $\mathfrak{S}$ can be in the state s at the instant τ.

Remark that in (4) we say that a certain value is in B only to follow the general use of keeping suggestive ways of speaking belonging to classical physics. Strictly speaking no value but only B is referred to in (4) from this quantistic point of view. Therefore one can call quantistic the corresponding expected values—cf. definition (14.1) below.

At last let us remark that the general notion (2) is not essential for quantum mechanics. However its avoidance requires redoubling the relation $Us[Y]$ into $Us^{(Q)}$ and $Us^{(P)}[Y^{(Q)}$ and $Y^{(P)}]$ in connection with $\mathcal{A}pp^{(Q)}$ and $\mathcal{A}pp^{(P)}$. Furthermore that general notion is useful in measure theory—cf. n. 20.

14. First axioms of $\mathcal{T}_1$.

The theory $\mathcal{T}_1$ being constructed has both classical and quantistic axioms. We want to write only the latter. Hence we simply consider any set of axioms on the classical primitive notions of $\mathcal{T}_1$ [N. 13] as

used to represent states is a separable Hilbert space, unlike e.g. Jauch—cf. [13].

We leave it for further steps to free our theory of something such as the assumption above (relevant for mixtures) and to combine it with theory [13] or some modern improvements of it.

(3) Even in connection with an extensional theory of physics, to define physical possibility as logical compatibility with physical laws is unsatisfactory—cf. [4].

included in $\mathcal{T}_1$. We start with some structural (or apartenance) axioms that reflect directly some semantical properties of the quantistic primitive concepts in $\mathcal{T}_1$ [N. 13]—i.e. a part of their intuitive characterizations.

It is useful to use the logical signs $\wedge$ (and) and $\vee$ (or), besides $\diamond$.

AXIOM 14.1. *If* $St_{\mathfrak{S},s,\tau}$ *(holds), then* $\mathfrak{S}$ *is a quantal system (formed with distinguishable spinless particles), is a state, and* τ *is an instant (or* $\tau \in \mathbb{R}$*).*

AXIOM 14.2. *If* $App_{\nu,\mathcal{A},\mathfrak{S}}$*, then* $\nu \in Z^+$*—i.e.* ν *is a positive integer—and* $\mathfrak{S}$ *is a quantal system.*

AXIOM 14.3. *If* $Us_{\nu,\mathcal{A},\mathfrak{S},\tau,B}$*, then* $App_{\nu,\mathcal{A},\mathfrak{S}}$*,* τ *is an instant, and* $B \in \mathfrak{B}_\nu$.

AXIOM 14.4. *If* $Y_{\nu,\mathcal{A},\mathfrak{S},\tau,B}$*, then* $Us_{\nu,\mathcal{A},\mathfrak{S},\tau,B}$.

AXIOM 14.5. *If* $App^{(Q)}_{n,\mathcal{A},\mathfrak{S}}$ *or* $App^{(P)}_{n,\mathcal{A},\mathfrak{S}}$*, then* $App_{N,\mathcal{A},\mathfrak{S}}$ *for* $N = 3n$.

If one prefers not to use the general notion (2) in N. 13 as a primitive, axioms 14.2 and 14.5 can be dispensed with; however one has to redouble axioms 14.3, 14.4, and the following.

AXIOM 14.6. *If* $App_{N,\mathcal{A},\mathfrak{S}}$*,* $B \in \mathfrak{B}_N$*, and* $\diamond\, St_{\mathfrak{S},s,\tau}$*, then* $\diamond\, (St_{\mathfrak{S},s,\tau} \wedge \wedge\, Us_{\nu,\mathcal{A},\mathfrak{S},\tau,B})$.

If s is a state for $\mathfrak{S}$ at τ, then it is certainly so also if we alter the external forces acting on $\mathfrak{S}$ after τ:

AXIOM 14.7. *If* $\mathfrak{S} = (\mathfrak{S}^{(i)}, H^{(e)})$ *and* $\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$ *are quantal systems with* $H^{(e)}_t = H'^{(e)}_t$ *for* $t \leqslant \tau$*, then* $\diamond\, St_{\mathfrak{S},s,\tau}$ *iff* $\diamond\, St_{\mathfrak{S}',s,\tau}$.

Now we state the main specific probability axioms for measurements of position variables or their conjugate momenta.

AXIOM 14.8 [14.9]. *If* $\diamond\, St_{\mathfrak{S},s,\tau}$*, there is a (unique) probability measure* μ *on* $\mathbb{R}^{3n}$ *where* n *is the number of particles forming* $\mathfrak{S}$*, such that if* $App^{(Q)}_{n,\mathcal{A},\mathfrak{S}}[App^{(P)}_{n,\mathcal{A},\mathfrak{S}}]$*,* $\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$ *is a quantal system,* $\diamond\, St_{\mathfrak{S}_,,s,\tau'}$*, and* $B \in \mathfrak{B}_{3n}$*, then* $\mu(B)$ *equals the probability* $\mathcal{P}^{(Q)}_{\mathcal{A},\mathfrak{S}',s,\tau',B}[\mathcal{P}^{(P)}_{\mathcal{A},\mathfrak{S}',s,\tau',B}]$ *that if* $St_{\mathfrak{S}',s,\tau'}$ *and* $Us_{3n,\mathcal{A},\mathfrak{S}',\tau',B}$*, then* $Y_{3n,\mathcal{A},\mathfrak{S}',s,\tau',B}$.

Note that the probability $\mathcal{P}^{(Q)}_{\mathcal{A},\mathfrak{S}',s,\tau',B}[\mathcal{P}^{(P)}_{\mathcal{A},\mathfrak{S}',s,\tau',B}]$ mentioned in axiom 14.8 [14.9] is independent of $\mathcal{A}$ and τ, and that it depends on $\mathfrak{S}$ only through $\mathfrak{S}^{(i)}$. Hence this probability, which equals the measure μ mentioned in axiom 14.8 [14.9] can be expressed by $\mathcal{P}^{(Q)}_{\mathfrak{S}^{(i)},s}[\mathcal{P}^{(P)}_{\mathfrak{S}^{(i)},s}]$. Now we can define the (*direct* or *quantistic*) *expected value*, in the state s

for $\mathfrak{S}$, of any (physical magnitude that is a) function f of the $q's[p's]$:

$$(14.1) \qquad \mathcal{E}_s(f) = \int_{\mathbf{R}^N} f d\mu \quad \text{where } N = 3n \text{ and } \mu \text{ is } \mathcal{T}^{(Q)}_{\mathfrak{S}^{(t)},s}[\mathcal{T}^{(P)}_{\mathfrak{S}^{(t)},s}],$$

The following reduced notions may be useful

DEF 14.1-4. *Let $\mathcal{A}pp_{\mathcal{A},\mathfrak{S}}$ mean $App_{n,\mathcal{A},\mathfrak{S}}$ for some n, and let $\mathcal{A}pp^{(Q)}$ $\mathcal{A}pp^{(P)}$, $\mathcal{U}s$, and $\mathcal{Y}$ be the analogues of $\mathcal{A}pp$ for the relations $App^{(Q)}$, $App^{(P)}$, Us, and Y respectively.*

For instance Def 14.2, 3 are useful in the following existence axioms which, together with the possibility axiom 14.6, implies the measurability of position and momentum

AXIOM 14.10 [14.11]. *If $\mathfrak{S}$ is a quantal system, then for some $\mathcal{A}$ $\mathcal{A}pp^{(Q)}_{\mathcal{A},\mathfrak{S}}[\mathcal{A}pp^{(P)}_{\mathcal{A},\mathfrak{S}}]$.*

* * *

At this point it is natural to ask for a general axiom system for probability because, it is true, several of them are well known—see e.g. the one in [16] which is also accepted in some text-books for engineers such as [11]. However most of them are based on extensional logic—e.g. in [16] Reichenbach speeks of extensional modalities—and this extensionality constitutes a handicap; furthermore from the rigorous point of view none of those axiom systems is fit for the present theory $\mathcal{T}_1$.

In addition a logical analysis of probability is planned to be performed in further works, e.g. in [7]. Therefore we now write axioms 14.12 to 14.17′ below on probability, which suffice for $\mathcal{T}_1$. Most of them, precisely Axioms 10.12-17, are rather natural analogues for the modal logical calculus MC^{ν} developed in [1], of the main axioms in [16]. Our proofs of the theorems of the probability calculus are also similar to the corrisponding proofs in [16], so that we can help writing them here. Axioms 14.16′-17′ are the analogues of some theorems in [16] deduced by use of the so called existence rule—see [16, p. 53] or [11, p. 66]. We postulate them instead of the rule because we don't need other consequences of this rule; furthermore the rule is stated in the books [16] and [11] without referring to any determined formal system (4), and is considered in them as metalogic or metaprobabilistic. In [5] an axiom is proposed as a substitute for the existence rule, which formally seems somewhat more general. Its applications made in [7] work. However it is preferable to wait for further tests.

The reader not interested in logico-mathematical analyses can disregard axiom 14.12 below, which as far as I know, has no counterpart in the preceding literature and is tighly connected with MC^{ν}.

Let α, β, and γ be propositions. Then, following Reichenbach partially—cf. [16, § 72]—by $\alpha \ni_p \beta$ we mean that α implies β with the probability p. Furthermore we use the symbols $\supset$ and $\equiv$ [$\supset^{\cap}$ and $\equiv^{\cap}$] for extensional [physically or causally necessary] implication and equivalence respectively; and sometimes we write e.g. $\alpha \cdot \beta$ for $\alpha \wedge \beta$.

Axiom 14.12. *$\Diamond (\alpha \ni_p \beta)$ implies $\sim \Diamond \sim (\alpha \ni_p \beta)$ and that p is an absolute real number in the sense of* [1].

Axiom 14.13. $p \neq q \supset [(\alpha \ni_p \beta) \cdot (\alpha \ni_q \beta) \equiv \sim \Diamond \alpha]$,

Axiom 14.14. $(\Diamond \alpha) \cdot (\alpha \supset^{\cap} \beta) \supset (\alpha \ni_1 \beta)$,

Axiom 14.15. $(\Diamond \alpha) \cdot (\alpha \ni_p \beta) \supset p \geqslant 0$,

Axiom 14.16. $(\alpha \cdot \beta \supset^{\cap} \sim \gamma) \cdot (\alpha \ni_p \beta) \cdot (\alpha \ni_q \gamma) \supset (\alpha \ni_{p+q} \beta \vee \gamma)$,

Axiom 14.17. $(\alpha \ni_p \beta) \cdot (\alpha \cdot \beta \ni_q \gamma) \supset (\alpha \ni_{pq} \beta \cdot \gamma)$,

Axiom 14.16′. $(\alpha \cdot \beta \supset^{\cap} \sim \gamma) \cdot (\alpha \ni_p \beta) \cdot (\alpha \ni_{p+q} \beta \vee \gamma) \supset (\alpha \ni_q \gamma)$,

Axiom 14.17′. $p, q \neq 0 \wedge (\alpha \ni_{pq} \beta \cdot \gamma) \supset [(\alpha \cdot \beta \ni_q \gamma) \equiv (\alpha \ni_p \beta)]$.

Of course, the *probability $pr_{\alpha,\beta}$ of β relative to α*—or that β occurs when α holds—is defined as the p such that $\alpha \ni_p \beta$ (in case there is exactly one such p).

From axioms 14.12-14 the following theorems can be derived—as is shown in [7]

$$pr_{\alpha,\beta} \in \mathbb{R} \supset \Diamond \alpha, \qquad 0 \neq pr_{\alpha,\beta} \in \mathbb{R} \supset \Diamond (\alpha \cdot \beta), \tag{14.2}$$

$$(\alpha \equiv^{\cap} \gamma) \cdot (\beta \equiv^{\cap} \delta) \supset [(\alpha \ni_p \beta) \equiv^{\cap} (\gamma \ni_p \delta)]. \tag{14.3}$$

Let us only remark that $pr_{\alpha,\beta} \in \mathbb{R}$ is equivalent to the existence

(4) The existence rule in [16] or [11] must be given a reasonable interpretation, since the litteral one may appear unsatisfactory and even false.

of the probability of β relative to α, and that (14.3) is a main contribution to the proof that $\ni$ can be manipulated extensionally in most situations.

15. Orthogonal apparata and an analogue of Theor. 4.1 in $\mathcal{T}_1$.

Now we define an analogue for $\mathcal{T}_1$ of the notions of an apparatus orthogonal for the state s or the wave function ψ [Defs 7.1-2 in Part 1]. This analogue—cf. Def. 15.2—has a more complex logical structure and a more elaborated definition, because now wave functions cannot be used in that a goal of $\mathcal{T}_1$ is just to define them. Set $\mathcal{H} = L^2(\mathbf{R}^N)$.

DEF 15.1. *Assume that s is a state for $\mathfrak{S}$ at τ* (i.e. $St_{\mathfrak{S},s,\tau}$), $\psi \in \mathcal{H}$, $\|\psi\| = 1$, *and that* $\mathcal{A}pp^{(Q)}_{\mathcal{A}_Q,\mathfrak{S}}$, i.e. *$\mathcal{A}_Q$ is an apparatus to measure the above system q (of inertial co-ordinates) on $\mathfrak{S}$. We shall say that ψ q-p-represents s with respect to $\mathcal{A}_Q$ if, in addition, the conditions (a) and (b) below hold for all $B, B' \in \mathfrak{B}_N$.*

(*a*) *The expected value $\mathcal{E}_s[\chi_B(q)]$ of $\chi_B(q)$ in s (for $\mathfrak{S}$) is*—cf. (4.1)

$$(15.1) \qquad \mathcal{E}_s[\chi_B(q)] = \langle\psi|E_B^{\{Q_i\}}\psi\rangle = \int_B |\psi(q)|^2 dq = \int_{\mathbf{R}^N} \psi^*(q)(\theta_B\psi)(q)\,\mathrm{d}q\,,$$

(*b*) *For $\mathcal{N}_B = \|\theta_B\psi\| \neq 0$, the quantity $\mathcal{N}_B^{-1}\langle\theta_B\psi|E^{(P)}(B')\,\theta_B\psi\rangle$ equals the probability $\mathcal{P}^{\mathcal{A}_Q}_{s,B,B'}$ that if* (i) *we measure q (on $\mathfrak{S}$) in s (at τ) with the apparatus $\mathcal{A}_Q$, if* (ii) *the result of this measurement is in B, and if* (iii) *immediately after we measure p, then* (iv) *we obtain a value in B'.*

In order to explain rigorously our use of « immediately after » in (iii), let $\mathcal{P}^{\mathcal{A}_Q}_{s,B,B',\varepsilon}$ be the probability that if (i) and (ii) occur and we measure p on $\mathfrak{S}$ at the instant $\tau + \varepsilon$ $(\varepsilon > 0)$, then (iv) holds. Now we can set

$$(15.2) \qquad \mathcal{P}^{\mathcal{A}_Q}_{s,B,B'} = \lim_{\varepsilon\to 0^+} \mathcal{P}^{\mathcal{A}_Q}_{s,B,B',\varepsilon}\,.$$

For the state $s' = \theta_B s$ of $\mathfrak{S}$ immediately after the above measurement of q and for every $B \in \mathfrak{B}_N$

$$(15.3) \qquad \mathcal{E}_{s'}[\chi_{B'}(p)] = \mathcal{P}^{\mathcal{A}_Q}_{s,B,B'} = \frac{1}{\mathcal{N}_B}\int_{B'} |(\theta_B\psi)^\wedge(p)|^2\,dp \qquad (s' = \theta_B s)\,.$$

By the definition of expected value and well known theorems on the Fourier transform

$$(15.4)\qquad \mathcal{N}_B \mathcal{E}_{s'}(p_h) = \int_{\mathbf{R}^N} p_h |(\theta_B \psi)^\wedge(p)|^2 \, dp = \frac{\hbar}{i} \int_B \psi^*(q) \frac{\partial \psi}{\partial q_h} \, dq ,$$

provided the integrals above exist.

Remark that condition (*b*) in Def. 14.1 includes the equality of a certain real number with the probability $\mathcal{P}^{\mathcal{A}_Q}_{s,B,B'}$ that if (i) to (iii) hold, so does (iv); hence $\mathcal{P}^{\mathcal{A}_Q}_{s,B,B'} \in \mathbb{R}$. By the remark below (14.3), this implies the existence of the probability $\mathcal{P}^{\mathcal{A}_Q}_{s,B,B'}$, which in turn requires that (i) to (iii) can happen (together). Remark further that equality (15.1), in Def 15.1 conforms with Axiom 14.8 in that it is (14.1) for $f = \chi_B$ and $d\mu = |\psi(q)|^2 dq$; however the true assertion that every pure state is q-p-represented by some $\psi \in \mathcal{H}$ is independent of all preceding axioms; nor can we postulate (or even make) it in $\mathcal{T}_1$, in that the notion of pure states is not yet available in $\mathcal{T}_1$. A first step towards its definition in $\mathcal{T}_1$ is included in the following

DEF 15.2. *If ψ q-p-represents s with respect to $\mathcal{A}_Q$, we say that $\mathcal{A}_Q$ is q-p-orthogonal for s and ψ, (and that s is q-p-pure).*

It is obvious how to define (i) ψ *p-q-represent* s with respect to the apparatus $\mathcal{A}_P$ to measure p on $\mathfrak{S}$, (ii) $\mathcal{A}_P$ is *p-q-orthogonal* for s and ψ, (and (iii) s is *p-q-pure*).

THEOR 15.1 (15.2). *Assume that the continuous functions ψ_1 and ψ_2 represent two vectors in $\mathcal{H}$, they q-p-represent [p-q-represent] the same state s of $\mathfrak{S}$ with respect to the same apparatus $\mathcal{A}_Q[\mathcal{A}_P]$, and their first partial derivatives exist a.e. and are square integrable [the analogue holds for $\hat{\psi}_1$ and $\hat{\psi}_2$]. Then for $B \in \mathcal{B}_N$ the restrictions of ψ_1 and ψ_2 [$\hat{\psi}_1$ and $\hat{\psi}_2$] to any (connex) component of* Supp (ψ_1) [Supp $(\hat{\psi}_1)$] *are proportional.*

PROOF OF THEOR 15.1. For $B \in \mathcal{B}_N$ (4.2) holds by definition (4.1). For $r = 1, 2$ ψ_r is assumed to q-p-represent s, so that by def 15.1 (15.1) holds for every $B \in \mathcal{B}_N$ and for $\psi = \psi_r$. Then $|\psi_1| = |\psi_2|$ a.e.; hence Supp $(\psi_1) =$ Supp (ψ_2) and (4.5) holds for some real function φ differentiable a.e.. By definitions (4.6), (4.5) implies (4.7) again.

Assume $r \in \{1, 2\}$ and $\mathcal{N}_{1,B} \neq 0$, hence $\mathcal{N}_{2,B} \neq 0$ by $(4.7)_2$. Now, since ψ_r q-p-represents s, by $(4.7)_{1,2}$ assertion (*b*) in Def 15.1 holds for $\psi = \psi_r$. Hence so does $(15.3)_2$ and, as a consequence, (15.4). This yields $(4.8)_1$. Furthermore $(4.8)_2$ follows again from (4.5). From this point—*i.e.* formula (4.8)—on, the proofs of theorems 4.1 and 15.1 coincide. q.e.d.

A proof of Theor 15.2 can be substantially obtained from the above one by replacement of ψ_r with $\hat{\varphi}_r$.

Remark that Theor 15.1 can be regarded as independent of the postulates of quantum physics, and more in particular as a logical consequence of Def 15.1.

16. The first axiom on ψ^+ in $\mathcal{C}_1$. Orthogonal apparata.

Let $\mathcal{A}_{Qr}$ be q-p-orthogonal for s and ψ_r $(r = 1, 2)$ [Def 15.2]; and let s be *connex* in the sense that so is Supp (ψ_1) = Supp (ψ_2) [Theor. 15.1]. Can we assert that e.g. $\mathcal{A}_{Q1}$ is q-p-orthogonal also for s and ψ_2?

The answer is no or at least that this assertion is not satisfactorily grounded. Indeed remember the functions ψ and $f(= \psi^*)$ introduced by (10.3). They were shown to fulfil (10.1) and not to be proportional [Theor 10.1]. Now suppose that ψ_1 is the (true) wave-function of s, so that $\mathcal{A}_{Q1}$ is truly orthogonal, i.e. performs ideal (position) measurements. No basis seems to exist, to deny satisfactorily the existence of an apparatus $\mathcal{A}_{Q2}$ that for every $B \in \mathcal{B}_N$ is capable to send the state s of $\mathfrak{S}$ into the state $\theta_B \psi_1^*$—cf. (4.1). *If $\mathcal{A}_{Q2}$ is such an apparatus, then it is q-p-orthogonal for s and ψ_2 where $\psi_2 = \psi_1^*$; and of course $\mathcal{A}_{Q1}[\mathcal{A}_{Q2}]$ is not* q-p-orthogonal for s and $\psi_2[\psi_1]$.

Furthermore *the analogue holds for p-q-orthogonality.* Hence we cannot define the wave functions of s to be those $\psi \in \mathcal{H}$ that q-p-represent and p-q-represent s with respect to some apparata $\mathcal{A}_Q$ and $\mathcal{A}_P$ respectively.

To overcome the difficulty above let us consider the solution $\psi_t^{(r)}$ of the Schrödinger equation for $\mathfrak{S}$ that initially reduces to ψ_r:

$$(16.1) \qquad i\hbar \frac{\partial \psi^{(r)}}{\partial t} = H\psi^{(r)}\,, \qquad \psi_\tau^{(r)} = \psi_r \ (r = 1, 2)\,.$$

We know that in the example above $\psi_t^{(1)}$ certainly represents correctly the state s_t of $\mathfrak{S}$ at every $t > \tau$, so that $\psi_t^{(1)}$ q-p- (and p-q-) represents s_t with respect to suitable apparata (called ideal), whereas, in general, $\psi_t^{(2)}$ will not q-p (or p-q-) represent s_t; more, it will not give us the correct statistical distribution of position for s_t.

The considerations above suggest a way to overcome the aforementioned difficulty in general. However there are exceptions. For instance, if in the same example s is a stationary state, then $\psi_t^{(2)}$ q-p-represents s_t for every $t \geqslant \tau$. To cope with such exceptional cases too we shall now take into account the possibility of varying the external forces acting on $\mathfrak{S} = (\mathfrak{S}^{(i)}, H^{(e)})$.

DEF 16.1. *We say that $\psi \, (\in \mathcal{H})$ represents the state s (possible) for $\mathfrak{S}$ at (the instant) τ with respect to (the orthogonal apparatus) $\mathcal{A}_Q$ (to measure q* on *$\mathfrak{S}^{(i)}$) in the case when, first, for $B \in \mathfrak{B}_N$*

$$\mathcal{E}_s[\chi_B(q)] = \int_B |\psi(q)|^2 dq \,, \qquad \mathcal{E}_s[\chi_B(p)] = \int_B |\hat{\psi}(p)|^2 dp \,, \tag{16.2}$$

second, ψ q-p-represents s with respect to the apparatus $\mathcal{A}_Q$, and third, if

(i) *$\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$ is any quantal system obtained from $\mathfrak{S}$ by varying the external forces after τ*—i.e. *with $H_t'^{(e)} = H_t^{(e)}$ for $t \geqslant \tau$—,*

(ii) *the process $t \to s_t$ is (physically) possible for $\mathfrak{S}'$ in $\tau^{\vdash}\infty$ and fulfils the initial condition $s_\tau = s$ (which determines it uniquely), and*

(iii) *$t \to \psi_t$ is the continuous solution of a Schrödinger equation* (5.1) *for $\mathfrak{S}'$, that fulfils the initial condition $\psi_\tau = \psi$,*

then ψ_t q-p-represents s_t for every $t \geqslant \tau$, with respect to $\mathcal{A}_Q$.

Incidentally, the first two conditions in the definiens of Def 16.1 follow from the third.

The considerations above Def. 16.1 push us to accept the following axiom.

AXIOM 16.1. *If $\psi_r \, (\in \mathcal{H})$ represents the state s for $\mathfrak{S}$ at τ with respect to (the orthogonal apparatus) $\mathcal{A}_{Qr}$ (to measure q on $\mathfrak{S}^{(i)}$) ($r = 1, 2$), then ψ_2 represents s with respect to $\mathcal{A}_{Q1}$.*

DEFINITION 16.2. *We say that ψ represents s, or is the wave-function of s, if ψ represents s with respect to some (orthogonal) apparatus $\mathcal{A}_Q$ (to measure q on $\mathfrak{S}^{(i)}$)* [Def 16.1].

DEFINITION 16.3. *The state s is called pure if it is represented by some ψ* ($\in \mathcal{H}$).

By Defs 16.1-3 Axiom 16.1 concerns only pure states.

DEFINITION 16.4. *We say that $\mathcal{A}_Q$ is orthogonal for (the pure state) s, if some ψ ($\in \mathcal{H}$) represents s with respect to $\mathcal{A}_Q$.*

Incidentally $\mathcal{A}_Q$ can be called *ideal*—in the sense of [13], p. 166—if it is orthogonal for every state s.

By Axiom 16.1 we can assert the following

THEOR 16.1. *If $\mathcal{A}_Q$ is orthogonal for s and ψ represents s, then $\mathcal{A}_Q$ is orthogonal for s and ψ.*

Remark that by Defs 15.1 and 16.1-2, Axiom 16.1 is a postulate on ψ^+. It is a weak substitute in $\mathcal{T}_1$ for Post 6.1 (von Neumann), or Post 6.2 (Lüders), or any of Posts 7.1-3, and it differs deeply from all of them in that in it the properties of the apparatus are linked essentially with the evolution of $\mathfrak{S}$, more with its evolution under varied external forces.

The same can be said of the other axioms on ψ^+ in $\mathcal{T}_1$, i.e. Axioms 17.1 and 18.1-3 below. The first is the analogue for $\mathcal{T}_1$ of Post 5.1 and allows us to complete the proof of Theor 3.1. Axioms 16.1 and 18.2 express substantially uniqueness properties, whereas axioms 18.1 and 18.3 express substantially two different existence properties.

17. Joinability and evolution axioms in $\mathcal{T}_1$ and a proof of Theor 2.1.

Remark that the definition of the *joinability* of the pure state s of $\mathfrak{S}$ with the state s' (of $\mathfrak{S}^{(i)}$) under varied external forces, given in N. 5 is meaningful also in $\mathcal{T}_1$. The same holds for Post 5.1 that involves a connex pure state. A possibly non-pure state s can be called *connex* if $B \to \mathcal{T}^{(Q)}_{\mathcal{A},\mathfrak{S},\tau,s,B}$—cf. Axiom 14.8—, which by Axiom 16.1 is a probability measure $\mu_{\mathfrak{S},s}$ on $\mathbb{R}^N$ independent of $\mathcal{A}$ and τ, has a connex support. We state Post 5.1 in $\mathcal{T}_1$.

AXIOM 17.1. *Every pure state for $\mathfrak{S}$ is joinable with a connex state.*

Given e.g. the above system q of Lagrangian co-ordinates for $\mathfrak{S}$, there is a unique natural choice for the system p of their conjugate momenta, if the external electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ vanishes. Not even in this case are the corresponding classical and relativistic Hamiltonians H_c and H completely determined. Two admissible choices of $H_c[H]$ give rise to equivalent [generally non-equivalent] canonical [Schrödinger] equations. If $(\boldsymbol{E}, \boldsymbol{H}) \not\equiv 0$, the known indetermination of the 4-potential induces an indetermination of the system p. More generally p can always be considered as determined by the system q *only* up to a canonical transformation that leaves q invariant. We shall call such q's and p's *canonical variables* for $\mathfrak{S}$.

Of course we say that the function $t \to \psi_t$ (defined on e.g. $\tau \vdash + \infty$) *represents* the process $t \to s_t$ possible for $\mathfrak{S}$ in $\tau \vdash t'$, if ψ_t represents s_t for $\tau \leqslant t \leqslant t'$ [Def 16.2]. After H has been chosen, a function $t \to \psi_t$ representing that process, generally fails to solve $\mathfrak{S}$'s Schrödinger equation (2.1). However, on the one hand, it solves (2.1) for another choice of the Hamiltonian H of $\mathfrak{S}$, and on the other hand, for every such choice the process $t \to s_t$ is represented by a solution of (2.1). In stating Axiom 17.2 below the considerations above will be kept in mind.

Let $\Gamma_{H,\tau}$ be the class formed with the mappings of $\tau \vdash + \infty$ into $L^2(\mathbb{R}^N)$ such that

(α) *$t \to \psi_t$ solves the Schrödinger equation $(2.1)_1$ for $\mathfrak{S}$ constructed with the Hamiltonian H,*

(β) *the function $(q, t) \to \psi_t(q)$ is continuous, and*

(γ) *$\partial_1 \psi_t$ to $\partial_N \psi_t$ are in $L^2(\mathbb{R}^N)$, so that they exist a.e.*

We shall say that the *existence and uniqueness theorems hold for (the quantal system) $\mathfrak{S}$ in $\Gamma_{H,\tau}$* if H is a Hamiltonian of $\mathfrak{S}$ and conditions (δ) and (ε) below hold respectively:

(δ) *For every $\psi' \in L^2(\mathbb{R}^N)$ representing a state for $\mathfrak{S}$ at some instant $t' \geqslant \tau$, $\Gamma_{H,\tau}$ contains a function $t \to \psi_t$ with $\psi_{t'} = \psi'$.*

(ε) *If two functions in $\Gamma_{H,\tau}$ have the same value at some $t' \geqslant \tau$, then they coincide (in $\tau \vdash + \infty$).*

The following axiom substantially states that $\mathfrak{S}$ obeys its Schrödinger equation and states some regularity properties of this equation.

AXIOM 17.2. *If $\mathfrak{S}$ is a quantal system (formed with distinguishable spinless particles) and a system of canonical variables has been fixed for it, then a corresponding Hamiltonian H can be chosen in such a way that* (*a*) *if* $t \rightarrow s_t$ *is a process possible for* $\mathfrak{S}$ *in* $\tau \vdash\!\dashv t'$ *for some* $t' > \tau$, *then it can be represented by a function* $t \rightarrow \psi_t$ *in* $\Gamma_{H,\tau}$, *and*

(*b*) *the existence and uniqueness theorems hold for* $\mathfrak{S}$ *in* $\Gamma_{H,\tau}$.

PROOF OF THEOR 2.1 IN $\mathcal{T}_1$ (5). Let the continuous functions ψ_1 and ψ_2 represent the (pure) state s for $\mathfrak{S}$ at τ. Hence by Def 16.2 $\psi_1[\psi_2]$ represents s with respect to some (orthogonal) apparatus $\mathcal{A}_{Q1}[\mathcal{A}_{Q2}]$ to measure q on $\mathfrak{S}^{(i)}$. Then by Axiom 16.1 both ψ_1 and ψ_2 represent s with respect to $\mathcal{A}_Q = \mathcal{A}_{Q1}$.

Since ψ_1 represents s, by Def. 16.3 s is pure; hence by Axiom 17.1 it is joinable with a connex state s'. This means that the system $\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$ can be chosen in such a way that it can undergo a process $t \rightarrow s_t$ in some interval $\tau \vdash\!\dashv t'$, for which $s_\tau = s$ and $s_{t'} = s'$. By (the regularity part of) Axiom 17.2, the Hamiltonian $H' = H^{(i)} + H'^{(e)}$ of $\mathfrak{S}'$ can be chosen in such a way that the existence and uniqueness theorems hold for $\mathfrak{S}'$ in $\Gamma_{H',\tau}$; hence conditions (α) to (ε) (above Axiom 17.2) in $\mathfrak{S}$ and H (and in equation (2.1)) hold for $\mathfrak{S}'$ and H' (and for the Schrödinger equation (5.1) of $\mathfrak{S}'$). In particular, by (δ) there is a function $t \rightarrow \psi_t^{(r)}$ in $\Gamma_{H',\tau}$ that (solves (5.1) and) fulfils the initial condition $\psi_\tau^{(r)} = \psi_r$ $(r = 1, 2)$.

At this point the validity of all the conditions (i) to (iii) in Def 16.1 has been proved for $\psi = \psi_r$ $(r = 1, 2)$. Since, in addition, ψ_r represents s with respect to $\mathcal{A}_Q$ $(= \mathcal{A}_{Q1})$, by (the third point in) Def 16.1, $t \rightarrow \psi_t^{(r)}$ *q-p*-represents s_t for every $t \geqslant \tau$ with respect to $\mathcal{A}_Q$ $(r = 1, 2)$. In particular this holds for $t = t'$. Furthermore $s_{t'}$ is the connex state s', so that by Theor 15.1 $\psi_{t'}^{(2)} = c\psi_{t'}^{(1)}$ for some $c \in \boldsymbol{C}$. In addition, since $t \rightarrow \psi_t^{(2)}$ is in $\Gamma_{H',\tau}$, so does the function $t \rightarrow c\psi_t^{(2)}$, besides $t \rightarrow \psi_t^{(1)}$. Hence the last two functions are in $\Gamma_{H',\tau}$ and have the same value at t' $(\geqslant \tau)$. Since the uniqueness condition (ε) in H holds for H', those two functions coincide in $\tau \vdash\!\dashv t'$, so that $\psi_2 = \psi_\tau^{(2)} = c\psi_\tau^{(1)} = c\psi_1$. We conclude that Theor 2.1 *holds in* $\mathcal{T}_1$. q.e.d.

18. On pure states and mixtures.

Up to now no answers can be given on the basis of $\mathcal{T}_1$ to the following questions: are there pure states? And if there are, how many are they? Therefore, first, we state the following axiom, having care

(5) This proof of Theor 2.1 in $\mathcal{T}_1$ differs considerably from the one in n. 5 (Part I) because axioms and definitions have been deeply changed.

not to require e.g. that every unit vector in $\mathcal{H}$ should represent a state of $\mathfrak{S}$.

Axiom 18.1. *The vectors of $\mathcal{H}$ determined (i.e. represented) by wave functions of (pure) states for $\mathfrak{S}$ at the instant τ, generate algebrically a possibly non-closed subspace S_τ of $\mathcal{H}$ that is dense in $\mathcal{H}$.*

This axiom (used only for Def. 19.1) holds also when superselection rules are present.

Incidentally the reading of paper [17(*a*)], where a solution of well known paradoxes in quantum mechanics is presented, pushed me to consider, briefly speaking, the following notion of pure states: s is said to be *the pure state of $\mathfrak{S}$ at t relative to (the recognition instant) t_0* ($\leqslant t$) if it is the statistical state $s^{\mathfrak{S}}_{t_0 t}$ of $\mathfrak{S}$ at t that can be recognized by the maximum set of observations made up to t_0. As far as I know this notion of state ($s^{\mathfrak{S}}_{t_0 t}$) has not yet been considered in any pubblications—not even in [17]—; and most writers identify the pure state of $\mathfrak{S}$ at t with our (*absolute*) *pure state* $s^{\mathfrak{S}}_t = s^{\mathfrak{S}}_{t_0 t}$ of $\mathfrak{S}$ at t; perhaps some of them prefer (at least at first sight) to consider $s^{\mathfrak{S}}_{t_0 t}$ as independent of t_0 ($\leqslant t$), whereas on the basis of [17] $s^{\mathfrak{S}}_t \neq s^{\mathfrak{S}}_{t_0 t}$ can be shown to hold in some important cases. Therefore references to the recognition instant t_0 are put withing square brackets in Axs 18.2-3 and some other passages, and will be omitted later.

Now it is natural to ask what can be said in $\mathcal{C}_1$ of an arbitrary state s of $\mathfrak{S}$. Of course we must postulate that s is either pure or a statistical mixture of some pure (different) states $s_1, s_2, \ldots$ [relative to the recognition instant t_0] with the respective probabilities $p_1, p_2, \ldots$. These probabilities are frequentistic, so that they depend partially on the observer's ignorance—c.f. footnote 5. Hence the assumption that $\mathfrak{S}$ is in the state s at t implies that exactly one s_i is the pure state of $\mathfrak{S}$ at t. This leads us to assert the following two axioms.

Axiom 18.2. *At every instant $\mathfrak{S}$ is in exactly one pure state [relative to the recognition instant t_0].*

Axiom 18.3. *If s is a state for $\mathfrak{S}$, there is a finite or infinite sequence of pure states $s_1, s_2, \ldots$ [relative to the recognition instant t_0] and a similar sequence of non-negative real numbers $p_1, p_2, \ldots$ with $p_1 + p_2 + \ldots = 1$, for which p_i is the probability that, if $\mathfrak{S}$ is in the state s (at τ), then $\mathfrak{S}$ is in the state s_i ($i = 1, 2, \ldots$).*

Since the notion of pure states involves ψ^+ through Defs. 15.1 and 16.1-3, and it occours in Axioms 18.2-3 as well as in Axiom 16.1, these axioms are on ψ^+. In connection with this let us emphasize that, within $\mathcal{T}_1$, « $\mathfrak{S}$ is in the pure state s » means that $\mathfrak{S}$ has had a certain preparation $\mathcal{P}$ (at an earlier instant t_0) for which there is a unit vector $\psi \in \mathcal{H}$ with the following properties:

(i) if $B \in \mathcal{B}_N$ and we measure $\chi_B(q)$ on $\mathfrak{S}$ at τ with an apparatus $\mathcal{A}_Q$ that is ideal (or at least orthogonal for s according to Def. 16.4), then the expected value $\mathcal{E}_s[\chi_B(q)]$ is $\|\theta_B\psi\|^2$—see (4.1),

(ii) if the above measurement gives the result 1 and immediately after we measure $\chi_{B'}(p)$ on $\mathfrak{S}$ with $B' \in \mathcal{B}_N$, then the expected value is $\|\theta_{B'}(\theta_B\psi)^\wedge\|^2/\|\theta_B\psi\|^2$, and

(iii) if we change the external forces acting on $\mathfrak{S}$ after τ, thus turning $\mathfrak{S}$ into $\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$, if $t' > \tau$, and if $t \to \psi_t$ is the solution (in $\Gamma_{H,\tau}$) of the Schrödinger equation (5.1) for $\mathfrak{S}'$, that fulfils the initial condition $\psi_\tau = \psi$, then the conditions (i) and (ii) in $\mathfrak{S}$, τ, and ψ hold for $\mathfrak{S}'$, t', and $\psi_{t'}$.

Note that the evolution condition (iii) is essential also for (the meaning of) connex states, whereas in Part 1 evolution properties were taken into account only to prove Theor 2.1 also for non-connex states.

Pure states are defined as states that have certain statistical properties. The existence of non-pure states is compatible with the above axioms but does not follow from them.

It is a controversial problem whether or not a pure state s_i has a (partially) subjective statistical character. The second alternative is in harmony with Born's views and is equivalent to the conditions (i) and (ii) below:

(i) O_1 is the class of all observations that we are able to make on the preparations of states for $\mathfrak{S}$, and

(ii) no observation in the class O_1, on the preparation $\mathcal{P}_i$ of the pure state s_i, can improve the predictions about the results of measurements on $\mathfrak{S}$ in s_i, that are afforded by the wave functions of s_i.

It is rather in harmony with Einstein's views to accept (ii) but to replace (i) with the condition that O_1 is a special class of observations, adding that O_1 probably has a natural but presently unknown satisfactory characterization, and that it contains all observations similar to those made up to now.

Axiom 18.3 seems to me compatible with both views (and with the existence of hidden variables), but with Born's views it is compatible in a more natural way by the following reasons. A particular performance of the preparation $\mathfrak{J}$ mentioned in Axiom 18.3 sends $\mathfrak{S}$ in some pure state s_i. On the one hand, according to Born s_i has an objective meaning. Hence that performance of $\mathfrak{J}$ is also a preparation $\mathfrak{J}_i$ of s_i. This fact is unknown at least during the performance of $\mathfrak{J}$. However a maximal set of observations could have revealed it.

On the other hand the analogue for Einstein's views of the above assertions on $\mathfrak{J}_i$ is e.g. that $\mathfrak{J}$ contains (or implies) a preparation $\mathfrak{J}'$ of $\mathfrak{S}$, that determines the values of all hidden variables related to $\mathfrak{S}$. Incidentally, not only is $\mathfrak{J}'$ ($\neq \mathfrak{J}_i$) unknown in performing $\mathfrak{J}$, but presently we see no experiments capable to measure any hidden variable. Now the main point is that we can conjecture again that during the performance of $\mathfrak{J}$ some observations could be made that, if registered, would turn this performance into one of $\mathfrak{J}_i$. However these observations cannot be defined precisely and simply any longer (they are not yet all possible observations on $\mathfrak{J}$); and they belong to or constitute the class O_1, which for this reason was said not to be satisfactorily characterizable in connection with Einstein's views.

It is worth while adding some further discussion on Einstein's point of view and axiom 18.3, because we said that states are characterized by preparations, so that the preparation $\mathfrak{J}'$ above ought to characterize an objective state s' that does not comply with axiom 18.3. We can observe that from the present point of view states must be meant in a (partially) subjective sense—cf. footnote 2: we can say e.g. that s_i is determined by the above preparation $\mathfrak{J}_i$ only if, whenever we perform $\mathfrak{J}_i$, we are aware of this, i.e. we know that we realize all defining features of $\mathfrak{J}_i$. The above $\mathfrak{J}'$ is not available now-a-days, because, even if we perform it, we cannot know this.

However, if we admit Einstein's point of view in its particular version of a theory of hidden variables—as we are doing to fix ideas—we must admit that in the future we shall be able to recognize the performances of $\mathfrak{J}'$–otherwise its objective existence could not be accepted from the scientific point of view. This considerations may push one to try and define the preparations of states to be considered in $\mathfrak{T}_1$ as those recognizable [or ideally recognizable] now-a-days. However this is too little [vague]. We conclude that we have to use axiom 18.3 itself to determine which preparations are to be considered in $\mathfrak{T}_1$. Thus we are able to characterize a priori the notion of states from the Born-Bohr point of view but not from Einstein's. Hence the theory $\mathfrak{T}_1$ has a better epistomological status—more informative content—when the first of the above points of views is accepted.

19. On observables and Born's rule in $\mathcal{T}_1$.

Positions and their conjugate momenta are primitive notions in $\mathcal{T}_1$. Especially from the operational point of view this is equivalet to assuming as undefined (in $\mathcal{T}_1$) the ideal apparata (considered explicitly in ordinary text-books) to measure them. We now want to define the (other) observables.

Let A be any bounded linear operator in $\mathcal{H}$, briefly $A \in \mathcal{B}(\mathcal{H})$. We fix a system γ of canonical variables q_h, p_h $(h = 1, \ldots, N)$ for $\mathfrak{S}$ and consider a pure state s for $\mathfrak{S}$ at τ. Since Theor 2.1 holds in $\mathcal{T}_1$, by substantially the considerations made in Part 1 in connection with (2.6), we can call $\langle A \rangle_s$—where $\langle A \rangle_s = \langle A \rangle_\psi = \langle \psi | A \psi \rangle$ for some (hence every) wave function ψ of s—*the* (*indirect*) *expectation of* (*the operator*) A *in the pure state* s (*and the system* γ *of canonical variables*).

DEF 19.1. *If* $A \in \mathcal{B}(\mathcal{H})$ *and* ω *is the above function* $s \to \langle A \rangle_s$ *connected with the system* γ *and the instant* τ, *and defined on pure states, we write* $\omega = \omega_{A,\tau} = \omega_{A,\gamma,\tau}$ *and we say that* ω *is the observable corresponding to* A (, γ, *and* τ).

Remark that in the case of a time-dependent electromagnetic field no privileged choices of γ exist and if the canonical systems γ and γ' coincide only before $\tau + 1 < \tau'$, we cannot say (in general) whether or not $\omega_{A,\gamma,\tau} (= \omega_{A,\gamma',\tau})$ and $\omega_{A,\gamma,\tau'} (\neq \omega_{A,\gamma',\tau'})$ are the same observable.

THEOR 19.1. *If* $A, B \in \mathcal{B}(\mathcal{H})$ *and* $\omega_{A,\tau} = \omega_{B,\tau}$, *then* $A = B$.

Indeed $\omega_{A,\tau} = \omega_{B,\tau}$ implies $\langle A \rangle_s = \langle B \rangle_s$ for every pure state s, hence $\langle \psi | (A - B) \psi \rangle = 0$ for every ψ in the subspace S of $\mathcal{H}$ that is generated by all wave functions. In addition S is dense in $\mathcal{H}$ by Axiom 18.1. Hence $A = B$. q.e.d.

DEF 19.2. *If* $\mathcal{A}$ *is a* (*possibly unbounded*) *self-adjoint operator in* $\mathcal{H}$, *we denote by* $\bar{\omega}_{A,\tau}$ *or* $\bar{\omega}_{A,\gamma,\tau}$, *the function* $(s, B) \to \langle \chi_B(A) \rangle_s$ *defined for* $B \in \mathcal{B}_1'$ (*B bounded*) [N. 2] *and every pure state* s (*for* $\mathfrak{S}$ *at* τ); *and we call it a* (*general*) *observable for* $\mathfrak{S}$ (*at* τ).

THEOR 19.2. *If* A_1 *and* A_2 *are self-adjoint operators in* $\mathcal{H}$, *and* $\omega_{A_1,\tau} = \omega_{A_2,\tau}$, *then* $A = B$.

Indeed, since $\bar{\omega}_{A_1,\tau} = \bar{\omega}_{A_2,\tau}$, for all $B \in \mathcal{B}_1'$ $\chi_B(A_1) = \chi_B(A_2)$; hence $A_1 = A_2$. q.e.d.

By Theors 19.1-2 we can write

$$(19.1)\qquad \begin{aligned} &A = A_{\omega,\tau}\ (\text{or } A = A_{\omega,\gamma,\tau}) &&\text{if } \omega = \omega_{A,\tau}\,,\\ &A = \bar{A}_{\omega,\tau}\ (\text{or } A = \bar{A}_{\omega,\gamma,\tau}) &&\text{if } \omega = \bar{\omega}_{A,\tau}\,. \end{aligned}$$

Furthermore, whenever $\langle A\rangle_s$ exists, we can set

$$(19.2)\qquad \langle\omega\rangle_s = \langle A\rangle_s \quad \text{for } A = A_{\omega,\tau} \textit{ or } A = \bar{A}_{\omega,\tau}\,.$$

It is natural to call this $\langle\omega\rangle_s$ *expected value* (*in the indirect sense*) of the (general or bounded) observable ω in the (pure) state s. On the basis of Axiom 18.3 we can represent every state s by a positive definite operator $\varrho = \varrho_s$ of the trace class. This ϱ is called the statistical operator of s—cf. e.g. [15] Axiom S, p. 386. When the trace $Tr\varrho A$ exists, we can extend definition (19.2) to this general case by setting, as is usually done,

$$(19.3)\qquad \langle\omega\rangle_s = TrA\varrho \qquad (\varrho = \varrho_s\,,\ A = A_\omega)\,.$$

If ω is a bounded observable, (19.2) and (19.3) are certainly meaningful. Hence by Theor 19.2 *two bounded observables* ω *and* ω' *coincide in case* $\langle\omega\rangle_s = \langle\omega'\rangle_s$ *for every state* s, which is postulate (3.3.4) in [10], p. 192.

DEF 19.3. (*a*) *If* (i) A_0 *to* A_ν *are self-adjoint operators in* $\mathcal{H}$, (ii) *they are compatible,* (iii) $A_0 = f(A_1, \dots, A_\nu)$ *for some measurable mapping* f *of* $\mathbf{R}^\nu$ *into* $\mathbf{R}$, *and* (*iv*) $\omega_\alpha = \omega_{A_\alpha}$ ($\alpha = 0, \dots, \nu$), *then we say that* $\omega_0 = f(\omega_1, \dots, \omega_\nu)$.

(*b*) *If* (i) *and* (iv) *hold for* $\nu = 2$ *and* $A_0 = A_1 + A_2$ [$= A_1 \circ A_2$ (*Jordan product*)], *then we say that* $\omega_0 = \omega_1 + \omega_2$ [$= \omega_1 \circ \omega_2$].

* * *

REMARK Post 2.1—a dual of (3.3.4) in [10]—or better its version

$$(19.4)\qquad \langle\omega\rangle_s = \langle\omega\rangle_{s'} \quad \textit{for all bounded variables } \omega,\ \textit{yields } s = s'$$

cannot be deduced from the preceding axioms of $\mathcal{T}_1$ when s is not pure. It can be included in $\mathcal{T}_1$. However *neither this postulate nor axiom* 18.2 *need be asserted in* $\mathcal{T}_1$.

Indeed (semantically) since a state s for $\mathfrak{S}$ is a short for a preparation $\mathcal{T}$ of $\mathfrak{S}$, $\mathcal{T}$ may contain superabundant features. To emphasize this, call *pre-state* what was called state up to now. Furthermore, instead of asserting Axiom 18.2, call the (pure) pre-states s and s' *E_1-equivalent* if they have the same wave functions; and call the E_1-equivalence classes *pure states. Thus Axiom* 18.2 *is a theorem in* $\mathcal{T}_1$.

Similarly, on the basis of (19.3) call the pre-states s and s' *E_2-equivalent* if $\langle\omega\rangle_s = \langle\omega\rangle_{s'}$ for every bounded observable; and call the E_2-equivalence classes *states.* Now Post 2.1 *with* e.g. $\mathcal{E}_s(\omega)$ *replaced by* $\langle\omega\rangle_s$ *is a theorem in* $\mathcal{T}_1$.

Incidentally Post 2.1 is equivalent to postulate (3.3.1) in [10], p. 192 in that this has the version: the states s and s' coincide if $\mathcal{E}_s[\chi_B(\omega)] = \mathcal{E}_{s'}[\chi_B(\omega')]$ for all bounded observables and all $B \in \mathcal{B}_1$.

20. On fundamental observables in $\mathcal{T}_1$ and related notions.

Strictly speaking the subject briefly dealt with in this section does not belong to quantum mechanics but rather to measure theory. It is essentially based on the general notion of apparatus (App)—see (2) in N. 13.

The first two definitions below are preliminaries to define, in $\mathcal{T}_1$, fundamental observables for $\mathfrak{S}$ at the instant τ. Two versions of this notion, of increasing strengths are considered. It is worth while writing these definitions explicitly because it is useful to give examples how to deal in $\mathcal{T}_1$ with notions belonging to measure theory, and in particular because certain possibility assumptions or axioms written explicitly below are usually implicit in ordinary treatments.

Def 20.1. *Assume that A_1 to A_n are self-adjoint operators, that $\mathcal{A}$ is an apparatus to measure v (scalar) magnitudes on $\mathfrak{S}$ and that, for $B \in \mathcal{B}_v$ and every state s for $\mathfrak{S}$ at τ.*

(i) *we have the probability $\mathcal{T}_{\mathcal{A},\mathfrak{S},s}$ that if, at τ, $\mathfrak{S}$ has the state s and $\mathcal{A}$ is used on $\mathfrak{S}$ and B, then the result is in B, and*

(ii) *this probability is the measure μ on $\mathbf{R}^v$ for which*

$$(20.1) \qquad \mu(B_1 \times \ldots \times B_v) = \prod_{r=1}^{v} \langle \chi_{B_r}(A_r) \rangle_s \qquad \textit{for all } B_1, \ldots, B_v \in \mathcal{B}_1'.$$

Then we say that $\mathcal{A}$ *is an apparatus to measure (simultaneously)* $\bar{\omega}_{A_1,\tau}$ *to* $\bar{\omega}_{A_\nu,\tau}$ *on* $\mathfrak{S}$.

By von Neumann's theorem, if A_1 to A_ν are commuting self-adjoint operators, then there is a self-adjoint operator A and $\nu + 1$ measurable functions $f_1, \dots, f_\nu, f \in \mathbb{R}_{\mathbb{R}}$ for which

$$(20.2) \qquad A = f(A_1, \dots, A_\nu) \quad A_i = f_i(A) \qquad (i = 1, \dots, \nu);$$

hence an apparatus $\mathcal{A}$ can (i.e. is to) measure (simultaneously) $\bar{\omega}_{A_1,\tau}$ to $\bar{\omega}_{A_\nu,\tau}$ [Def. 20.1] iff it can measure $\bar{\omega}_{A,\tau}$. This remark allows us to simplify our speach and in particular Def 20.1. Its version above was written explicitly, because in many physically interesting cases—e.g. when $\nu = N = 3n$ and $A_i = Q_i$ $(i = 1, \dots, N)$—the above operator A has a non-practical physical meaning.

DEF 20.2. *Assume that* (i) *for* $t \in \tau \vdash\!\dashv \tau_1$ A_t *is a self-adjoint operator*, (ii) $\mathcal{A}$ *is an apparatus to measure* $\omega_t = \bar{\omega}_{A_t,t}$ *on* $\mathfrak{S}$ *for* $t \in \tau \vdash\!\dashv \tau_1$, (iii) *we can use* $\mathcal{A}$ *on* $\mathfrak{S}$ *twice, at* τ *and at* $t \in \tau \dashv \tau_1$ *without performing other measurements on* $\mathfrak{S}$, (iv) $p_{s,\tau,t}$ *is the probability that, if* $\mathfrak{S}$ *has the state s at* τ *and the result of the use of* $\mathcal{A}$ *at* τ *is in* B, *then the result of the use of* $\mathcal{A}$ *at t is again in* B, *and* (v) *for every state s for* $\mathfrak{S}$ *at* τ

$$(20.3) \qquad \lim_{t \to \tau^+} p_{s,\tau,t} = 1 .$$

Then we say that $\mathcal{A}$ *is an apparatus of the first kind to measure* ω_τ *on* $\mathfrak{S}$, *or that* $\mathcal{A}$ *can perform measurements of* ω_τ *on* $\mathfrak{S}$, *of the first kind* —cf. [13], p. 165.

In the literature possibility conditions such as (iii) in Def 20.2. are usually lacking. An outstanding example of this occurred in the axiomatization of classical mechanics according to *E.* Mach and P. Painlevé—see e.g. [1]—, whereas, from a rigorous point of view in several physical theories possibility assumptions or postulates are as relevant as existence postulates in Euclidean geometry.

Let us add that possibility conditions such as (iii) in Def. 20.2 are likely not to be written ordinarily because the following axiom is accepted, again implicitly.

AXIOM 20.1. *If* (i) $\tau_1 < \tau_2 < \dots$, (ii) $B_1, B_2, \dots \in \mathfrak{B}_1$, (iii) $A_1, A_2, \dots$ *are self-adjoint operators, and* (iv) $\mathcal{A}_i$ *is an apparatus to measure* $\omega_i = \bar{\omega}_{A_i,\tau}$ *on the quantal system* $\mathfrak{S}$ $(i = 1, 2, \dots)$, *then we can use* $\mathcal{A}_i$ *on* $\mathfrak{S}$ *and* B_i $(1 = 1, 2, \dots)$ *and help making other measurements on* $\mathfrak{S}$.

If this axiom is accepted, Def 20.1 and many other definitions in measure theory can be simplified.

DEF 10.3. *We say that the general observable* $\omega = \bar{\omega}_{A,\tau}$ *is fundamental for* $\mathfrak{S}$ *at* τ [*fundamental for* $\mathfrak{S}$], *briefly* $\omega \in FOb_{\mathfrak{S},\tau}$ [$\omega \in FOb_{\mathfrak{S}}$], *if there is some apparatus* $\mathcal{A}$ *to measure it at* τ [*if* $\omega \in FOb_{\mathfrak{S},\tau}$ *for every* τ].

Remark that e.g. axiom 14.8 directly implies that, by (20.2) with $A_i = Q_i$ or $A_i = P_i$, for $\tau \in \mathcal{A}$ and every measurable $f \in \mathbb{R}^{\mathbb{R}}$, $\bar{\omega}_{A,\tau}$ is a fundamental observable (for $\mathfrak{S}$ at τ) in a certain strong sense. Since this was essential in proving our main theorems, we now hint at a corresponding definition of *strong apparatus to measure* $\omega = \bar{\omega}_{A,\tau}$. We say that $\mathcal{A}$ is such an apparatus in case, for every choice of $H'^{(e)}$ with $H_t'^{(e)} = H_t^{(e)}$ for $t \leqslant \tau$, for every $B \in \mathcal{B}_1'$, and for every state s for $\mathfrak{S}$ at τ—which by Axiom 14.7 is also a state for $\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$ at τ—, $\langle \chi_B(A) \rangle_s$ is the probability that if, at τ, $\mathfrak{S}'$ has the state s and $\mathcal{A}$ is used on $\mathfrak{S}'$ and B, then the result is in B.

The corresponding strengthenings of Defs 10.2-3 give us the notions of *strong apparatus of the first kind* to measure ω_τ on $\mathfrak{S}$ and *strong fundamental observable for* $\mathfrak{S}$ (at τ)—briefly $SFOb_{\mathfrak{S},\tau}$ (or $SFOb_{\mathfrak{S}}$).

Remark that in case A_1 to A_ν are commuting self-adjoint operators, $\bar{\omega}_{A_i,\tau} \in SFOb_{\mathfrak{S},\tau}$ ($i = 1, \dots, \nu$), and $(20.2)_1$ holds, the preceding postulates do not imply that $\omega_{A,\tau} \in FOb_{\mathfrak{S},\tau}$ for f bounded [Defs, 19.1-2].

A definition of apparata for *ideal measurements* of $\omega = \bar{\omega}_{A,\tau}$ (on $\mathfrak{S}$ at τ) of the first kind—cf. [13] p. 166—need not be written explicitly in $\mathcal{T}_1$. Likewise, on the basis of the preceding framework it is a matter of routine to define other notions of measure theory, e.g. the one of compatible observables in the sense of [10], p. 195.

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Manoscritto pervenuto in redazione il 26 febbraio 1979.