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Minkowski Asymptotic Spaces in General Relativity.

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SUMMARY - In this note a class of asymptotically Minkowskian space-times is defined rigorously by means of certain conditions endowed with a direct physical meaning. On the one hand these conditions are not too restrictive—for instance they do not limit us to particular cases such as the stationary one—, on the other hand they are sufficiently strong to imply the existence (of the right number) of *asymptotic inertial spaces* whose proof is the main aim of this note.

1. Introduction.

In the classical gravitational theory we consider a system of masses confined to a bounded region of Euclidean space, such as, for example, the solar system taken in isolation. However complicated the distribution of masses may be, we can always say that the gravitational force and the gravitational potential approach zero asymptotically. This concept of a physical gravitating system—often called an *insular* system—is obviously noteworthy in any theory of gravitation unless we are concerned with the cosmological universe.

In relativity the gravitational field is described by the Riemannian geometry of space-time, and its absence, or rather negligibility, is

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expressed by the flatness of the Minkowski space. Thus we come to assume that space-time is « asymptotically Minkowksian » (1).

The first aim of this note is to define this concept rigorously in the sense of determining a class of space-times, to be called *μ-Minkowskian*, satisfying certain conditions endowed with a direct physical meaning. On the one hand these conditions are not too restrictive (for instance they do not limit us to particular cases such as the stationary one); on the other hand they are sufficiently strong to imply the existence of *asymptotic inertial spaces* that are in a one-to-one correspondence with the inertial spaces of special relativity whose proof is the second and main aim of this note.

Bressan pointed out that up to now one lacks any natural absolute concept of event point in general relativity—see [7] and N. 9 in [2]. Moreover he showed that this concept would be available if something like Fock's conjecture on the existence of inertial spaces in general relativity (in a suitable sense) were proved—see N. 10 in [2] and N. 93 in [4].

Hopefully, among other things, this note will be useful in working on the problem of inertial spaces and hence in determining the aforementioned absolute concept.

* * *

In this note a space-time S_4 is called μ-Minkowskian if certain frames called μ-Minkowskian exist in it—Def 5.1. In Def 5.1 the aforementioned classical behaviour of the gravitational potential and of the gravitational force is « translated » in terms of suitable properties of the components of the metric tensor and the Christoffel symbols. In addition, each μ-Minkowskian frame satisfies a condition related to the 4-velocity field associated to it—subclause (i′) in Def 5.1. This property assures the boundedness of certain fields which are to be considered; it is defined in N. 3 where its main implications, used later repeatetly, are developed (2).

(1) Incidentally we might point out that the aforementioned boundedness of the region occupied by the gravitational sources in the classical theory cannot be translated into a corresponding property in general relativity where electromagnetic radiation is also a source of gravity.

(2) This property is relative to a suitable region of space-time. In N. 3 we consider it with respect to an arbitrary region.

In N. 8 asymptotic Minkowski spaces, asymptotic Minkowski frames, and asymptotic inertial spaces are introduced; and these spaces are proved to be ∞^3—Theor 8.2. Let us give more detail about this conclusion. Consider a space-time which admits a class of frames such that if (x) and (x') belong to it, then along every asymptotic geodesic—in the sense of (h) in N. 4—the limits of the components, in both frames, of the metric tensor coincide with the diagonal matrix $(-1, 1, 1, 1)$, and the limit of the field $(\partial x'^{\alpha}/\partial x^{\beta})$ is a Lorentz matrix. Therefore we can say that (x) and (x') are asymptotically equivalent if $(\partial x'^{\alpha}/\partial x^{\beta})$ approaches the identity matrix in the limit along every asymptotic geodesic, and we can call the equivalence classes of this relation asymptotic Minkowski frames. Consequently the asymptotic inertial spaces are picked out.

Using our (μ-Minkowskian) conditions, the limit of $(\partial x'^{\alpha}/\partial x^{\beta})$ is not only a Lorentz matrix but it is also independent of the asymptotic geodesic on which it is calculated. This result—Theor. 7.3—makes it possible to think of asymptotic mutual velocities of the ideal fluids $\mathscr{F}$ and $\mathscr{F}'$ to which (x) and (x') are joined respectively. In fact, in our case, the limits of the components in (x) of the 4-velocity of $\mathscr{F}'$ relative to $\mathscr{F}$ exist and are independent of the particular asymptotic geodesic on which they are calculated—Remark in N. 8.

If the thesis of Theor 7.3 were not valid, the existence of a single asymptotic velocity would not be guaranteed and the number of asymptotic inertial spaces would be much greater than ∞^3 (and therefore they could not be called in this way).

Lastly we observe that in N. 2 several concepts, well known from the literature on relativity—see e.g. [3]—, are referred to, and in [6] the validity of our hypothesis in the case of the Schwarzschild universe is verified.

2. Preliminaries on space-time.

Let the functions $x^{\alpha} = x^{\alpha}(\mathscr{E})$, with $\alpha \in \{0, 1, 2, 3\}$ (3), define the *co-ordinate system* (x) in the *space-time* S_4 whose arbitrary point—or *event*—is denoted by $\mathscr{E}$.

As is well known, S_4 is a Riemannian manifold of four dimensions

(3) Latin [Greek] indices range from 1 [0] to 3, and Einstein's convention on dummy indices is used.

whose *metric tensor* $g_{\alpha\beta}$ is pointwise reducible to the diagonal matrix $(-1, 1, 1, 1)$ called the Minkowski matrix.

We denote the scalar product of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, defined at $\mathscr{E}$, by $(\boldsymbol{v}\cdot\boldsymbol{w})(\mathscr{E})$, and the norm of $\boldsymbol{v}$ by $\|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})$.

For the infinitesimal vector $\boldsymbol{dx}$, we set $ds = (e\, \boldsymbol{dx}\cdot\boldsymbol{dx})^{\frac{1}{2}}$, where e, the indicator, equals -1 for $\boldsymbol{dx}$ time-like and 1 for $\boldsymbol{dx}$ space-like.

It can be easily shown that

LEMMA. *The scalar product of two time-like future-oriented unit vectors is less than or equal to* -1.

We shall consider a congruence of ∞^3 regular time-like lines as the motion of an *ideal fluid* $\mathscr{F}$.

We say that the coordinate system (x) is joined to the ideal fluid $\mathscr{F}$ if

(i) the coordinate lines $x^0 = \text{var.}$ of (x) coincide with the world lines for the elements of $\mathscr{F}$;

(ii) the variable x^0 increases toward the future along every coordinate line $x^0 = \text{var.}$;

(iii) the coordinate lines $x^i = \text{var.}$, with $i \in \{1, 2, 3\}$ (3), are space-like.

We say that a coordinate system is *physically admissible* if it is joined to an ideal fluid.

We assume that it is possible to define in S_4 a single system of physically admissible coordinates onto $\mathbf{R}^4$, and we call such systems *frames* (*of reference*).

It is well known that the physical admissibility of (x) is equivalent to the validity of the inequalities

$$g_{00} < 0\,, \qquad g_{ij}\omega^i\omega^j > 0\,, \tag{2.1}$$

where ω^1, ω^2, and ω^3 are any three numbers, not all zero.

Two frames (x) and (x') are joined to the same ideal fluid $\mathscr{F}$ if they are related by a transformation of the form

$$x'^0 = x'^0(x^0, x^1, x^2, x^3)\,, \qquad x'^i = x'^i(x^1, x^2, x^3)\,,$$

called an *internal transformation.*

The ideal fluid $\mathscr{F}$ is considered to be uniquely determined by the congruence above and hence by the field of the unit vectors $\boldsymbol{\gamma}$ tangent to the world-lines for its elements and pointing into the future. If the frame (x) is joined to $\mathscr{F}$

(2.2) $$\gamma^0 = (-g_{00})^{-\frac{1}{2}}\,, \qquad \gamma^i = 0\,.$$

Given the ideal fluid $\mathscr{F}$, for each vector $\boldsymbol{v}$ we set

$$\overset{\parallel}{\boldsymbol{v}} = (-\boldsymbol{v}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma} \quad \text{and} \quad \overset{\perp}{\boldsymbol{v}} = \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}\,,$$

where $\overset{\parallel}{\boldsymbol{v}}[\overset{\perp}{\boldsymbol{v}}]$ is called *the temporal [spatial] projection of* $\boldsymbol{v}$. The norm of $\overset{\parallel}{\boldsymbol{v}}[\overset{\perp}{\boldsymbol{v}}]$ is called *the temporal [spatial] norm of* $\boldsymbol{v}$, and is denoted by $\|\boldsymbol{v}\|_{\boldsymbol{\gamma}}[\|\boldsymbol{v}\|_{\perp\boldsymbol{\gamma}}]$. Obviously

(2.3) $$\|\boldsymbol{v}\|_{\boldsymbol{\gamma}} = -(\boldsymbol{\gamma}\cdot\boldsymbol{v})^2 \leqslant 0\,, \qquad \|\boldsymbol{v}\|_{\perp\boldsymbol{\gamma}} = \|\boldsymbol{v}\| - \|\boldsymbol{v}\|_{\boldsymbol{\gamma}} \geqslant 0\,.$$

Furthermore we set

(2.4) $$|\boldsymbol{v}|_{\boldsymbol{\gamma}} \equiv (-\|\boldsymbol{v}\|_{\boldsymbol{\gamma}})^{\frac{1}{2}} = |\boldsymbol{\gamma}\cdot\boldsymbol{v}|\,, \qquad |\boldsymbol{v}|_{\perp\boldsymbol{\gamma}} \equiv (\|\boldsymbol{v}\|_{\perp\boldsymbol{\gamma}})^{\frac{1}{2}}\,.$$

Let c be the speed of light in vacuo. If $\boldsymbol{dx}$ is future-oriented and tangent to the world-line of a moving point, then $\boldsymbol{dx}/dT$, where $dT = -c^{-1}(\boldsymbol{\gamma}\cdot\boldsymbol{dx})$, is called its *relative 4-velocity*, and the spatial projection of $\boldsymbol{dx}/dT$ is called its *relative standard velocity*. Obviously $v = |\boldsymbol{dx}/dT|_{\perp\boldsymbol{\gamma}}$ is called the *relative standard speed* of the moving point (and it equals c along a null world-line). Along a time-like world-line

(2.5) $$-(\boldsymbol{\gamma}\cdot\boldsymbol{dx}/ds) = (1 - v^2/c^2)^{-\frac{1}{2}}$$

* * *

We shall denote

(i) the Christoffel symbols of the second kind by $\left\{\begin{matrix}\alpha\\ \beta\;\sigma\end{matrix}\right\}$;

(ii) the components of the Minkowski matrix $(-1, 1, 1, 1)$ by $m_{\alpha\beta}$ $(= m^{\alpha\beta})$;

(iii) a regular geodesic joining the events $\mathscr{E}_0$ and $\mathscr{E}_1$ by $l[\mathscr{E}_0, \mathscr{E}_1]$ (or briefly by l); and, if $l[\mathscr{E}_0, \mathscr{E}_1]$ is oriented, $\mathscr{E}_0$ is its origin.

3. Ideal fluids with bounded intrinsic speed-oscillation in a space-time region.

Consider the ideal fluid $\mathscr{F}$ in the space-time S_4, and the field of unit vectors $\boldsymbol{\gamma}$ tangent to the world-lines for the elements of $\mathscr{F}$ and pointing into the future.

For any given space-like geodesic $l[\mathscr{E}_0, \mathscr{E}_1]$, let $\boldsymbol{u}_{\mathscr{E}_0}(\mathscr{E}_1)$ $[\boldsymbol{u}_{\mathscr{E}_1}(\mathscr{E}_0)]$ represent the vector obtained in $\mathscr{E}_1[\mathscr{E}_0]$ by parallel transport of $\boldsymbol{\gamma}(\mathscr{E}_0)$ $[\boldsymbol{\gamma}(\mathscr{E}_1)]$ along $l[\mathscr{E}_0, \mathscr{E}_1]$. By well known properties of parallel transport $\boldsymbol{u}_{\mathscr{E}_0}(\mathscr{E}_1)$ and $\boldsymbol{u}_{\mathscr{E}_1}(\mathscr{E}_0)$ are time-like future-oriented unit vector, and

$$(\boldsymbol{u}_{\mathscr{E}_0}\cdot\boldsymbol{\gamma})(\mathscr{E}_1) = (\boldsymbol{u}_{\mathscr{E}_1}\cdot\boldsymbol{\gamma})(\mathscr{E}_0)\,.$$

Let

$$\gamma(l) = -(\boldsymbol{u}_{\mathscr{E}_1}\cdot\boldsymbol{\gamma})(\mathscr{E}_0)\,. \tag{3.1}$$

By the Lemma in N. 2

$$\gamma(l) \geqslant 1\,. \tag{3.2}$$

DEFINITION 3.1. – *For the ideal fluid $\mathscr{F}$ and the space-like geodesic $l[\mathscr{E}_0, \mathscr{E}_1]$ consider*

$$v(l) = [\gamma^2(l) - 1]^{\frac{1}{2}} c/\gamma(l)\,. \tag{3.3}$$

We call $v(l)$ the intrinsic difference, along l (4), of the speeds of $\mathscr{F}$ in $\mathscr{E}_0$ and $\mathscr{E}_1$.

By (3.2) $0 \leqslant v(l) < c$. Moreover we see at once that $v(l)$ is the standard speed—see N. 2—of the ideal fluid $\mathscr{F}$, at one of the two endpoints of l, relative to the observer represented there by the unit vector obtained there by parallel transport (along l) of the value taken by $\boldsymbol{\gamma}$ at the other endpoint.

DEFINITION 3.2. *In connection with the ideal fluid $\mathscr{F}$ and the region $\mathfrak{D}$ of S_4, consider*

$$\Delta v(\mathfrak{D}) = \sup\{v(l) : l \in L(D)\}\,, \tag{3.4}$$

where $L(D)$ is the set of the space-like regular geodesics contained by D

(4) This is not necessarily the only geodesic joining $\mathscr{E}_0$ and $\mathscr{E}_1$.

and having endpoints. We call $\Delta v(\mathfrak{D})$ the intrinsic speed-oscillation of $\mathscr{F}$ in $\mathfrak{D}$.

DEFINITION 3.3. *We say that the ideal fluid $\mathscr{F}$ has a bounded intrinsic speed-oscillation in $\mathfrak{D}$—briefly is $\mathfrak{D}$-BISO—if there exists a constant, V, with $0 \leqslant V < c$, such that*

$$\Delta v(\mathfrak{D}) \leqslant V . \tag{3.5}$$

Let $\Gamma = [1 - V^2/c^2]^{-\frac{1}{2}}$. From (3.3) and (3.4) it follows that (3.5) is equivalent to

$$\gamma(l) \leqslant \Gamma , \qquad \forall l \in L(\mathfrak{D}) \text{ (and } \Gamma \geqslant 1) . \tag{3.6}$$

THEOREM 3.1. *Let the ideal fluid $\mathscr{F}$ be $\mathfrak{D}$-BISO in that (3.6) holds. Then for any space-like geodesic l in $\mathfrak{D}$, not necessarily bounded, but having an origin, $\mathscr{E}_0$, and for any vector field $\boldsymbol{v}$ undergoing parallel transport along l*

$$|\boldsymbol{v}(\mathscr{E})|_{\boldsymbol{\gamma}} \leqslant \Gamma\big(|\boldsymbol{v}(\mathscr{E}_0)|_{\boldsymbol{\gamma}} + |\boldsymbol{v}(\mathscr{E}_0)|_{\perp\boldsymbol{\gamma}}\big) , \qquad \forall \mathscr{E} \in l\text{—see (2.4)} . \tag{3.7}$$

PROOF. Consider the space-like geodesic $l \subseteq \mathfrak{D}$. Let $\mathscr{E}_0$ be its origin, and let $\boldsymbol{v}$ be a vector field transported by parallelism along it. Moreover let $\mathscr{E}$ denote the arbitrary point of l.

Case 1. $\boldsymbol{v}$ is parallel to $\boldsymbol{\gamma}$ at $\mathscr{E}_0$, i.e. $\boldsymbol{v}(\mathscr{E}_0) = -(\boldsymbol{v}\cdot\boldsymbol{\gamma})(\mathscr{E}_0)\boldsymbol{\gamma}(\mathscr{E}_0)$. Let $\boldsymbol{u}(\mathscr{E})$ be the unit vector obtained by parallel transport of $\boldsymbol{\gamma}(\mathscr{E}_0)$ along l. Then $\boldsymbol{v}(\mathscr{E}) = -(\boldsymbol{v}\cdot\boldsymbol{\gamma})(\mathscr{E}_0)\,\boldsymbol{u}(\mathscr{E})$, which yields (3.7) by (3.6).

Case 2. $|\boldsymbol{v}(\mathscr{E}_0)|_{\perp\boldsymbol{\gamma}} \neq 0$. Let $\boldsymbol{c}(\mathscr{E}_0) = \{[\boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\gamma})\boldsymbol{\gamma}]/|\boldsymbol{v}|_{\perp\boldsymbol{\gamma}}\}(\mathscr{E}_0)$, and let $\boldsymbol{c}(\mathscr{E})$ be the unit vector obtained from it by parallel transport along l. Then

$$\boldsymbol{v}(\mathscr{E}) = -(\boldsymbol{v}\cdot\boldsymbol{\gamma})(\mathscr{E}_0)\boldsymbol{u}(\mathscr{E}) + |\boldsymbol{v}(\mathscr{E}_0)|_{\perp\boldsymbol{\gamma}}\boldsymbol{c}(\mathscr{E}) . \tag{3.8}$$

Let $\{\boldsymbol{c}_{(\alpha)}\}_{\alpha=0,1,2,3}$, with $\boldsymbol{c}_{(0)} = \boldsymbol{u}$ and $\boldsymbol{c}_{(1)} = \boldsymbol{c}$, be an orthonormal tetrad at $\mathscr{E}$. With respect to it $-1 = \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = m^{\alpha\beta}[\boldsymbol{\gamma}\cdot\boldsymbol{c}_{(\alpha)}][\boldsymbol{\gamma}\cdot\boldsymbol{c}_{(\beta)}]$. Therefore $|\boldsymbol{\gamma}\cdot\boldsymbol{c}| < |\boldsymbol{\gamma}\cdot\boldsymbol{u}|$, which, together with (3.8) yields $|\boldsymbol{v}\cdot\boldsymbol{\gamma}|(\mathscr{E}) \leqslant [|\boldsymbol{v}\cdot\boldsymbol{\gamma}|(\mathscr{E}_0) + |\boldsymbol{v}(\mathscr{E}_0)|_{\perp\boldsymbol{\gamma}}]\,|\boldsymbol{u}\cdot\boldsymbol{\gamma}|(\mathscr{E})$. By the last inequality and by (3.6), the theorem is proved. q.e.d.

If $\boldsymbol{v}$ is a time-like or null vector, $|\boldsymbol{v}|^2_{\perp\boldsymbol{\gamma}} - |\boldsymbol{v}|^2_{\gamma} \leqslant 0$, (3.7) yields

$$|\boldsymbol{v}\cdot\boldsymbol{\gamma}|(\mathscr{E}) \leqslant 2\Gamma|\boldsymbol{v}\cdot\boldsymbol{\gamma}|(\mathscr{E}_0) \qquad \big(\text{for } \|\boldsymbol{v}\| \leqslant 0\big) . \tag{3.7'}$$

In the following we are also concerned with the case where $\boldsymbol{v}$ is a space-like unit vector. For such vectors (3.7) leads to

$$(3.7'') \qquad |\boldsymbol{v}\cdot\boldsymbol{\gamma}|(\mathscr{E})\leqslant\Gamma(2|\boldsymbol{v}\cdot\boldsymbol{\gamma}|(\mathscr{E}_0)+1) \quad (\text{for } \|\boldsymbol{v}\|=1)\,.$$

LEMMA 3.2. *If $\boldsymbol{a}$ is a time-like unit vector and $\boldsymbol{b}_{(i)}$—$i=1,2$—are non-space-like vectors (all defined at $\mathscr{E}$), then*

$$(3.8) \qquad |\boldsymbol{b}_{(1)}\cdot\boldsymbol{b}_{(2)}|\leqslant 2|\boldsymbol{b}_{(1)}\cdot\boldsymbol{a}||\boldsymbol{b}_{(2)}\cdot\boldsymbol{a}|\,.$$

PROOF. Decomposing $\boldsymbol{b}_{(1)}$ into the sum $-(\boldsymbol{b}_{(i)}\cdot\boldsymbol{a})\boldsymbol{a}+\overset{\perp}{\boldsymbol{b}}_{(i)}$, where $\overset{\perp}{\boldsymbol{b}}_{(i)}$ is orthogonal to $\boldsymbol{a}$, we have

$$(3.10) \qquad (\boldsymbol{b}_{(1)}\cdot\boldsymbol{b}_{(i)})=-(\boldsymbol{b}_{(1)}\cdot\boldsymbol{a})(\boldsymbol{b}_{(2)}\cdot\boldsymbol{a})+\overset{\perp}{\boldsymbol{b}}_{(1)}\cdot\overset{\perp}{\boldsymbol{b}}_{(2)}\,.$$

If we prove that $|\overset{\perp}{\boldsymbol{b}}_{(1)}\cdot\overset{\perp}{\boldsymbol{b}}_{(2)}|\leqslant|\boldsymbol{b}_{(1)}\cdot\boldsymbol{a}||\boldsymbol{b}_{(2)}\cdot\boldsymbol{a}|$, from (3.10) we have the proof by increasing the modulus of the sum to the sum of the moduli.

In the subspace orthogonal to the time-like vector $\boldsymbol{a}$, the Cauchy-Schwarz inequality holds. Therefore $|\overset{\perp}{\boldsymbol{b}}_{(1)}\cdot\overset{\perp}{\boldsymbol{b}}_{(2)}|\leqslant|\overset{\perp}{\boldsymbol{b}}_{(1)}||\overset{\perp}{\boldsymbol{b}}_{(2)}|$, which, together with the hypothesis $\|\boldsymbol{b}_{(i)}\|\leqslant 0$, gives the proof. q.e.d.

In the following theorems $\boldsymbol{\gamma}[\boldsymbol{\gamma}^{(\prime)}]$ is the field of future-oriented unit vectors tangent to the world-lines for the elements of the ideal fluid $\mathscr{F}[\mathscr{F}']$.

THEOREM 3.3. *If $\mathscr{F}$ is $\mathfrak{D}$-BISO, a sufficient condition for $\mathscr{F}'$ to be $\mathfrak{D}$-BISO is that the standard speed of the two ideal fuids with respect to one another, be bounded in $\mathfrak{D}$ by a constant less than c.*

PROOF. By hypothesis and by (2.5), for some constant $K\geqslant 1$

$$(3.11) \qquad -[\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}^{(\prime)}](\mathscr{E})\leqslant K\,, \qquad \forall\mathscr{E}\in\mathfrak{D}\,.$$

Let us consider $l[\mathscr{E}_0,\mathscr{E}_1]\in L(\mathfrak{D})$—see Def. 3.2. By Theor 3.1—which holds for the $\mathfrak{D}$-BISO fluid $\mathscr{F}$—by (3.7′), and by (3.11)

$$|\boldsymbol{u}^{(\prime)}_{\mathscr{E}_0}\cdot\boldsymbol{\gamma}|(\mathscr{E}_1)\leqslant 2\Gamma|\boldsymbol{\gamma}\cdot\boldsymbol{\gamma}^{(\prime)}|(\mathscr{E}_0)\leqslant 2\Gamma K\,,$$

where $\boldsymbol{u}^{(\prime)}_{\mathscr{E}_0}$ is the unit vector obtained by parallel transport of $\boldsymbol{\gamma}^{(\prime)}(\mathscr{E}_0)$

along l. By Lemma 3.2, (3.11), and the preceding inequality

$$\gamma^{(')}(l) \equiv |\boldsymbol{u}_{\mathscr{E}_0}^{(')} \cdot \boldsymbol{\gamma}^{(')}|(\mathscr{E}_1) \leqslant 2|\boldsymbol{u}_{\mathscr{E}_0}^{(')} \cdot \boldsymbol{\gamma}|(\mathscr{E}_1)) \cdot |\boldsymbol{\gamma}^{(')} \cdot \boldsymbol{\gamma}|(\mathscr{E}_1) \leqslant 2\Gamma K^2 .$$

Setting $\Gamma' = 2\Gamma K^2$, we have

$$(3.6') \qquad \gamma^{(')}(l) \leqslant \Gamma' \quad \forall l \in L(\mathfrak{D}) . \quad \text{q.e.d.}$$

In N. 7 we shall need the following theorem which involves two regions $\mathcal{A}$ and $\mathfrak{D}$ satisfying

CONDITION 3.1. *For some compact subset $\mathfrak{B}$ of $\mathcal{A}$, any $\mathscr{E} \in \mathcal{A}$ is joined to (at least one) $\hat{\mathscr{E}} \in \mathfrak{B}$ by the space-like geodesic $l[\hat{\mathscr{E}}, \mathscr{E}]$ contained by $\mathfrak{D}$. Furthermore $A \subseteq \mathfrak{D}$.*

THEOREM 3.4. *Let $\mathscr{F}$ and $\mathscr{F}'$ be ideal* $\mathfrak{D}$-BISO *fluids, and let $\mathcal{A}$ and $\mathfrak{D}$ satisfy the condition above. Then the standard speed of the two fluids with respect to one another is bounded in $\mathcal{A}$ by a constant less than c.*

PROOF. Formulae (3.6) and (3.6′)—in the proof of Theor 3.3—hold by hypothesis. Let $\mathfrak{B}$ be a compact subset of $\mathcal{A}$ with respect to which Condition 3.1 holds for $\mathcal{A}$ and $\mathfrak{D}$. Then, since $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{(')}$ is continuous, for some constant K_1

$$(3.12) \qquad |\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{(')}|(\mathscr{E}) \leqslant K_1 \quad \forall \mathscr{E} \in \mathfrak{B} .$$

By Condition 3.1, each $\mathscr{E} \in \mathcal{A}$ is joined to some $\hat{\mathscr{E}} \in \mathfrak{B}$ by some space-like geodesic l. By Theor 3.1, formula (3.7) in $\mathscr{F}$ holds for $\mathscr{F}'$. Hence, by the instance (3.7′) of this—with respect to $\mathscr{F}'$ and where Γ is substituted by the constant Γ' in (3.6′)—and by (3.12)

$$|\boldsymbol{u}_{\mathscr{E}_0} \cdot \boldsymbol{\gamma}^{(')}|(\mathscr{E}) \leqslant 2\Gamma' |\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{(')}|(\hat{\mathscr{E}}) \leqslant 2\Gamma' K_1 .$$

Finally, by Lemma 3.2, (3.6) and the preceding inequality

$$|\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{(')}|(\mathscr{E}) \leqslant 2|\boldsymbol{\gamma} \cdot \boldsymbol{u}_{\mathscr{E}_0}|(\mathscr{E})|\boldsymbol{\gamma}^{(')} \cdot \boldsymbol{u}_{\mathscr{E}_0}|(\mathscr{E}) = 2\gamma(l)|\boldsymbol{\gamma}^{(')} \cdot \boldsymbol{u}_{\mathscr{E}_0}|(\mathscr{E}) \leqslant 4\Gamma\Gamma' K_1 \equiv K$$

where K is independent of $\mathscr{E} \in \mathcal{A}$. Therefore (2.5), with $\boldsymbol{\gamma}^{(')} = \boldsymbol{dx}/ds$, yields the thesis. q.e.d.

4. Preliminaries for μ-Minkowskian space-time.

The following assumptions and conventions will be in force.

(a) Any two (distinct) events $\mathscr{E}_0$ and $\mathscr{E}_1$ in space-time S_4 are joined by at least one geodesic $l[\mathscr{E}_0, \mathscr{E}_1]$.

(b) Given a frame, (x), and a real number, a, we denote the hypersurface $x^0 = a$ by $\Sigma(a)$ (and in connection with (x') we use $\Sigma'(a)$).

THEOREM 4.1. *Any two events of $\Sigma(a)$ are joined by a space-like geodesic.*

PROOF. Consider a geodesic joining them (which exists by assumption (a)). Along it $\int (dx^0/d\lambda)\, d\lambda = 0$ where λ is a special parameter. By the continuity of $dx^0/d\lambda$, at some event on this geodesic $(dx^0/d\lambda) = 0$. Therefore $\|\boldsymbol{dx}/d\lambda\| > 0$ at such event—see (2.1). Hence by a well known theorem the same holds at any event on the geodesic. q.e.d.

(c) We say that the region $\mathcal{U}$ of S_4 is *spatially-bounded*, if for every choice of the frame (x) and the real numbers a and b, with $a \leqslant b$, the set $\{\mathscr{E} \in \mathcal{U}\colon x^0(\mathscr{E}) \in [a, b]\}$ is bounded (5). Obviously the union of two spatially-bounded regions is spatially-bounded.

(d) Let us consider some time-like regular lines unbounded in both senses. If their union $\mathcal{W}$ is spatially-bounded, we call it a *world-bundle.* Obviously the union of two world-bundles is a world-bundle

An example of a world-bundle is the set of events of the coordinate lines $x^0 =$ var. of a frame (x), which satisfy the following inequality

$$\delta_{ij} x^i x^j \leqslant 1 \text{ (6)} . \tag{4.1}$$

In fact, given the frame (x') arbitrarily, the equalities $x''^0 = x'^0(x^\alpha)$ and $x''^i = x^i$, define a frame, (x''). For the events which satisfy (4.1) and are such that $x'^0 \in [a, b]$, $\delta_{\alpha\beta} x''^\alpha x''^\beta \leqslant 1 + a^2 + b^2$. q.e.d.

(e) Any space-like geodesic l of the form $\{\mathscr{E}(s)\colon 0 \leqslant s < \infty\}$, where s is the arc parameter, will be called a *space-like semigeodesic* (with

(5) I.e. its image under some frame-which is onto $\mathbb{R}^4$—is bounded. Then this holds with respect to any frame.

(6) We shall often denote $(\delta_{ij} x^i x^j)^{\frac{1}{2}}$ by r.

origin $\mathscr{E}(0)$). If we have fixed the above line l, we say that the event $\mathscr{E}_2$ follows $\mathscr{E}_1$ on l—briefly $\mathscr{E}_1 <_l \mathscr{E}_2$ or $\mathscr{E}_1 < \mathscr{E}_2$—if for some s_1 and s_2, with $0 \leqslant s_1 < s_2$, $\mathscr{E}_1 = \mathscr{E}(s_1)$ and $\mathscr{E}_2 = \mathscr{E}(s_2)$.

(f) The space-like semigeodesic l, is called an *asymptotic semigeodesic*—briefly AS—if (i) for every spatially-bounded region $\mathcal{U}$, and for some point $\mathscr{E}_0$ of l, $\{\mathscr{E} \in l\colon \mathscr{E}_0 \leqslant_l \mathscr{E}\} \subseteq \mathcal{U}^c (= \mathbf{S}_4 - \mathcal{U})$, and (ii) there exists $\mathscr{E}_1 \in l$ such that if $\mathscr{E}_1 \leqslant_l \mathscr{E}_2 <_l \mathscr{E}_3$, then $\mathscr{E}_2$ and $\mathscr{E}_3$ are not in causal relation, i.e. they are not joinable by a future-oriented line consisting of n_1 regular time-like arcs and n_2 regular null arcs, where n_1 or n_2 may vanish.

(g) We assume that $\mathbf{S}_4$ admits some asymptotic semigeodesics.

(h) We say that two AS are asymptotically equivalent if one of them contains the other. This relation is an equivalence and we call its equivalence classes *asymptotic geodesics*—briefly AG. The arbitrary AG will be denoted by l_∞.

(i) If f is a field defined on the AS l, and if $\lim_{s\to\infty} f(s)$ along it exists, this remains unchanged if we substitute for the above AS, another asymptotically equivalent to it. Therefore it makes sense to use for this limit the notation $l_\infty - \lim f$, where $l \in l_\infty$.

We say that a tensor field is defined on the AG l_∞, if it is defined on some AS l belonging to l_∞.

(j) Given the frame (x) and the events $\mathscr{E}_0$ and $\mathscr{E}_1$, the line $\Gamma[\mathscr{E}_0, \mathscr{E}_1]$ joining them and having in (x) the equations

$$x^\alpha = x^\alpha(\mathscr{E}_0)(1-\omega) + x^\alpha(\mathscr{E}_1)\omega \qquad \text{with } \omega \in [0, 1]\,,$$

will be called a *coordinate segment*—of the frame (x).

We say that the coordinate segment $\Gamma[\mathscr{E}_0, \mathscr{E}_1]$ is *spatial* (that is not the same as space-like) if $\mathscr{E}_0$ and $\mathscr{E}_1$ are joined by a space-like geodesic.

Remark One might think that every space-like semigeodesic is asymptotic, and that any two distinct events of a space-like geodesic are not in casual relation. To give an idea of how in the presence of a «very strong» curvature this is not necessarily true, we consider the Schwarzschild universe and assume that the external metric is defined for values of the radial coordinate less than $3m$ (where $2m$

is the value for which the external metric becomes singular). Then there exist circular orbits for photons—see e.g. in [1] p. 674.

We introduce now some notations and properties useful in N. 6.

(k) Consider the world-bundle $\mathcal{W}$, the frame (x), and the real number a. For each $\mathscr{E} \in \mathcal{W}$ such that $x^0(\mathscr{E}) \geqslant a$ $[x^0(\mathscr{E}) \leqslant a]$, consider the set of all events $\mathscr{E}'$ belonging to the coordinate line $x^0 =$ var. through $\mathscr{E}$, with $x^0(\mathscr{E}') \geqslant x^0(\mathscr{E})$ $[x^0(\mathscr{E}') \leqslant x^0(\mathscr{E})]$. We denote the union of all the sets above and $\mathcal{W}$ itself by $\mathrm{S}_+\{\mathcal{W}, (x), a\}[\mathrm{S}_-\{\mathcal{W}, (x), a\}]$ and we set $\mathrm{S}\{\mathcal{W}, (x), a\} = \mathrm{S}_+\{\mathcal{W}, (x), a\} \cup \mathrm{S}_-\{\mathcal{W}, (x), a\}$.

LEMMA 4.2. *Consider the world-bundle* $\mathcal{W}$, *the frames* (x) *and* (x'), *and the real numbers* a *and* a'. *The set*

$$\mathfrak{J}_{a,a'} = \{\mathscr{E} \in \mathcal{W} \colon x^0(\mathscr{E}) \geqslant a, \, x'^0(\mathscr{E}) \leqslant a'\}$$

is bounded.

PROOF. $\mathcal{W} \cap \Sigma'(a')$—see ($b$)—is bounded because $\mathcal{W}$ is spatially-bounded. Therefore we can consider $b = \sup\{x^0(\mathscr{E}) \colon \mathscr{E} \in \mathcal{W} \cap \Sigma'(a')\}$. By definition—see ($d$)—, each $\mathscr{E} \in \mathcal{W}$, with $x'^0(\mathscr{E}) \leqslant a'$, belongs to a regular time-like line in $\mathcal{W}$ that is unbounded in both senses. If the intersection of this line with $\Sigma'(a')$ would be empty, $\mathcal{W}$ could not be spatially-bounded; moreover the intersection above is a single event, $\hat{\mathscr{E}}$ (the proof is similar to the one of Theor 4.1). Thus

$$x^0(\mathscr{E}) \leqslant x^0(\hat{\mathscr{E}}) \leqslant b \qquad \text{for each } \mathscr{E} \in \mathcal{W} \text{ with } x'^0(\mathscr{E}) \leqslant a' , \tag{4.2}$$

because, along the line above, x^0 is a (strictly) increasing function of x'^0 (which can be assumed as a parameter because the line is regular and time-like). By (4.2) $\{\mathscr{E} \in \mathcal{W} \colon x'^0(\mathscr{E}) \leqslant a'\} \subseteq \{\mathscr{E} \in \mathcal{W} \colon x^0(\mathscr{E}) \leqslant b\}$, hence $\mathfrak{J}_{a,a'} \subseteq \{\mathscr{E} \in \mathcal{W} \colon x^0(\mathscr{E}) \geqslant a\} \cap \{\mathscr{E} \in \mathcal{W} \colon x^0(\mathscr{E}) \leqslant b\}$. If $b < a$ then $\mathfrak{J}_{a,a'}$ is empty; otherwise $\mathfrak{J}_{a,a'} \subseteq \{\mathscr{E} \in \mathcal{W} \colon x^0(\mathscr{E}) \in [a, b]\}$, hence our thesis holds because $\mathcal{W}$ is spatially-bounded.

THEOREM 4.3. *The aforementioned region* $\mathrm{S}\{\mathcal{W}, (x), a\}$ *is spatially bounded.*

PROOF. We shall prove that $\mathrm{S}_+\{\mathcal{W}, (x), a\}$ is spatially-bounded. The analogue for $\mathrm{S}_-\{\mathcal{W}, (x), a\}$—and thus the thesis—has a similar proof. Let us fix the frame (x') and the real numbers b' and d' with $b' \leqslant d'$, arbitrarily. We must prove that $\mathcal{G} = \{\mathscr{E} \in \mathrm{S}_+\{\mathcal{W}, (x), a\} \colon$

$x'^0(\mathscr{E}) \in [b', d']\}$ is bounded. $\mathscr{K} = \{\mathscr{E} \in \mathcal{W}: x'^0(\mathscr{E}) \in [b', d']\} (\subseteq \mathcal{G})$ is bounded because $\mathcal{W}$ is spatially-bounded. If $\mathscr{E} \in \mathcal{G} - \mathscr{K}$, by the definition of $S_+\{\mathcal{W}, (x), a\}$, there exists $\hat{\mathscr{E}} \in \mathcal{W}$ such that

$$x^i(\hat{\mathscr{E}}) = x^i(\mathscr{E}) , \tag{4.3}$$

and

$$a \leqslant x^0(\hat{\mathscr{E}}) \leqslant x^0(\mathscr{E}) \tag{4.4}$$

Moreover

$$x'^0(\hat{\mathscr{E}}) \leqslant x'^0(\mathscr{E}) \leqslant d' , \tag{4.5}$$

because x'^0 is an increasing function of x^0 along every line $x^0 =$ var. By (4.4), (4.5), and Lemma 4.2, the set $\mathcal{J} = \{\hat{\mathscr{E}}: \mathscr{E} \in \mathcal{G} - \mathscr{K}\}$ is bounded. Hence we can consider

$$r_0^2 = \sup\{\delta_{ij} x^i(\hat{\mathscr{E}}) x^j(\hat{\mathscr{E}}): \hat{\mathscr{E}} \in \mathcal{J}\} = \sup\{\delta_{ij} x^i(\mathscr{E}) x^j(\mathscr{E}): \mathscr{E} \in \mathcal{G} - \mathscr{K}\}$$

where we have used (4.3). In the frame (x''), defined by $x''^0 = x'^0(x^\alpha)$ and $x''^i = x^i$, $\delta_{\alpha\beta} x''^\alpha x''^\beta \leqslant r_0^2 + b'^2 + d'^2$ for each $\mathscr{E} \in \mathcal{G} - \mathscr{K}$. q.e.d.

5. μ-Minkowskian frames.

Let us consider a frame that satisfies the following (weak)

CONDITION 5.1. *For any choice of the spatially-bounded region* $\mathcal{U}_1$ [*the world-boundle* $\mathcal{W}_1$]—see (*c*) and (*d*) in N. 4—, *there exists another*, $\mathcal{U}_2$, [$\mathcal{W}_2$], *with* $\mathcal{U}_1 \subseteq \mathcal{U}_2 [\mathcal{W}_1 \subseteq \mathcal{W}_2]$, *and such that*

(i) *if a spatial coordinate segment* (*of the considered frame*)—*see* (*j*) in N. 4 — *is contained by* $\mathcal{U}_2^c[\mathcal{W}_2^c]$, *then every space-like geodesic with the same endpoints is a subset of* $\mathcal{U}_1^c[\mathcal{W}_1^c]$;

(ii) *if a space-like geodesic is contained by* $\mathcal{U}_2^c[\mathcal{W}_2^c]$, *then the spatial coordinate segment with the same endpoints is a subset of* $\mathcal{U}_1^c[\mathcal{W}_1^c]$.

DEFINITION 5.1. *If* $\mu > 0$, *we say that the frame* (x), *satisfying the condition above, is* μ-*Minkowskian—briefly* μ-*Mink.—if*

(i) *for some world-bundle* $\mathcal{W}_0$

(i′) *the ideal fluid to which* (x) *is joined is* $\mathcal{W}_0^c$-BISO—see Def. 3.3—, (i″) *there exists the positive function* $f_{(x)}$, *continuous in* $\mathfrak{W}_0^c$, *such that along any space-like geodesic with origin* $\mathscr{E}$ *and belonging to* $\mathfrak{W}_0^c$

$$(5.1)\qquad \left|\begin{Bmatrix}\alpha\\ \beta\sigma\end{Bmatrix}(s)\right| \leqslant f_{(x)}(\mathscr{E})\, s^{-1-\mu}\,, \qquad 0 < s < \infty\,,$$

where s *is the arc parameter*;

(ii) *for every choice of the positive number* ε, *there exists the world-bundle* $\mathfrak{W}_1$, *such that*

$$(5.2)\qquad |g_{\alpha\beta}(\mathscr{E}) - m_{\alpha\beta}| \leqslant \varepsilon\ (^7)\,, \qquad \forall \mathscr{E} \in \mathfrak{W}_1^c\,.$$

Let (x) be a μ-Mink. frame. By (ii) of Def 5.1 we can consider the world-bundle $\mathfrak{W}$ such that for any $\mathscr{E} \in \mathfrak{W}^c$

(i) the matrices $(g_{ij})(\mathscr{E})$ and $(g^{ij})(\mathscr{E})$ have eigenvalues [8] larger than a positive number independent of $\mathscr{E}$;

(ii) $-g_{00}(\mathscr{E})$ and $-g^{00}(\mathscr{E})$ [8] are larger than a positive number independent of $\mathscr{E}$;

(iii) all $|g_{\alpha\beta}(\mathscr{E})|$ and $|g^{\alpha\beta}(\mathscr{E})|$ are less than a number independent of $\mathscr{E}$.

Furthermore let $\mathfrak{W}_0$ be a world-bundle with respect to which the clause (i) in Def 5.1 holds. All the properties above hold also with respect to any world-bundle containing the world-bundle $\mathfrak{W}_0 \cup \mathfrak{W}$.

DEFINITION 5.2. *We say that the world-bundle* $\mathfrak{W}$ *covers the irregularities* (*if any*) *of the* μ-*Mink. frame* (x); *if for it*

(ii) *the preceding clauses* (i) *to* (iii) *hold, and*

(ii) *the clause* (i) *in* Def 5.1 *holds with* $\mathfrak{W}_0 = \mathfrak{W}$.

In the sequel we shall consider only μ-Mink. space-times according to the following

DEFINITION 5.3. *We say that* S_4 *is* μ-*Minkowskian if it admits some* μ-*Mink. frames.*

(7) Remember that $(m_{\alpha\beta})$ is the Minkowski matrix, i.e. the diagonal matrix $(-1, 1, 1, 1)$.

(8) They are positive by the physical admissibility of the frame—see (2.1) which imply similar inequalities involving the $g^{\alpha\beta}$.

REMARK. Condition (i) in Def 5.1 is given in the complement of a suitable world-bundle and not in the whole of space-time, because we wish not to exclude phenomena related to a « very strong » curvature of S_4—see Remark in N. 4. Such a curvature may be caused by e.g. the presence of « very dense » matter. On the other hand, the assumption that such « exceptional » phenomena take place (if at all) inside a world-bundle is consistent with the Minkowskian asymptotic behaviour of the « gravitational potentials », i.e. with (5.1) and (5.2). For this to occur, in fact, we must exclude e.g. the existence of « very dense » matter outside arbitrarily large spatially-bounded regions.

We note also that we have not excluded the possibility that « very dense » masses move away from one another indefinitely. In fact, even in this case, we can find a (suitably large) world-bundle that contains the world-tubes of these masses and outside of which the curvature is suitably small—even using a world-bundle of the type which, in a suitable frame, satisfies (4.1); in fact such a frame is not necessarily (said to be) μ-Minkowskian and its behaviour can be very different from the one of a rigid frame.

Lastly we note that we have not required that outside a suitable region the Ricci tensor should vanish (9) (field equations in vacuo); i.e. we have not excluded sources of gravitational field in the whole of space—such as e.g. electromagnetic radiation.

6. Asymptotic geodesics in a μ-Minkowskian space-time.

THEOREM 6.1. *If* (x) *is a* μ*-Mink. frame, and* l_∞ *is an asymptotic geodesic*—see (h) and (i) in N. 4—, *then*

$$(6.1) \qquad l_\infty - \lim g_{\alpha\beta} = m_{\alpha\beta} \qquad \text{—see } fnt(7) \text{ in N. 5}.$$

The proof immediately follows from (ii) in Def. 5.1.

THEOREM 6.2. *Assume that* (x) *is a* μ*-Mink. frame,* l_∞ *is an asymptotic geodesic, and* $\boldsymbol{v}$ *is a vector field undergoing parallel transport along* l_∞. *Then the limit* $l_\infty - \lim v^\alpha$ *exists (finite).*

PROOF. Let us consider a world-bundle, $\mathcal{W}$, that covers (x)'s irregularities—see Def. 5.2—, and an asymptotic semigeodesic—see (f) in

(9) A condition of this type is required in the definition of « asymptotically empty and simple space » in [5], p. 222.

N. 4—, $l \in l_\infty$ contained by $\mathcal{W}^c$. Furthermore let $\boldsymbol{v}$ be transported by parallelism along l. Firstly let us prove that all $v^\alpha(s)$—weere s is the arc parameter along l—are bounded. By definition the ideal fluid to which (x) is joined is $\mathcal{W}^c$-BISO, so that by Theor. 3.1

$$|(\boldsymbol{v}\cdot\boldsymbol{\gamma})(s)| \leqslant \Gamma\big(|\boldsymbol{v}(0)_{\boldsymbol{\gamma}} + |\boldsymbol{v}(0)|_{\perp\boldsymbol{\gamma}}\big) \equiv K_1 , \qquad 0 \leqslant s < \infty ,$$

where $\boldsymbol{\gamma}$ is tangent to the world-lines for the elements of the ideal fluid, and Γ is a suitable constant—see (3.6) with $D = \mathcal{W}^c$. By this, (2.2), and (i) in Def. 5.2.

$$|v_0(s)| \leqslant K_1[-g_{00}(s)]^{\frac{1}{2}} \leqslant K_2 . \tag{6.2}$$

In (6.2) and in the sequel we take as understood that K_i—$i = 2, 3, \ldots$— are suitable constants, and $s \in [0, +\infty[$. Remembering that g^{00} is bounded in $\mathcal{W}^c$, and that $\|\boldsymbol{v}(s)\|$ is constant, (6.2) yields

$$\varphi(s) \equiv |(2g^{0i}v_0v_i + g^{ij}v_i\,v_j)(s)| = |(\|\boldsymbol{v}\| - g^{00}v_0v_0)(s)| \leqslant K_3 . \tag{6.3}$$

If we denote the smallest eigenvalue of the matrix (g^{ij}) by ϱ, by (i) in Def 5.2 we have

$$(g^{ij}v_iv_j)(s) \geqslant (\varrho\delta^{ij}v_iv_j)(s) \geqslant K_4(\delta^{ij}v_iv_j)(s) . \tag{6.4}$$

Furthermore, by (6.2) and the boundedness of g^{0i} in $\mathcal{W}^c$

$$|(2g^{0i}v_0v_i)(s)| \leqslant K_5[(\delta^{ij}v_iv_j)(s)]^{\frac{1}{2}} \tag{6.5}$$

By (6.3) to (6.5)

$$\begin{aligned} K_3 \geqslant \varphi(s) \geqslant |(g^{ij}v_iv_j)(s)| - |(2g^{0i}v_0v_i)(s)| \geqslant \\ \geqslant K_4(\delta^{ij}v_iv_j)(s) - K_5[(\delta^{ij}v_iv_j)(s)]^{\frac{1}{2}} . \end{aligned} \tag{6.6}$$

Now, let us suppose that some $v_i(s)$ be unbounded. Then we can consider the sequence $\{s_n\}_{n=1,\ldots}$ such that $[(\delta^{ij}v_iv_j)(s_n)]^{\frac{1}{2}} \geqslant n$, for each value of n. Therefore by (6.6) $K_3 \geqslant \varphi(s)n^{-1} \geqslant K_4\,n - K_5$, $n = 1, 2, \ldots$, which is absurd. Then remembering also (6.2), we have

$$|v_\alpha(s)| \leqslant K_6 . \tag{6.7}$$

From this and from the boundedness of all $g^{\alpha\beta}$ in $\mathcal{W}^c$, we have that all $v^\alpha(s)$ are bounded. By this result, that holds in particular for the unit tangent vector, the condition

$$\frac{dv^\alpha}{ds} + \begin{Bmatrix} \alpha \\ \beta\sigma \end{Bmatrix} v^\beta \frac{dx^\sigma}{ds} = 0$$

for parallel transport, and (5.1)—see Def 5.2—

$$|dv^\alpha/ds|(s) \leqslant K_7 s^{-1-\mu}, \tag{6.8}$$

which is sufficient condition for the existence of our limit. q.e.d.

In particular the preceding theorem holds for the unit tangent vector $\boldsymbol{\lambda}$. In this case we denote the aforementioned limit by λ_∞^α. By Theor 6.1

$$m_{\alpha\beta}\lambda_\infty^\alpha \lambda_\infty^\beta = 1 \qquad \text{where } \lambda_\infty^\alpha = l_\infty - \lim \lambda^\alpha . \tag{6.9}$$

In Lemma 6.3 below and in Theor 7.3 we consider an asymptotic semigeodesic l. Then we denote its arbitrary point by P and its origin by P_0. Furthermore in connection with the frame (x) we denote by Z the arbitrary point of the line given by

$$x^0[Z(P)] = x^0(P_0), \qquad x^i[Z(P)] = x^i(P), \qquad P \in l \tag{6.10}$$

(that is the «image» of l on the hypersurface $x^0 = x^0(P_0)$).

LEMMA 6.3. *Assume that* (x) *is a* μ*-Mink. frame,* l_∞ *is an asymptotic geodesic, the world-bundle* $\mathcal{W}$ *covers* (x)'s *irregularities—see* Def 5.2—, *and* x^0 *is unbounded on some asymptotic semigeodesic* (AS) *of* l_∞. *Then there exist an AS* $l \in l_\infty$, *a positive constant* K, *and a sequence* $\{P_n\}_{n\in\mathbb{N}}$ *formed with points on* l *that satisfy*

$$\lim_{n\to\infty} s(P_n) = +\infty \text{—where } s \text{ is the arc parameter on } l\text{—} \tag{6.11}$$

and the following condition.

For every fixed P_n, *if* Q_n *is the arbitrary point of the coordinate segment*—see (j) in N. 4—$\Gamma[Z(P_n), P_n]$—cf. (6.10)—, *then*

(i) *the geodesic* $l[P_0, Q_n]$ (*where* P_0 *is the origin of* l) *belongs to* $\mathcal{W}^c$ *and is space-like, and*

(ii) *denoting the length of* $l[P_0, Q_n]$ *by* $s(Q_n)$

$$s(Q_n) \geqslant K s(P_n)\,. \tag{6.12}$$

PROOF. If we denote by λ^α the components in (x) of the unit vector tangent to l_∞, then by Theors 6.1-2 (6.9) holds. Hence easy continuity considerations show that for some positive ε

$$(1-4\varepsilon)\Big[\delta_{ij}\lambda^i_\infty \lambda^j_\infty - 2\varepsilon \sum_{i=1}^{3} |\lambda^i_\infty| - 3\varepsilon^2\Big] - (1+4\varepsilon)[\varepsilon + |\lambda^0_\infty|]^2 > 0\,. \tag{6.13}$$

We can choose $l_1 \in l_\infty$ satisfying

$$|\lambda^\alpha(\mathscr{E}) - \lambda^\alpha_\infty| \leqslant \varepsilon \qquad \forall \mathscr{E} \in l_1\,. \tag{6.14}$$

Furthermore let $\mathfrak{W}_1$ be a world-bundle containing $\mathfrak{W}$ and such that (5.2) holds with respect to the aforementioned ε. Since the frame (x) is μ-Mink., there is $\mathfrak{W}_2$, with $\mathfrak{W}_2 \supseteq \mathfrak{W}_1$, such that the clause (i) in Condition 5.1 holds with respect to $\mathfrak{W}_1$ and $\mathfrak{W}_2$.

Now let us consider the region $\mathrm{S}\{\mathfrak{W}_2, (x), a\}$—see (k) in N. 4—where a is the value of x^0 at the origin of l_1. We denote it by $\mathfrak{U}_1$ because, by Theor 4.3, it is spatially-bounded. For $\mathfrak{U}_1$ and for some $\mathfrak{U}_2$ containing it, the clause (ii) in Condition 5.1 holds.

Let $l_2 \in l_\infty$ be such that $l_2 \subseteq l_1$ and $l_2 \subseteq \mathfrak{U}_2^c$.

CASE 1. For some $P_0 \in l_2$: $x^0(P_0) = a$. In this case let us denote by l the AS of l_∞ with origin P_0 $(l \subseteq l_2 \subseteq l_1)$. Let P denote its arbitrary point, and let $l[P_0, P]$ be its arc joining P_0 and P. By our choice of $\mathfrak{U}_2$, the inclusion $l[P_0, P] \subseteq l_2 \subseteq \mathfrak{U}_2^c$ yields

$$\Gamma[P_0, P] \subseteq \mathfrak{U}_1^c = \big(\mathrm{S}\{\mathfrak{W}_2, (x), x^0(P_0)\}\big)^c \qquad \forall P \in l\,. \tag{6.15}$$

For every fixed $P \in l$, let us consider the arbitrary point Q on $\Gamma[Z(P), P]$ where $Z(P)$ satisfies (6.10). As an hypothesis for reduction ad absurdum, let there be some $\mathscr{E} \in \Gamma[P_0, Q] \cap \mathfrak{U}_1$. Then, since $\mathfrak{U}_1 = \mathrm{S}\{\mathfrak{W}_2, (x), x^0(P_0)\}$, by the definition of such regions, the intersection of the coordinate line $x^0 =$ var. through $\mathscr{E}$ with $\Gamma[P_0, P]$ would belong to $\mathfrak{U}_1$. This contrasts to (6.15). Therefore $\Gamma[P_0, Q] \subseteq \subseteq \mathfrak{U}_1^c \subseteq \mathfrak{W}_2^c$. Moreover every geodesic $l[P_0, Q]$—obviously with $Q \neq P_0$—is space-like (in fact, by the contrary assumption, we could join the distinct points P_0 and P by the future-orientable line consisting of the non-space-like geodesic $l[P_0, Q]$ and the time-like segment $\Gamma[Q, P]$ and hence P_0 and P would be in causal relation against clause (ii) in N. 4, (f)). Then, by our choice of $\mathcal{W}_2$,

$$l[P_0, Q] \subseteq \mathcal{W}_1^c \subseteq \mathcal{W}^c . \tag{6.16}$$

CASE 2. The assumption of Case 1 does not hold: i.e. $x^0(\mathscr{E}) \neq a$ for every $\mathscr{E} \in l_2$. We denote l_2 by l, its origin by P_0 and its arbitrary point by P. By hypothesis, x^0 is unbounded on l. Then we can consider a sequence of points on l, $\{P_n\}_{n\in\mathbf{N}}$, satisfying (6.11) and either

$$x^0(P_n) \geqslant x^0(P_0) \quad \forall n \in \mathbf{N}, \qquad \text{and } x^0(P) > a \quad \forall P \in l,$$

or

$$x^0(P_n) \leqslant x^0(P_0) \quad \forall n \in \mathbf{N}, \qquad \text{and } x^0(P) < a \quad \forall P \in l.$$

By this, and by reasonings similar to those of Case 1, we prove that for every P_n

$$l[P_0, Q_n] \text{ is space-like and } l[P_0, Q_n] \subseteq \mathcal{W}_1^c \subseteq \mathcal{W}^c, \qquad \forall Q_n \in \Gamma[Z(P_n), P_n] . \tag{6.17}$$

By (6.16) also in Case 1 we can choose a sequence, $\{P_n\}_{n\in\mathbf{N}}$, satisfying (6.11) and (6.17).

Now let us fix (arbitrarily) the space-like geodesic $l[P_0, Q_n]$. Our theorem is completely proved if we show that thesis (ii) holds for it.

By our choice of $\mathcal{W}_1$, the inclusion $l[P_0, Q_n] \subseteq \mathcal{W}_1^c$—cf. (6.17)—yields

$$|g_{\alpha\beta}(\mathscr{E}) - m_{\alpha\beta}| \leqslant \varepsilon \ \big(\text{where } \varepsilon \text{ satisfies (6.13)}\big) \quad \forall \mathscr{E} \in l[P_0, Q_n] . \tag{6.18}$$

Now, let us suppose that $(dx^0/ds) \neq 0$ along $l[P_0, Q_n]$, and let l_3 denote $l[P_0, Q_n]$ oriented in the increasing sense of x^0. Then (6.18) yields

$$\left(\frac{ds}{dx^0}\right)^2 = g_{\alpha\beta}\frac{dx^\alpha}{dx^0}\frac{dx^\beta}{dx^0} \geqslant m_{\alpha\beta}\frac{dx^\alpha}{dx^0}\frac{dx^\beta}{dx^0} - \varepsilon\left(\sum_{\alpha=0}^{3}\left|\frac{dx^\alpha}{dx^0}\right|\right)^2 \geqslant$$
$$\geqslant (1-4\varepsilon)\,\delta_{ij}\frac{dx^i}{dx^0}\frac{dx^j}{dx^0} - (1+4\varepsilon)$$

at every point on l_3. Thus

$$\int_{l_3}\left(\frac{ds}{dx^0}\right)^2 dx^0 \geqslant (1-4\varepsilon)\int_{l_3}\delta_{ij}\frac{dx^i}{dx^0}\frac{dx^j}{dx^0}dx^0 - (1+4\varepsilon)|\Delta x^0(Q_n)|\,,$$

where we have set

$$(6.19)\qquad \Delta x^\alpha(\mathscr{E}) = x^\alpha(\mathscr{E}) - x^\alpha(P_0)\,.$$

If we multiply each term of the inequality above by $|\Delta x^0(Q_n)|$, and we use

$$\left[\int_{l_3}\frac{dx^i}{dx^0}dx^0\right]^2 \leqslant |\Delta x^0(Q_n)|\int_{l_3}\left(\frac{dx^i}{dx^0}\right)^2 dx_0\,,$$

which follows from the Cauchy-Schwarz inequality, we have

$$|\Delta x^0(Q_n)|\int_{l_3}(ds/dx^0)^2 dx^0 \geqslant (1-4\varepsilon)(\delta_{ij}\Delta x^i\Delta x^j)(Q_n) - (1+4\varepsilon)[\Delta x^0(Q_n)]^2\,.$$

Let $\boldsymbol{t}$ be the unit vector tangent to $l[P_0, Q_n]$. By definition, $x^i(Q_n) = x^i(P_n)$, and $|\Delta x^0(Q_n)| \leqslant |\Delta x^0(P_n)|$. Then

$$[\Delta x^0(P_n)]^2[t^0(\hat{\mathscr{E}})]^{-2} \geqslant (1-4\varepsilon)(\delta_{ij}\Delta x^i\Delta x^j)(P_n) - (1+4\varepsilon)[\Delta x^0(P_n)]^2$$

where $\hat{\mathscr{E}}$ is a suitable point of $l[P_0, Q_n]$. By (6.14)—with $\alpha = 1, 2, 3$— and the inclusion $l \subseteq l_1$, we have

$$(6.20)\qquad (\delta_{ij}\Delta x^i\Delta x^j)(P_n) \geqslant \left[\delta_{ij}\lambda^i_\infty\lambda^j_\infty - 2\varepsilon\sum_{i=1}^3|\lambda^i_\infty| - 3\varepsilon^2\right]s^2(P_n)\,.$$

By (6.20) and (6.14)—with $\alpha = 0$—the preceding inequality yields

$$[t^0(\hat{\mathscr{E}})]^{-2} \geqslant (1-4\varepsilon)\left[\delta_{ij}\lambda^i_\infty\lambda^j_\infty - 2\varepsilon\sum_{i=1}^3|\lambda^i_\infty| - 3\varepsilon^2\right]\left[|\lambda^0_\infty| + \varepsilon\right]^{-2} - (1+4\varepsilon)\,.$$

By (6.13), the term on the right hand side of the above inequality is positive. Therefore for some positive constant K_1 independent of the particular $l[P_0, Q_n]$ which we consider

$$(6.21)\qquad |t^0(\hat{\mathscr{E}})| \leqslant K_1 \ \big(\text{at the suitable point } \hat{\mathscr{E}} \text{ on } l[P_0, Q_n]\big)\,.$$

We have deduced (6.21) in the case where $t^0 \neq 0$ along $l[P_0, Q_n]$, but obviously it holds also in the opposite case.

In the sequel we take as understood that K_i—$i = 2, 3, \ldots$—are suitable positive constants independent of the particular $l[P_0, Q_n]$.

Since $l[P_0, Q_n] \subseteq \mathfrak{W}^c$—see (6.17)—, and $\mathfrak{W}$ covers (x)'s irregularities, the result in (6.21) yields

$$K_2 \geqslant |-g_{00}t^0t^0 + 1| = |2g_{0i}t^0t^i + g_{ij}t^it^j| \geqslant K_3(\delta_{ij}t^it^j) - K_4(\delta_{ij}t^it^j)^{\frac{1}{2}}$$

at $\hat{\mathscr{E}}$—cf. (6.3) to (6.6). By this $|t^\alpha(\hat{\mathscr{E}})| \leqslant K_5$. Then $|\boldsymbol{\gamma}\cdot\boldsymbol{t}|(\hat{\mathscr{E}}) = |(g_{\alpha 0}t^\alpha)(-g_{00})^{-\frac{1}{2}}|(\hat{\mathscr{E}}) \leqslant K_6$. The ideal fluid to which (x) is joined is $\mathfrak{W}^c$-BISO—see (ii) in Def 5.2. Furthermore $l[P_0, Q_n] \subseteq \mathfrak{W}^c$. Hence by Theor 3.1, formula (3.7) holds. By the instance (3.7)″ of this $|\boldsymbol{\gamma}\cdot\boldsymbol{t}| \leqslant K_7$ at every point on $l[P_0, Q_n]$. By this and (i) in Def 5.2, $|t_0| \leqslant K_7(-g_{00})^{\frac{1}{2}} \leqslant K_8$. Thence we prove in the usual way that

$$(6.22)\qquad |t^\alpha| \leqslant K_9 \quad \text{at every point on } l[P_0, Q_n]\,.$$

Let us observe now that by (6.13) the term on the right hand side in (6.20) is positive. Moreover it is independent of $l[P_0, Q_n]$. Then by (6.20) and (6.22) we have

$$K_{10}^2\, s^2(P_n) \leqslant (\delta_{ij}\Delta x^i\,\Delta x^j)(P_n) = (\delta_{ij}\Delta x^i\,\Delta x^j)(Q_n) \leqslant 3K_9^2\, s^2(Q_n)\,,$$

where $s(Q_n)$ is the length of $l[P_0, Q_n]$ and (6.19) is used. Lastly if we set $K = K_{10}/\sqrt{3}\,K_9$ we have (6.12). q.e.d.

7. Existence of the asymptotic Lorentz matrix joined to an ordered couple of μ-Minkowskian frames.

In this number we consider some properties involving two μ-Mink. frames (x) and (x'). We always assume that $\mathscr{F}[\mathscr{F}']$ is the ideal fluid to which the frame $(x)[(x')]$ is joined, and $\boldsymbol{\gamma}[\boldsymbol{\gamma}^{(\prime)}]$ is the field of unit vectors tangent to the world-lines of the elements of $\mathscr{F}[\mathscr{F}']$, and pointing into the future.

LEMMA 7.1. *Let* (x) *and* (x') *be* μ*-Mink. frames, let the world-bundle* $\mathcal{W}$ *cover their irregularities* (10)—see Def 5.2—, *and let Condition* 3.1 *on* $\mathfrak{D}$ *and* $\mathcal{A}$ ($\mathcal{A} \subseteq \mathfrak{D}$) *hold with* $\mathfrak{D} = \mathcal{W}^c$. *Then all the functions* $\partial x'^\alpha/\partial x^\beta$ *are bounded in* $\mathcal{A}$.

PROOF. At any event $\mathscr{E} \in \mathcal{A}$ let us consider an orthonormal tetrad, $\{\boldsymbol{c}_{(\alpha)}\}_{\alpha=0,\dots,3}$, with $\boldsymbol{c}_{(0)} = \boldsymbol{\gamma}^{(')}$. Let us prove, firstly, that the components, in (x'), of each $\boldsymbol{c}_{(\alpha)}$ are bounded in $\mathcal{A}$. By (2.2) $c'^{\alpha}_{(0)} = \delta^\alpha_0(-g'_{00})^{-\frac{1}{2}}$ and $c'_{(0)\alpha} = g'_{\alpha 0}(-g'_{00})^{-\frac{1}{2}}$. Then, by (i) in Def 5.2 and the inclusion $\mathcal{A} \subseteq \mathcal{W}^c$ $(= \mathfrak{D})$, they are bounded in $\mathcal{A}$. Let us consider now the spacelike vectors $\boldsymbol{c}_{(i)}$. Since they are orthogonal to $\boldsymbol{\gamma}^{(')}$ $(= \boldsymbol{c}_{(0)})$, we have

$$1 = \boldsymbol{c}_{(i)} \cdot \boldsymbol{c}_{(i)} = g'^{lm} c'_{(i)l} c'_{(i)m} \geqslant \varrho' \, \delta^{lm} c'_{(i)l} c'_{(i)m} ,$$

where ϱ' is the smallest eigenvalue of the matrix (g'^{ij}). Therefore, by (i) in Def 5.2, all the functions $c'_{(i)l}$ are bounded in $\mathcal{A}$, and such are also all $c'^{\alpha}_{(i)} = g'^{\alpha j} c'_{(i)j}$. Thus the boundedness of $c'^{\alpha}_{(\beta)}$ has been proved.

Now let us write the components of $\boldsymbol{\gamma}$ in (x') in the forms

$$\gamma'^\alpha = [\partial x'^\alpha/\partial x^0](-g_{00})^{-\frac{1}{2}} \qquad \text{and} \qquad \gamma'^\alpha = m^{\sigma\varrho}[\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(\sigma)}] c'^{\alpha}_{(\varrho)} .$$

By them

$$\partial x'^\alpha/\partial x^0 = m^{\sigma\varrho}[\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(\sigma)}] c'^{\alpha}_{(\varrho)}(-g_{00})^{\frac{1}{2}} . \tag{7.1}$$

In this expression $c'^{\alpha}_{(\varrho)}$ are bounded in $\mathcal{A}$, and such is also $(-g_{00})^{\frac{1}{2}}$. By hypothesis the ideal fluids $\mathscr{F}$ and $\mathscr{F}'$ are $\mathcal{W}^c$-BISO—see clause (ii) in Def. 5.2—, and Condition 3.1 on $\mathfrak{D}$ and $\mathcal{A}$ holds with $\mathfrak{D} = \mathcal{W}^c$. Then, by Theor 3.4, the function $|\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}^{(')}| \equiv |\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(0)}|$ is bounded in $\mathcal{A}$. But from the equalities $-1 = \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = m^{\alpha\beta}[\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(\alpha)}][\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(\beta)}]$ we have $|\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(\alpha)}| \leqslant |\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(0)}|$; then the functions $[\boldsymbol{\gamma} \cdot \boldsymbol{c}_{(\sigma)}]$ are all bounded in $\mathcal{A}$. We have thus proved the boundedness of the term on the right hand side in (7.1), i.e. the thesis for $\partial x'^\alpha/\partial x^0$.

Let us consider now the expression

$$g'^{\alpha\alpha} = \left(\frac{\partial x'^\alpha}{\partial x^0}\right)^2 g^{00} + \frac{\partial x'^\alpha}{\partial x^i}\frac{\partial x'^\alpha}{\partial x^j} g^{ij} + 2\, g^{0i}\frac{\partial x'^\alpha}{\partial x^0}\frac{\partial x'^\alpha}{\partial x^i} .$$

(10) Remember that the union of two world-bundles is a world-bundle. Moreover if a world-bundle covers the irregularities of a frame, then any world-bundle containing it has the same property. Therefore it makes sense to consider $\mathcal{W}$ covering the irregularities of both frames.

The first term on the right hand side and $g'^{\alpha\alpha}$ are bounded in $\mathcal{A}$. Then for some positive constant K and for each $\alpha \in \{0, 1, 2, 3\}$

$$\psi^\alpha(\mathscr{E}) \equiv \left| g^{ij} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\alpha}{\partial x^j} + 2g^{0i} \frac{\partial x'^\alpha}{\partial x^0} \frac{\partial x'^\alpha}{\partial x^i} \right| \leqslant K, \qquad \forall \mathscr{E} \in \mathcal{A} .$$

By this we easily prove that $\partial x'^\alpha/\partial x^i$ are bounded in $\mathcal{A}$—see the reasonings in Theor 6.2 from (6.3) to (6.7), where we had the function φ instead of each ψ^α. q.e.d.

In the sequel we shall often denote by x the event with coordinates x^α in the frame (x).

For any function $f(x)$ on $\mathbf{S}_4$ let $[a, b]$-lim $f(x)$ express the limit of $f(x)$ when $x = (x^0, x^i)$ approaches ∞ fulfilling the condition $a \leqslant x^0 \leqslant b$.

LEMMA 7.2. *If (x) and (x') are μ-Mink. frames, and a and b $(\geqslant a)$ are real numbers, then the limit*

$$[a, b] - \lim\, (\partial x'^\alpha/\partial x^\beta)(x) \tag{7.2}$$

is a matrix independent of a and b.

PROOF. We shall prove that

(7.3) $\quad |\partial^2 x'^\alpha/\partial x^\beta\, \partial x^\sigma|(x) \leqslant K_1(r - r_0)^{-1-\mu}$

with $x^0 \in [a, b]$ and $r \equiv (\delta_{ij} x^i x^j)^{\frac{1}{2}} > r_0$,

where K_1 and r_0 are suitable positive constants. By (7.3) it is easy to prove that

(i) $\lim\limits_{x\to\infty} (\partial x'^\alpha/\partial x^\beta)(x)$ along each coordinate line $r =$ var.—i.e. a line having the equations $x^\alpha = A^\alpha r$ with $0 < r < \infty$ and the constants A^α satisfy $\delta_{ij} A^i A^j = 1$ and $A^0 = 0$—exists (finite); and

(ii) $\lim\limits_{r\to\infty} \omega^\alpha_\beta = 0$, where ω^α_β is the oscillation of $\partial x'^\alpha/\partial x^\beta$ on $\{x\colon x^0 \in [a, b]$ and $r =$ const.$\}$.

The existence of the limit in (7.2) follows immediately from these results. Moreover its uniqueness—i.e. its independence of a and b—is trivial. Therefore let us prove (7.3). To this extent we can use the well known formula

$$\frac{\partial^2 x'^\alpha}{\partial x^\beta \partial x^\sigma} = \begin{Bmatrix} \nu \\ \beta\sigma \end{Bmatrix} \frac{\partial x'^\alpha}{\partial x^\nu} - \begin{Bmatrix} \alpha \\ \nu\varrho \end{Bmatrix}' \frac{\partial x'^\nu}{\partial x^\beta} \frac{\partial x'^\varrho}{\partial x^\sigma}. \tag{7.4}$$

Let us consider a world-bundle, $\mathfrak{W}_1$, covering the irregularities of both our frames—see fnt. (10). Since (x) is μ-Mink., for $\mathfrak{W}_1$ and for some $\mathfrak{W}_2$ containing it, the clause (i) of Condition 5.1 holds (with respect to (x)). By definition $\mathfrak{W}_2$ is spatially-bounded—see (c) and (d) in N. 4. Therefore we can consider a positive constant, r_0, such that the region $\mathcal{A} = \{x: x_0 \in [a, b] \text{ and } r \geqslant r_0\}$ is contained by $\mathfrak{W}_2^c$. Let $\mathcal{B} = \{x \in \mathcal{A}: r = r_0\}$. For each $\mathcal{E} \in \mathcal{A}$ let $\hat{\mathcal{E}}$ be the event of the compact $\mathcal{B}$ belonging to the coordinate segment—see (j) in N. 4— joining $\mathcal{E}$ and the event having the coordinates $x^0(\mathcal{E}), 0, 0, 0$, in (x). The segment above is spatial—in fact, by Theor 4.1, its endpoints are joined by a space-like geodesic—and is contained by $\mathfrak{W}_2^c$. Then, by our choice of $\mathfrak{W}_2$, there exists a space-like geodesic, $l[\hat{\mathcal{E}}, \mathcal{E}]$, contained by $\mathfrak{W}_1^c$. Condition 3.1 is thus satisfied by $\mathfrak{D} = \mathfrak{W}_1^c$ and its subset $\mathcal{A}$. Therefore, by the preceding Lemma, all $\partial x'^\alpha/\partial x^\beta$ are bounded in $\mathcal{A}$. If we prove that for some positive constant K_2, at each x in $\mathcal{A}$-$\mathcal{B}$,

$$\left|\begin{Bmatrix} \alpha \\ \beta\sigma \end{Bmatrix}(x)\right| \leqslant K_2(r - r_0)^{-1-\mu} \quad \text{and} \quad \left|\begin{Bmatrix} \alpha \\ \beta\sigma \end{Bmatrix}'(x)\right| \leqslant K_2(r - r_0)^{-1-\mu}, \tag{7.5}$$

then from (7.4) and from the boundedness of $\partial x'^\alpha/\partial x^\beta$ we have (7.3).

Let $\boldsymbol{t}$ be the unit vector tangent to $l[\hat{\mathcal{E}}, \mathcal{E}]$. By continuity there exists an event of $l[\hat{\mathcal{E}}, \mathcal{E}]$, where $t^0 = 0$—see the proof of Theor 4.1. By this—see the proof of Lemma 6.3 from (6.21) to (6.22)—for some constant K_3, independent of the particular $l[\hat{\mathcal{E}}, \mathcal{E}]$ being considered, $|t^\alpha| \leqslant K_3$ at every point on $l[\hat{\mathcal{E}}, \mathcal{E}]$. This yields

$$r - r_0 \leqslant \sqrt{3}\, K_3 s \tag{7.6}$$

where s is the length of $l[\hat{\mathcal{E}}, \mathcal{E}]$, and $r = [\delta_{ij} x^i(\mathcal{E}) x^j(\mathcal{E})]^{\frac{1}{2}}$.

The geodesic $l[\hat{\mathcal{E}}, \mathcal{E}]$ is contained by $\mathfrak{W}_1^c$. Then by clause (ii) in Def 5.2, for some positive continuous functions $f_{(x)}$ and $f_{(x')}$ on $\mathfrak{W}_1^c$, condition (5.1) in (x) holds for (x) and (x') respectively. The aforementioned functions are bounded in $\mathcal{B}$ because it is compact, thus (5.1) and (7.6) imply (7.5). q.e.d.

THEOREM 7.3. *If (x) and (x') are μ-Mink. frames, and l_∞ is an asymptotic geodesic, then the limit*

$$(7.7)\qquad l_\infty\text{—}\lim(\partial x'^\alpha/\partial x^\beta)\text{—see(i) in } N4\text{—}$$

exists, and is a Lorentz matrix independent of the choice of l_∞.

PROOF. Assume that the world-bundle $\mathcal{W}$ covers the irregularities of both our frames—see fnt (10)—, l is an asymptotic semigeodesic of l_∞ contained by $\mathcal{W}^c$ (remember that this choice is possible because $\mathcal{W}$ is spatially-bounded).

Condition 3.1 holds with $\mathcal{D} = \mathcal{W}^c$ and $\mathcal{A} = l$ (the compact set $\mathcal{B}$ mentioned in Condition 3.1 can consist of the origin of l). Therefore, by Lemma 7.1, $\partial x'^\alpha/\partial x^\beta$ is bounded in l. Moreover, by Theor 6.2, such is also dx^α/ds—where s is the arc parameter on l. Thus, by (7.4) and (5.1), we have along l

$$(7.8)\qquad \left|\frac{d}{ds}\frac{\partial x'^\alpha}{\partial x^\beta}\right|(s) = \left|\frac{\partial^2 x'^\alpha}{\partial x^\beta\,\partial x^\sigma}\frac{dx^\sigma}{ds}\right|(s) \leqslant M_1 s^{-1-\mu}\,,\qquad 0 < s < \infty\,,$$

where M_1 is a suitable constant. This proves the existence of the limit in (7.7). Furthermore, since

$$g_{\alpha\beta} = \frac{\partial x'^\sigma}{\partial x^\alpha}\frac{\partial x'^\varrho}{\partial x^\beta}g'_{\sigma\varrho}\,,$$

by Theor 6.1, our limit is a Lorentz matrix.

Let us prove now the uniqueness of our limit, i.e. its independence of l_∞. To this extent it is sufficient to prove that it equals the matrix, (N^α_β), given by Lemma 7.2. By this lemma and the limit (along l) $\lim_{s\to\infty} r(s) = +\infty$—that follows from Theors 6.1-2—, our result is obvious if x^0 is bounded on l. Therefore let us consider the opposite case. In it, by Lemma 6.3, we can assume that l, and some sequence $\{P_n\}_{n\in\mathbf{N}}$ formed with points on it, satisfy the conditions of this lemma. Now let us observe that we prove the theorem if we show that our limit equals $\lim_{s\to\infty}(\partial x'^\alpha/\partial x^\beta)_{Z[P(s)]}$ where $Z(P)$ is given by (6.10). In fact, by Lemma 7.2 and $\lim_{s\to\infty} r(s) = +\infty$ along l, the latter coincides with (N^α_β).

If P_0 is the origin of l, then by Lemma 6.3 we have that $l[P_0, Q_n] \subseteq \mathcal{W}^c$ —we refer to Lemma 6.3 for the meaning of $l[P_0, Q_n]$. Moreover Condition 3.1 holds if $\mathcal{D} = \mathcal{W}^c$ and $\mathcal{A}$ is the union of all $l[P_0, Q_n]$ (this holds with e.g. $\mathcal{B} = \{P_0\}$). Therefore, by Lemma 7.1, the derivatives $\partial x'^{\alpha}/\partial x^{\beta}$ are bounded in $\mathcal{A}$. By this, (7.4), (5.1), (6.12), and by the boundedness of dx^0/ds in l, we have for some positive constants M_2 and M_3

$$\left| \frac{\partial x'^{\alpha}}{\partial x^{\beta}} (P_n) - \frac{\partial x'^{\alpha}}{\partial x^{\beta}} [Z(P_n)] \right| \leqslant M_2[s(P_n)]^{-1-\mu} \, |x^0(P_n) - x^0(P_0)| \leqslant M_3[s(P_n)]^{-\mu}$$

for every n. Thus, by (6.11), our limit equals (N^{α}_{β}). q.e.d.

By the preceding theorem we can introduce the following

DEFINITION 7.1. *Let* (x) *and* (x') *be two* μ*-Mink. frames. We call the limit in* (7.7)—which is independent of the choice of l_∞—*the asymptotic Lorentz matrix joined to* (x) *and* (x').

Obviously the asymptotic Lorentz matrix joined to (x') and (x) is the inverse of the one joined to (x) and (x').

DEFINITION 7.2. *We say that the* μ*-Mink. frames* (x) *and* (x') *are asymptotically equivalent, if the asymptotic Lorentz matrix joined to them coincides with the identity matrix.*

The relation introduced by Def 7.2 is an equivalence. Its equivalence class that contains (x) will be denoted by $[(x)]$. If $\varphi = [(x)]$ and $\varphi' = [(x')]$, the asymptotic Lorentz matrix joined to (x) and (x') is a function of φ and φ'.

DEFINITION 7.3. *We call the aforementioned matrix, the Lorentz matrix joined to* φ *and* φ'.

8. Asymptotic Minkowski space.

We consider a μ-Minkowskian space-time—see Def 5.3—. We call every couple $(l_\infty, \boldsymbol{v})$, where l_∞ is an asymptotic geodesic—see (*h*) in N. 4—and $\boldsymbol{v}$ is a vector field undergoing parallel transport along it, an (asymptotic) *pre-vector.*

By Theor. 6.2, if $(l_\infty, \boldsymbol{v})$ is a pre-vector and (x) is a μ-Mink. frame, then the limit $l_\infty - \lim v^{\alpha}$ exists. Furthermore it does not change if we replace (x) with any μ-Mink. frame asymptotically equivalent to it—see Def 7.2. Therefore it makes sense to call $l_\infty - \lim v^\alpha$ *the components of the pre-vector* $(l_\infty, \boldsymbol{v})$ *in* $[(x)]$.

By Theor 7.3, if two pre-vectors have the same components in $\varphi = [(x)]$, this property is independent of the particular class φ being considered. Through this property, we introduce an equipollence between prevectors, and we call its equivalence classes *asymptotic vectors*. In the sequel we shall denote by V the arbitrary asymptotic vector.

If $\varphi = [(x)]$, φ defines a function, to be denoted by the same symbol, that maps every asymptotic vector V in the common quadruple formed by the components in the class φ, of the pre-vectors belonging to V. We say that the function φ is an *asymptotic Minkowski frame* and we call the elements $\varphi^\alpha(V)$ of $\varphi(V)$, *the components of the asymptotic vector* V in φ.

We see at once that the asymptotic Minkowski frames φ and φ' are related by

$$\varphi'^\alpha = L^\alpha_\beta \varphi^\beta \tag{8.1}$$

where L is the Lorentz matrix joined to φ and φ'—see Def 7.3.

Let (x) be a μ-Mink. frame and let L be an orthocronous—i.e. one that does not reverse time ($L^0_0 > 0$)—Lorentz matrix. By (ii) in Def 5.1, the coordinate system $(x)_L$ obtained from (x) by L is physically admissible in the complement of some world-bundle $\mathfrak{W}_1$. Let (x') be a frame coinciding with $(x)_L$ in $\mathfrak{W}_1^c$. By (i′) in Def 5.1 there is a world-bundle, $\mathfrak{W}_0$, such that the ideal fluid joined to (x) is $\mathfrak{W}_0^c$-BISO By Theor 3.3 the ideal fluid joined to (x') is $(\mathfrak{W}_0 \cup \mathfrak{W}_1)^c$-BISO. Now we see at once that (x') is μ-Mink. Therefore, if φ is an asymptotic Minkowski frame, and L is an orthocronous Lorentz matrix, the function $\varphi' = (\varphi'^\alpha)_{\alpha=0,1,2,3}$ given by (8.1) is an asymptotic Minkowski frame (i.e. it is defined by a class of asymptotically equivalent μ-Mink. frames). We have thus proved the following

THEOREM 8.1. *Exactly the whole class of the asymptotic Minkowski frames, can be obtained from one of them, by means of the transformation* (8.1) *in connection with all orthocronous Lorentz matrices.*

Using this result we see at once that if $V_{(i)}$—$i = 1, 2$—are asymptotic vectors, and $\alpha \in \mathbb{R}$ then *the sum* $V_{(1)} + V_{(2)} = \varphi^{-1}[\varphi(V_{(1)}) + \varphi(V_{(2)})]$ and *the product* $\alpha V_{(1)} = \varphi^{-1}[\alpha\varphi(V_{(1)})]$, where φ is an asymptotic Minkowski frame, are well defined, i.e. they are independent of φ. The set of the asymptotic vectors, with these operations is a vector space. If we define on it *the scalar product* $V_{(1)} \cdot V_{(2)} = m_{\alpha\beta}\varphi^{\alpha}(V_{(1)})\varphi^{\beta}(V_{(2)})$, which is invariant by Theor 8.1, we obtain a Minkowski space.

If L is a Lorentz matrix

$$(8.2) \qquad L_0^0 = 1 \quad \text{implies } L_\alpha^0 = \delta_\alpha^0 \text{ and } L_0^\alpha = \delta_0^\alpha .$$

Let φ and φ' be asymptotic Minkowski frames. We say that they are *co-moving* if the component L_0^0 of the Lorentz matrix joined to φ and φ' equals 1. Using (8.2) we see at once that the relation above is an equivalence: we call its equivalence classes *asymptotic inertial spaces*. The arbitrary one of them will be denoted by $S_{(a,i)}$.

The Lorentz matrices L with $L_0^0 = 1$ are in one-to-one correspondence with the orthogonal matrices of the third order. Then, remembering that the Lorentz matrices are ∞^6, we have the following

THEOREM 8.2. *In a μ-Minkowskian space-time there are ∞^3 asymptotic inertial spaces.*

Let us consider now the asymptotic Mink. frame $\varphi[\varphi']$, and the asymptotic inertial space $S_{(a,i)}[S'_{(a,i)}]$ joined to it. Moreover let L be the Lorentz matrix joined to φ and φ'. By (8.2) we see at once that L_α^0 are independent of the choice of φ' in $S'_{(a,i)}$, and, in particular, L_0^0 is a function of $S_{(a,i)}$ and $S'_{(a,i)}$ symmetric in these arguments. Therefore it makes sense to call the asymptotic vector, whose covariant components in φ are $(-cL_\alpha^0/L_0^0)$, *the 4-velocity of $S'_{(a,i)}$ relative to $S_{(a,i)}$*. Obviously we call the asymptotic vector whose components in φ are $-c[(L_\alpha^0/L_0^0) + m_{\alpha 0}]$—which vanish iff φ and φ' are co-moving—*the standard velocity of $S'_{(a,i)}$ relative to $S_{(a,i)}$*. Lastly we say that $c[1-(L_0^0)^{-2}]^{\frac{1}{2}}$ is *the standard speed of the two asymptotic inertial spaces with respect to one another.*

REMARK. Let the μ-Mink. frames (x) and (x') be joined to the ideal fluids $\mathscr{F}$ and $\mathscr{F}'$, and to the aforementioned asymptotic Mink. frames φ and φ'. Then the limit, on each asymptotic geodesic, of the components, in (x), of the 4-velocity [standard velocity] of $\mathscr{F}'$ relative to $\mathscr{F}$, equals the components, in φ, of the 4-velocity [standard velocity] of $S'_{(a,i)}$ relative to $S_{(a,i)}$. Moreover the corresponding property for the standard speed holds.

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