RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

FRANCESCO FERRO

Integral characterization of functionals defined on spaces of *BV* functions

Rendiconti del Seminario Matematico della Università di Padova, tome 61 (1979), p. 177-201

<http://www.numdam.org/item?id=RSMUP_1979__61__177_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 61 (1979)

Integral Characterization of Functionals Defined on Spaces of BV Functions.

FRANCESCO FERRO (*)

SUMMARY - In [6] we extended in a suitable way a class of functionals defined on $W^{1,1}(\Omega)$ to the space $BV_b(\Omega) \oplus L^1(\partial\Omega)$. Here we give an integral characterization of the extended functional which is related with the functional defined in [10] in the one-dimensional case.

Introduction.

Many recent papers deal with the problem of defining variational functionals on spaces of functions of bounded variation. In [10] an integral functional defined on absolutely continuous functions in $[0, 1]$ is extended to the space of functions of bounded variation by means of the recession function of the integrand; the main result given in [10] is the characterization of optimal arcs in terms of a « generalized Hamiltonian condition ».

In [1], [2], [3] the same integral functional is extended in an alternative method; however it is proved in [1] that under suitable hypothesis the extended functional agrees with the extension given in [10]; the same is proved in [3] by different hypothesis in the case of a non convex functional. In [1], [2] there are mainly given optimization

(*) Indirizzo dell'A.: Istituto di Matematica, Università di Genova, Italy.
This work was supported in part by Laboratorio per la Matematica Applicata del C.N.R., Italy.

theorems for the extended functional involving the boundedness of the level sets of the starting functional.

Analogous problems in the n-dimensional case have been studied in [4], [5], [6].

The main results are in [6] where an integral functional defined on $W^{1,1}(\Omega)$ is extended to the space $BV_b(\Omega) \oplus (C(\partial\Omega))^*$, where $BV_b(\Omega)$ is the space of functions in $L^1(\Omega)$ whose gradient is a measure with finite total variation in Ω. Optimization theorems and applications to minimal surface problems are also given in [6].

The aim of this work is to give an integral characterization of the extended functional defined in [6]. In this way we emphasize the strict analogy of our results with the onedimensional case.

In Section 1 we give a survey of the functional background we developed in our preceding works and state some preliminary results of topological nature.

In Section 2 we give our main results; we remark that the hypothesis and the proof of Theorem 2.1 are quite similar to that used in [3].

1. Definitions and topological properties of some functional spaces.

Throughout this paper Ω will be an open, bounded and connected subset of $\mathbb{R}^n$, whose boundary $\partial\Omega$ verifies the local Lipschitz condition (in the sense of [7]). Let

$$BV_b(\mathbb{R}^n) = \{u\colon\, u \in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ \nabla u \in (M_b(\mathbb{R}^n))^n\}\,,$$

where $M_b(\mathbb{R}^n)$ is the space of all real-valued measures whose total variation is finite in $\mathbb{R}^n$. Let $C_0(\mathbb{R}^n)$ be the space of all continuous functions which have a compact support in $\mathbb{R}^n$; if we endow $C_0(\mathbb{R}^n)$ with the uniform convergence topology, $M_b(\mathbb{R}^n)$ is its dual space (a Banach space) and

$$\|v\|_{M_b(\mathbb{R}^n)} = \sup\left\{\int_{\mathbb{R}^n} fv\colon\, f \in C_0(\mathbb{R}^n),\ |f(x)| \leqslant 1\right\}.$$

Then $\mathbb{R} \oplus (M_b(\mathbb{R}^n))^n$ is the dual space of $\mathbb{R} \oplus (C_0(\mathbb{R}^n))^n$ and may be endowed with the weak topology of dual space (the socalled w^* topology).

An element $u \in BV_b(\mathbf{R}^n)$ may be identified with the couple

$$\Big(\int_\Omega u, \nabla u\Big) \in \mathbf{R} \oplus (M_b(\mathbf{R}^n))^n ;$$

in this sense $BV_b(\mathbf{R}^n)$ is a subspace of $\mathbf{R} \oplus (M_b(\mathbf{R}^n))^n$.

As we proved in [4] $BV_b(\mathbf{R}^n)$ is w^*-closed in $\mathbf{R} \oplus (M_b(\mathbf{R}^n))^n$; hence it is closed also relative to the norm topology of $\mathbf{R} \oplus (M_b(\mathbf{R}^n))(^n$ (and so it is a Banach space relative to the norm topology); moreover we emphasize that the closed balls of $BV_b(\mathbf{R}^n)$ are w^*-compact and their topology is metrizable.

A net $\{u_\alpha\} \subset BV_b(\mathbf{R}^n)$ w^*-converges to $u \in BV_b(\mathbf{R}^n)$ if and only if

$$\lim_\alpha \int_\Omega u_\alpha = \int_\Omega u$$

and

$$\lim_\alpha \int_{\mathbf{R}^n} G\, \nabla u_\alpha = \int_{\mathbf{R}^n} G\, \nabla u\,, \qquad \text{for every } G \in (C_0(\mathbf{R}^n))^n\,.$$

Let

$$E = \{u \in BV_b(\mathbf{R}^n):\ u = 0 \text{ a.e. in } \Omega\}\,;$$

E is w^*-closed (see [4], [5]). Let w_q^* be the quotient topology induced on $BV_b(\mathbf{R}^n)/E$ by the w^* topology of $BV_b(\mathbf{R}^n)$, that is the finest topology on $BV_b(\mathbf{R}^n)/E$ such that the canonical mapping

$$\pi\colon\ BV_b(\mathbf{R}^n) \to BV_b(\mathbf{R}^n)/E$$

be continuous. It is well-known that π is an open mapping.

Now let

$$BV_b(\Omega) = \{u\colon\ u \in L^1(\Omega), \nabla u \in (M_b(\Omega))^n\}\,,$$

where $M_b(\Omega)$ is the dual space of the space $C_0(\Omega)$ of all continuous functions which have a compact support in Ω ($C_0(\Omega)$ has the uniform convergence topology).

$BV_b(\Omega)$ is a Banach space if we put

$$\|u\|_{BV_b(\Omega)} = \|u\|_{L^1(\Omega)} + \|\nabla u\|_{(M_b(\Omega))^n}\,.$$

Let $u \in BV_b(\mathbf{R}^n)$ and $[u]$ be its equivalence class in $BV_b(\mathbf{R}^n)/E$; we define

$$i\colon BV_b(\mathbb{R}^n)/E \to BV_b(\Omega)$$

in the following way:

$$i([u]) = r(u)\,,$$

where r is the restriction operator. In [6] we proved that if $BV_b(\mathbb{R}^n)/E$ is endowed with the strong quotient topology then i is an isomorphism between Banach spaces; so we may identify $BV_b(\Omega)$ and $BV_b(\mathbb{R}^n)/E$ and give the following definition (see [6]):

DEFINITION 1.1. A set $D \subset BV_b(\Omega)$ is w_q^*-open if and only if $i^{-1}(D)$ is w_q^*-open. ■

We remark that the closed balls of $BV_b(\Omega)$ are w_q^*-compact and their induced topology is metrizable.

It follows by [5, Proposition 3.1] that if a sequence $\{u_m\} \subset BV_b(\Omega)$ w_q^*-converges to $u \in BV_b(\Omega)$ then $\lim_{m\to+\infty} u_m = u$ in $L^1(\Omega)$.

Now let $f \in L^1(\partial\Omega)$; we may put

$$(1.1)\qquad \langle g, f\rangle_1 = \int_{\partial\Omega} fg\, dH_{n-1}\,, \qquad \text{for every } g \in C(\partial\Omega)\,,$$

where H_{n-1} is the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$ and $C(\partial\Omega)$ is the space of all continuous functions on $\partial\Omega$; moreover we define

$$(1.2)\qquad \langle G, f\rangle_2 = \int_{\partial\Omega} fG\nu\, dH_{n-1}\,, \qquad \text{for every } G \in \big(C(\partial\Omega)\big)^n\,,$$

where ν is the unit outer normal to $\partial\Omega$.

In the sense of (1.1) $L^1(\partial\Omega)$ is a subspace of $\big(C(\partial\Omega)\big)^*$ while in the sense of (1.2) $L^1(\partial\Omega)$ is a subspace of $\big((C(\partial\Omega))^n\big)^*$ (a more detailed approach is in [6]). Let w_1^* and w_2^* be the weak topologies of dual space of $\big(C(\partial\Omega)\big)^*$ and $\big((C(\partial\Omega))^n\big)^*$ respectively. We proved in [6] that $L^1(\partial\Omega)$ is w_1^*-dense in $\big(C(\partial\Omega)\big)^*$ if $\partial\Omega$ is of class C^1, while, without this hypothesis on $\partial\Omega$, we called $M(\partial\Omega)$ the w_2^*-closure of $L^1(\partial\Omega)$ in $\big((C(\partial\Omega))^n\big)^*$.

If $u \in W^{1,1}(\Omega) = \{u\colon u \in L^1(\Omega),\ \nabla u \in \big(L^1(\Omega)\big)^n\}$ and $\gamma(u)$ is its trace in the sense of Sobolev spaces, we have $\big(u, \gamma(u)\big) \in BV_b(\Omega) \oplus L^1(\partial\Omega)$.

In this sense we may write $W^{1,1}(\Omega)\subset BV_b(\Omega)\oplus L^1(\partial\Omega)$.

We proved (see [6]) that $W^{1,1}(\Omega)$ is $w_q^*\times w_1^*$-dense in $BV_b(\Omega)\oplus \oplus (C(\partial\Omega))^*$ if $\partial\Omega$ is of class C^1 and that, without this supplementary hypothesis, $W^{1,1}(\Omega)$ is $w_q^*\times w_2^*$-dense in $BV_b(\Omega)\oplus M(\partial\Omega)$.

In what follows the regularity hypothesis « $\partial\Omega$ of class C^1 » will be implicitely assumed whenever we shall deal with w_1^*-topology.

PROPOSITION 1.1. Let $\{(u_m, f_m)\}\subset BV_b(\Omega)\oplus L^1(\partial\Omega)$ be a sequence and $(u, f)\in BV_b(\Omega)\oplus L^1(\partial\Omega)$.

Then $(u_m, f_m)\xrightarrow{w_q^*\times w_i^*}(u, f)$ for $i=1, 2$ if and only if the following conditions hold:

$$
(1.3)\quad\begin{cases}
\text{(i)}\ \lim\limits_{m\to+\infty} u_m = u \text{ in } L^1(\Omega);\\
\text{(ii) for every } u'\in BV_b(\mathbb{R}^n) \text{ such that } r(u')=u \text{ there exists a sequence } \{u'_m\}\subset BV_b(\mathbb{R}^n) \text{ such that } r(u'_m)=u_m \text{ and}\\
\qquad \lim\limits_{m\to+\infty}\int\limits_{\mathbb{R}^n} G\,\nabla u'_m=\int\limits_{\mathbb{R}^n} G\,\nabla u' \qquad \text{for every } G\in (C_0(\mathbb{R}^n))^n;\\
\text{(iii)}\ \lim\limits_{m\to+\infty}\int\limits_{\partial\Omega} f_m G\nu\, dH_{n-1}=\int\limits_{\partial\Omega} fG\nu\, dH_{n-1} \text{ for every } G\in (C(\partial\Omega))^n.
\end{cases}
$$

PROOF. The sufficiency of conditions (1.3) is obvious by the continuity of the canonical mapping π. As to the necessity (1.3) (i) is proved in [5] and (1.3) (iii) follows by the definition. Afterwards there exists a constant $c>0$ such that

$$\|u_m\|_{BV_b(\Omega)}\leqslant c\,, \qquad \text{for every } m\,,$$

by the uniform boundedness theorem, that is $\{u_m\}$ is contained in a closed ball (which is w_q^*-compact and whose induced w_q^* topology is metrizable) of $BV_b(\Omega)$. Since π is an open mapping $(\pi\circ i)^{-1}(\{u_m\})$ is w^*-relatively compact and contained in a closed ball (which is w^*-compact and whose induced w^* topology is metrizable) of $BV_b(\mathbb{R}^n)$.

Now (1.3) (ii) holds by standard topological arguments. ∎

Now we recall some results about traces of BV functions (see the References in [5], [6]).

If $u\in BV_b(\mathbb{R}^n)$ then there exist $\gamma^-(u),\ \gamma^+(u)\in L^1(\partial\Omega)$ such that

$$(1.4)\qquad \int\limits_{\bar\Omega} G\,\nabla u+\int\limits_{\Omega} u\,\operatorname{div} G=\int\limits_{\partial\Omega}\gamma^+(u)\,G\nu\, dH_{n-1}\,,\ \text{for every } G\in (C_0^1(\mathbb{R}^n))^n,$$

and

$$(1.5)\qquad \int_{\Omega} G\nabla u + \int_{\Omega} u \operatorname{div} G = \int_{\partial\Omega} \gamma^-(u) G\nu \, dH_{n-1}, \quad \text{for every } G \in \big(C_0^1(\mathbb{R}^n)\big)^n ;$$

$\gamma^+(u)$ and $\gamma^-(u)$ are called respectively the outer and inner trace of u on $\partial\Omega$.

If $u \in BV_b(\Omega)$ we may deal only with $\gamma^-(u)$; moreover if $u \in W^{1,1}(\Omega)$ we have $\gamma^-(u) = \gamma(u)$. We shall use the following notations: if $u \in BV_b(\Omega)$ and $f \in L^1(\partial\Omega)$ then u_f will be any function in $BV_b(\mathbb{R}^n)$ such that $u_f = u$ in Ω and $\gamma^+(u_f) = f$.

THEOREM 1.1. Conditions (1.3) hold if and only if the following conditions hold:

$$(1.6)\qquad \begin{cases} \text{(i)} \ \lim\limits_{m\to+\infty} u_m = u \ \text{in } L^1(\Omega); \\ \text{(ii)} \ \lim\limits_{m\to+\infty} \int_{\Omega} G\nabla(u_m)_{f_m} = \int_{\Omega} G\nabla u_f , \qquad \text{for every } G \in \big(C(\bar{\Omega})\big)^n ; \\ \text{(iii)} \ \lim\limits_{m\to+\infty} \int_{\partial\Omega} f_m G\nu \, dH_{n-1} = \int_{\partial\Omega} fG\nu \, dH_{n-1}, \quad \text{for every } G \in \big(C(\partial\Omega)\big)^n . \end{cases}$$

PROOF. Let (1.3) hold; then we must prove (1.6) (ii).

If $G \in \big(C^1(\bar{\Omega})\big)^n$ by (1.4) we have

$$\int_{\bar{\Omega}} G\nabla(u_m)_{f_m} = -\int_{\Omega} u_m \operatorname{div} G + \int_{\partial\Omega} f_m G\nu \, dH_{n-1} ;$$

hence by (1.3) (i) and (1.3) (iii) we obtain

$$(1.7)\qquad \lim_{m\to+\infty} \int_{\bar{\Omega}} G\nabla(u_m)_{f_m} = -\int_{\Omega} u \operatorname{div} G + \int_{\partial\Omega} fG\nu \, dH_{n-1} = \int_{\bar{\Omega}} G\nabla u_f$$

$$\text{for every } G \in \big(C^1(\bar{\Omega})\big)^n .$$

We have also

$$(1.8)\qquad \|\nabla(u_m)_{f_m}\|_{(C(\bar{\Omega}))^*} \leqslant \|u_m\|_{BV_b(\Omega)} + \int_{\partial\Omega} |f_m - \gamma^-(u_m)| \, dH_{n-1} \leqslant \|u_m\|_{BV_b(\Omega)} +$$

$$+ \int_{\partial\Omega} |f_m| \, dH_{n-1} + \int_{\partial\Omega} |\gamma^-(u_m)| \, dH_{n-1} \leqslant \text{const} \, \|u_m\|_{BV_b(\Omega)} + \|f_m\|_{L^1(\partial\Omega)} .$$

The right hand side of (1.8) is bounded by (1.3) (ii) and (1.3) (iii), then (1.6) (ii) is obtained by (1.7) using standard approximation techniques. Now let (1.6) hold; we must prove (1.3) (ii).

We have

$$\int_{\bar\Omega} G\,\nabla(u_m)_{f_m} = \int_{\Omega} G\,\nabla u_m + \int_{\partial\Omega} \big(f_m - \gamma^-(u_m)\big) G\nu\, dH_{n-1}\,, \tag{1.9}$$

for every $G \in \big(C(\bar\Omega)\big)^n$

and

$$\int_{\bar\Omega} G\,\nabla u_f = \int_{\Omega} G\,\nabla u + \int_{\partial\Omega} \big(f - \gamma^-(u)\big) G\nu\, dH_{n-1}\,, \tag{1.10}$$

for every $G \in \big(C(\bar\Omega)\big)^n$.

Using (1.6) (ii) and (1.6) (iii) in (1.9) we obtain by (1.10):

$$\lim_{m\to+\infty}\Big(\int_{\Omega} G\,\nabla u_m - \int_{\partial\Omega}\gamma^-(u_m)\,G\nu\,dH_{n-1}\Big) = \int_{\Omega} G\,\nabla u - \int_{\partial\Omega}\gamma^-(u)\,G\nu\,dH_{n-1}\,, \tag{1.11}$$

for every $G \in \big(C(\bar\Omega)\big)^n$.

Now we take $u' \in BV_b(\mathbf{R}^n)$ such that $r(u') = u$; let $u'_m \in BV_b(\mathbf{R}^n)$ such that $u'_m = u_m$ in Ω and $u'_m = u'$ in $\mathbf{R}^n - \Omega$. If $G \in \big(C_0(\mathbf{R}^n)\big)^n$ we have

$$\int_{\mathbf{R}^n} G\,\nabla u'_m = \int_{\mathbf{R}^n-\bar\Omega} G\,\nabla u' + \int_{\Omega} G\,\nabla u_m + \int_{\partial\Omega}\big(\gamma^+(u') - \gamma^-(u_m)\big) G\nu\, dH_{n-1}\,,$$

then by (1.11)

$$\lim_{m\to+\infty}\int_{\mathbf{R}^n} G\,\nabla u'_m = \int_{\mathbf{R}^n-\bar\Omega} G\,\nabla u' + \int_{\Omega} G\,\nabla u + \int_{\partial\Omega}\big(\gamma^+(u') - \gamma^-(u)\big)\, G\nu\, dH_{n-1} =$$

$$= \int_{\mathbf{R}^n} G\,\nabla u'\,, \qquad \text{for every } G \in \big(C_0(\mathbf{R}^n)\big)^n\,. \qquad \blacksquare$$

REMARK 1.1. It is easily seen that if a net $\{(u_\alpha, f_\alpha)\} \subset BV_b(\Omega) \oplus \oplus L^1(\partial\Omega)$ verifies (1.6) then it verifies (1.3) and if it verifies (1.3) then $(u_\alpha, f_\alpha) \xrightarrow{w_q^* \times w_i^*} (u, f)$; that is the «if part» in Proposition 1.1 and in Theorem 1.1 is true not only for sequences but also for nets. ∎

Theorem 1.1 characterizes the $w_q^* \times w_i^*$-convergence of sequences in $BV_b(\Omega) \oplus L^1(\partial\Omega)$. As easy consequences we obtain

COROLLARY 1.1. Let $\{u_m\} \subset BV_b(\Omega)$ be a sequence and $u \in BV_b(\Omega)$. Then $u_m \xrightarrow{w_q^*} u$ if and only if $\lim_{m\to+\infty} u_m = u$ in $L^1(\Omega)$ and for every $\varphi \in L^1(\partial\Omega)$ we have

$$\lim_{m\to+\infty} \int_{\bar\Omega} G\,\nabla(u_m)_\varphi = \int_\Omega G\,\nabla u_\varphi \qquad \text{for every } G \in \big(C(\bar\Omega)\big)^n . \qquad \blacksquare$$

COROLLARY 1.2. Let $u_m \xrightarrow{w_q^*} u$, then $u'_m \xrightarrow{w^*} u'$, where $u'_m = u_m$ in Ω, $u'_m = 0$ in $\mathbf{R}^n - \Omega$, $u' = u$ in Ω, $u' = 0$ in $\mathbf{R}^n - \Omega$.

PROOF. Let $G \in \big(C_0(\mathbf{R}^n)\big)^n$; we have

$$\lim_{m\to+\infty} \int_{\mathbf{R}^n} G\,\nabla u'_m = \lim_{m\to+\infty} \int_{\bar\Omega} G\,\nabla u'_m = \int_{\bar\Omega} G\,\nabla u' = \int_{\mathbf{R}^n} G\,\nabla u' . \qquad \blacksquare$$

In the final part of this Section we give an other characterization of the sequential w_q^*-convergence. However in the next Section we shall not use these results.
Let us consider the imbedding

$$j\colon BV_b(\Omega) \to \mathbf{R}^n \oplus \big(M_b(\Omega)\big)^n ,$$

where $j(u) = \big(\int_\Omega u, \nabla u\big)$. It is easily seen that j is an injective continuous linear mapping between Banach spaces. Moreover $\mathbf{R} \oplus \big(M_b(\Omega)\big)^n$ is the dual space of $\mathbf{R} \oplus \big(C_0(\Omega)\big)^n$; then it may be endowed with the weak topology of dual space (which will be noted $\tilde w^*$ topology) and so we may define the following induced topology on $BV_b(\Omega)$: a net $\{u_\alpha\} \subset BV_b(\Omega)$ $\tilde w^*$-converges to $u \in BV_b(\Omega)$ if and only if $j(u_\alpha)$ $\tilde w^*$-converges to $j(u)$, that is if and only if

$$\lim_\alpha \int_\Omega u_\alpha = \int_\Omega u \quad \text{and} \quad \lim_\alpha \int_\Omega G\,\nabla u_\alpha = \int_\Omega G\,\nabla u \quad \text{for every } G \in \big(C_0(\Omega)\big)^n .$$

It is obvious that the balls of $j\big(BV_b(\Omega)\big)$ are relatively $\tilde w^*$-compact; we shall prove that they are $\tilde w^*$-compact.

If $u \in BV_b(\Omega)$ and u_h are its integral averages (e.g. see [7]) we have $u_h \xrightarrow{\tilde{w}^*} u$; then

PROPOSITION 1.2. $W^{1,1}(\Omega)$, as a subset of $BV_b(\Omega)$, is $\tilde{w}^*$-dense in $BV_b(\Omega)$. ∎

Now we may prove:

THEOREM 1.2. Let $\{u_m\} \subset BV_b(\Omega)$ be a sequence which $\tilde{w}^*$-converges to $(a, \mu) \in \mathbf{R} \oplus (M_b(\Omega))^n$; then there exists $u \in BV_b(\Omega)$ such that $(a, \mu) = \left(\int_\Omega u, \nabla u\right)$ and $\lim_{m \to +\infty} u_m = u$ in $L^1(\Omega)$ (in particular the balls of $j(BV_b(\Omega))$ are $\tilde{w}^*$-compact).

PROOF. By Proposition 1.2 we may suppose $\{u_m\} \subset W^{1,1}(\Omega)$ and by the uniform boundedness theorem there exists $c > 0$ such that

$$\left|\int_\Omega u_m\right| \leqslant c \qquad \text{and} \qquad \|\nabla u_m\|_{(L^1(\Omega))^n} = \|\nabla u_m\|_{(M_b(\Omega))^n} \leqslant c\,.$$

By Poincaré's inequality we have also

$$\|u_m\|_{(L^1(\Omega))^n} \leqslant c_1\,,$$

for a suitable constant $c_1 > 0$.

Then, by a well-known strong compactness criterion in $L^1(\Omega)$ we may say that, given any subsequence $\{u_r\}$ of $\{u_m\}$, there exists a subsequence $\{u_s\}$ of $\{u_r\}$ and $u \in BV_b(\Omega)$ such that $\lim_{s \to +\infty} u_s = u$ in $L^1(\Omega)$; then $\int_\Omega u = a$, $\nabla u_s \to \nabla u$ in the sense of distributions and so $\nabla u = \mu$. Hence u is the same for every $\{u_r\}$ and $\{u_s\}$. The proof is complete since the $\tilde{w}^*$ topology is metrizable on the balls. ∎

THEOREM 1.3. Let $\{u_m\} \subset BV_b(\Omega)$ be a sequence and $u \in BV_b(\Omega)$. Then $\{u_m\}$ w^*_q-converges to u if and only if $\{u_m\}$ $\tilde{w}^*$-converges to u.

PROOF. The « only if part » is an obvious consequence of Corollary 1.1.

As to the « if part » let $G \in (C^1(\bar{\Omega}))^n$ and $\varphi \in L^1(\partial\Omega)$; we have by

Theorem 1.2:

$$\lim_{m\to+\infty}\int_{\bar\Omega} G\,\nabla(u_m)_\varphi = \lim_{m\to+\infty}\left(\int_{\partial\Omega}\varphi G\nu\, dH_{n-1} - \int_{\Omega} u_m \operatorname{div} G\right) =$$

$$= \int_{\partial\Omega}\varphi G\nu\, dH_{n-1} - \int_{\Omega} u \operatorname{div} G = \int_{\bar\Omega} G\nabla u_\varphi\,.$$

Afterwards, since $\|\nabla u_m\|_{(M_b(\Omega))^n}\leqslant$ const, we have also

$$\lim_{m\to+\infty}\int_{\bar\Omega} G\,\nabla(u_m)_\varphi = \int_{\bar\Omega} G\,\nabla u_\varphi\,, \qquad \text{for every } G\in(C(\bar\Omega))^n\,,$$

and the proof is complete by Theorem 1.1. ■

We wish to remark that Theorem 1.2 and Theorem 1.3 could also allow us to approach the problems considered in [4], [5], [6] by an alternative, and perhaps simpler, method.

2. Integral characterization.

Let

$$L\colon\ \bar\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$$

be a proper normal integrand, that is

(2.1)
- (i) $L(x,\cdot,\cdot)$ is lower semicontinuous for every $x\in\bar\Omega$;
- (ii) $L(x,\cdot,\cdot)$ is not identically $+\infty$;
- (iii) $E_L(x)=\{(u,v,\alpha)\colon L(x,u,v)\leqslant\alpha\}$ is a measurable multifunction, i.e. $E_L^{-1}(C)=\{x\colon E_L(x)\cap C\neq\emptyset\}$ is Lebesgue measurable for every $C\subset\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}$, C closed.

We remark that $L(x,u(x),v(x))$ is measurable whenever u and v are measurable (see [11] for an extensive study about normal integrands). We put

$$I_L(u)=\int_{\Omega} L(x,u(x),\nabla u(x))\,dx\,, \qquad u\in W^{1,1}(\Omega)\,.$$

$I_L(u)$ is well-defined if $L(x, u(x), \nabla u(x))$ is summable; otherwise we put $I_L(u) = -\infty$ if $L(x, u(x), \nabla u(x))$ is majorized by a summable function and $I_L(u) = +\infty$ in every other case.

We always suppose that there exists $u \in W^{1,1}(\Omega)$ such that $I_L(u) \in \mathbb{R}$. In [6] we defined the functionals

$$J_1(u, \mu) = \min\{\liminf_\alpha I_L(u_\alpha): \{u_\alpha\} \in W^{1,1}(\Omega), \{u_\alpha\} \text{ is a net } (u_\alpha, \gamma(u_\alpha)) \xrightarrow{w_q^* \times w_1^*} (u, \mu)\}, \quad (u, \mu) \in BV_b(\Omega) \oplus (C(\partial\Omega))^*$$

and

$$J_2(u, \mu) = \min\{\liminf_\alpha I_L(u_\alpha): \{u_\alpha\} \subset W^{1,1}(\Omega), \{u_\alpha\} \text{ is a net } (u_\alpha, \gamma(u_\alpha)) \xrightarrow{w_q^* \times w_2^*} (u, \mu)\}, \quad (u, \mu) \in BV_b(\Omega) \oplus M(\partial\Omega).$$

We have (see [6])

$$J_1(u, f) = J_2(u, f), \qquad \text{for every } (u, f) \in BV_b(\Omega) \oplus L^1(\partial\Omega).$$

Now let $H(x, u, \cdot)$ be the Fenchel conjugate of $L(x, u, \cdot)$, i.e.

$$H(x, u, p) = \sup\{pv - L(x, u, v): v \in \mathbb{R}^n\}$$

and

$$P_u(x) = \{p \in \mathbb{R}^n: H(x, u, p) < +\infty\}.$$

LEMMA 2.1. Let L be a proper normal integrand and $\sigma_0: \bar{\Omega} \times \mathbb{R}_+ \to \mathbb{R}$ such that $\sigma_0(\cdot, r) \in L^1(\Omega)$ and

$$\sup\{|L(x, u, v) - L(x, u_1, v)|: |u| \leqslant r, |u_1| \leqslant r, v \in \mathbb{R}^n\} \leqslant \sigma_0(x, r). \tag{2.2}$$

Then $P_u(x)$ is independent of u (in this case we shall write $P_u(x) = P(x)$).

PROOF. Let $p \in P_u(x)$, $u_1 \in \mathbb{R}$ and $r > 0$ such that $|u| \leqslant r$ and $|u_1| \leqslant r$. By (2.2) we have

$$L(x, u, v) \leqslant L(x, u_1, v) + \sigma_0(x, r),$$

$$pv - L(x, u_1, v) \leqslant pv - L(x, u, v) + \sigma_0(x, r)$$

and so

$$H(x, u_1, p) \leqslant H(x, u, p) + \sigma_0(x, r) < +\infty ;$$

then $p \in P_{u_1}(x)$. ■

If (2.2) holds we may put (see [3])

$$r_L(x, z) = \sup\{pz: p \in P(x)\} . \tag{2.3}$$

LEMMA 2.2. Let L be a proper normal integrand and (2.2) hold; if there exist $K_1 > 0$ and $\theta_1: \bar{\Omega} \to \mathbb{R}$ such that $\theta_1 \in L^1(\Omega)$ and

$$L(x, u, v) \geqslant K_1|v| - \theta_1(x) , \qquad \text{for every } (x, u, v) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n ,$$

then int $P(x) \neq \emptyset$.

PROOF. We have

$$pv - L(x, u, v) \leqslant pv - K_1|v| + \theta_1(x)$$

and

$$H(x, u, p) \leqslant \theta_1(x) , \qquad \text{if } |p| \leqslant K_1 ,$$

then

$$\{p: |p| \leqslant K_1\} \subset P(x) . \qquad ■$$

In what follows, if $\mu \in (C(\bar{\Omega}))^*$ we write $\mu = \mu_a + \mu_s$, where μ_a is absolutely continuous relative to Lebesgue measure and μ_s is the singular part of μ (relative to Lebesgue measure); $d\mu_a/dx$ will be the Radon-Nykodim derivative of μ_a relative to Lebesgue measure. If $(u, f) \in BV_b(\Omega) \oplus L^1(\partial\Omega)$ and $\mu = \nabla(u_f)$ we shall write $\mu_a = \nabla_p u(x)\, dx$ and $\mu_s = \nabla_s u$, where $\nabla_p u(x)$ is the gradient of u in the elementary sense ($\nabla_p u(x)$ exists a.e. in Ω); in this case we have $d\mu_a/dx = \nabla_p u(x)$; if $u \in W^{1,1}(\Omega)$ and $\gamma(u) = f$ we have $\mu_s = 0$.

The following theorem gives a comparison between J_i, $i = 1, 2$, and an integral functional related with the so-called recession function r_L.

THEOREM 2.1. Let L be a proper normal integrand and the following statements hold:

(2.4)

(i) there exists a summable function $\sigma\colon \bar{\Omega} \to \mathbb{R}$ such that $\sup\{|L(x, u, v) - L(x, u_1, v)|\colon u, u_1 \in \mathbb{R}, v \in \mathbb{R}^n\} \leqslant \sigma(x)$;

(ii) there exist $K_1 > 0$ and $\theta_1\colon \bar{\Omega} \to \mathbb{R}$ such that $\theta_1 \in L^1(\Omega)$ and $L(x, u, v) \geqslant K_1|v| - \theta_1(x)$, for every $(x, u, v) \in \bar{\Omega}\times\mathbb{R}\times\mathbb{R}^n$;

(iii) $G = \operatorname{int} \operatorname{cl} G$, where $G = \{(x, p)\colon p \in \operatorname{int} P(x)\}$;

(iv) there exists $u_0 \in \mathbb{R}$ such that $\int_V |H(x, u_0, p)|\, dx < +\infty$ if V is an open set and $p \in \mathbb{R}^n$ has a neighborhood U contained in $P(x)$;

(v) $\lim_{u\to\tilde{u}} \sup\{|L(x, u, v) - L(x, \tilde{u}, v)|\colon v \in \mathbb{R}^n\} = 0$, for every $x \in \Omega$ and $\tilde{u} \in \mathbb{R}$;

(vi) either the level sets $\{u\colon I_L(u) \leqslant z\}$ are bounded in $W^{1,1}(\Omega)$ or $L = L(x, v)$ and the sets $\left\{u\colon I_L(u) \leqslant z, \int_\Omega u = 0\right\}$ are bounded in $W^{1,1}(\Omega)$;

(vii) $L(x, u, \cdot)$ is convex for every $(x, u) \in \bar{\Omega}\times\mathbb{R}$.

Then

$$\int_\Omega L(x, u(x), \nabla_p u(x))\, dx + \int_{\bar{\Omega}} r_L\left(x, \frac{d\nabla_s(u_f)}{dq}(x)\right) q(dx) \leqslant J_i(u, f), \qquad i = 1, 2,$$

for every $(u, f) \in BV_b(\Omega) \oplus L^1_\omega(\partial\Omega)$, where q is a non-negative measure relative to which $\nabla_s u$ is absolutely continuous.

Proof. Since (2.4) (i) implies (2.2), by Lemma 2.1 $P(x)$ is independent of u and r_L is well-defined by (2.3).

Afterwards let $\bar{u}$ be a measurable function and define $\phi_{\bar{u}}(x, v) = L(x, \bar{u}(x), v)$. It is known ([11, Corollary 2P]) that $\phi_{\bar{u}}$ is a normal integrand on $\bar{\Omega}\times\mathbb{R}^n$.

Now we prove that there exists $v_1 \in (L^1(\Omega))^n$ such that

$$\int_\Omega \phi_{\bar{u}}(x, v_1(x))\, dx \in \mathbb{R};$$

by the general hypothesis made on I_L there exists $u_1 \in W^{1,1}(\Omega)$ such that

$$\int_\Omega L(x, u_1(x), \nabla u_1(x))\, dx \in \mathbb{R}$$

and by (2.4) (i) we have

$$\left|\int_\Omega L(x, \bar{u}(x), \nabla u_1(x))\, dx\right| \leqslant \int_\Omega \sigma(x)\, dx + \left|\int_\Omega L(x, u_1(x), \nabla u_1(x))\, dx\right| \in \mathbb{R}\,.$$

Then we may assume $v_1 = \nabla u_1$.

By [11, Proposition 2S] $\psi_{\bar{u}}(x, v) = H(x, \bar{u}(x), v)$ is a normal integrand. Then the hypothesis of [11, Theorem 3C] are fulfilled by $\phi_{\bar{u}}$ and $\psi_{\bar{u}}$ and we have

$$\text{(2.5)} \qquad \int_\Omega H(x, \bar{u}(x), f(x))\, dx = \\ = \sup\left\{\int_\Omega f(x) v(x)\, dx - \int_\Omega L(x, \bar{u}(x), v(x))\, dx \colon v \in (L^1(\Omega))^n\right\}$$

for every $f \in (L^\infty(\Omega))^n$ and measurable function $\bar{u}$.

In this case we are interested with $\bar{u} \in BV_b(\Omega)$ and $f \in (C(\bar{\Omega}))^n$. We put

$$\text{(2.6)} \quad F(\mu) = \begin{cases} \int_\Omega L(x, \bar{u}(x), \mu(x))\, dx\,, & \mu \in (L^1(\Omega))^n\,, \\ +\infty\,, & \mu \in ((C(\bar{\Omega}))^n)^* - (L^1(\Omega))^n\,. \end{cases}$$

The Fenchel conjugate function of F is

$$F^*(f) = \sup\left\{\int_{\bar{\Omega}} f\mu - F(\mu) \colon u \in ((C(\bar{\Omega}))^n)^*\right\}, \qquad f \in (C(\bar{\Omega}))^n\,,$$

and, by (2.5), (2.6),

$$F^*(f) = \int_\Omega H(x, \bar{u}(x), f(x))\, dx\,, \qquad f \in (C(\bar{\Omega}))^n\,.$$

We have also

$$(2.7)\qquad F^{**}(\mu) = \sup\Big\{ \int_{\bar\Omega} f\mu - \int_{\Omega} H(x, \bar u(x), f(x))\, dx;\ f \in \big((C(\bar\Omega))^n\big)\Big\}.$$

We observe that as an easy consequence of (2.4) (i) we have

$$|H(x, \bar u(x), p)| \leqslant |H(x, u_0, p)| \quad + \quad \sigma(x)\,,$$

where u_0 is given in (2.4) (iv), and so $\int_V |H(x, \bar u(x), p)|\, dx$ is finite whenever $\int_V |H(x, u_0, p)|\, dx$ is so. Then by (2.4) (ii), (iii), (iv), (vii) and Lemma 2.2 we may apply [9, Theorem 5]; we obtain by (2.7):

$$(2.8)\qquad F^{**}(\mu) = \int_{\Omega} L\Big(x, \bar u(x), \frac{d\mu_a}{dx}(x)\Big)\, dx + \int_{\bar\Omega} r_L\Big(x, \frac{d\mu_s}{dq}(x)\Big)\, q(dx)\,,$$

for every $\mu \in \big((C(\bar\Omega))^n\big)^*$, where q is a non-negative measure relative to which μ_s is absolutely continuous.

Since F is a convex functional on $\big((C(\bar\Omega))^n\big)^*$ and $F(\nabla u_1) \in \mathbb{R}$, by [8] and (2.6) we have

$$(2.9)\qquad F^{**}(\mu) = \min\Big\{\liminf_{\alpha} F(v_\alpha) \colon \{v_\alpha\} \subset (L^1(\Omega))^n, \{v_\alpha\} \text{ is a net,}$$
$$\int_{\bar\Omega} G v_\alpha \to \int_{\bar\Omega} G\mu \text{ for every } G \in (C(\bar\Omega))^n\Big\}.$$

We have also

$$(2.10)\qquad F^{**}(\mu) = \min\Big\{\liminf_{m\to+\infty} F(v_m) \colon \{v_m\} \subset (L^1(\Omega))^n,\ \{v_m\} \text{ is a}$$
$$\text{sequence,}\quad \int_{\bar\Omega} G v_m \to \int_{\bar\Omega} G\mu \text{ for every } G \in (C(\bar\Omega))^n\Big\}\,;$$

if $F^{**}(\mu) = +\infty$ (2.10) follows obviously by (2.9); if $F^{**}(\mu) < M < +\infty$, $\int_{\bar\Omega} G v_\alpha \to \int_{\bar\Omega} G\mu$ and $\lim_{\alpha} F(v_\alpha) < M$, then $F(v_\alpha) < M + 1$, whenever $\alpha > \bar\alpha$, for a suitable $\bar\alpha$. By (2.4) (ii) we have $\|v_\alpha\|_{(L^1(\Omega))^n} \leqslant \text{const}$; then the value $F^{**}(\mu)$ depends only on the elements of the ball whose radius is $M+1$. Since the topology we consider on this ball is metrizable, (2.10) holds.

Afterwards if $(u, f) \in BV_b(\Omega) \oplus L^1(\partial\Omega)$ and if we put $\bar{u} = u$, $\mu = \nabla(u_f)$ in (2.10), then we obtain by Theorem 1.1

$$(2.11)\quad F^{**}(\nabla(u_f)) \leqslant \min\Big\{\liminf_{m\to+\infty} \int_\Omega L(x, u(x), \nabla u_m(x))\, dx\colon\ \{u_m\} \subset W^{1,1}(\Omega),$$

$$\{u_m\} \text{ is a sequence, } (u_m, \gamma(u_m)) \xrightarrow{w_q^* \times w_i^*} (u, f)\Big\} \leqslant$$

$$\leqslant \min\Big\{\liminf_{m\to+\infty}\Big(\int_\Omega L(x, u_m(x), \nabla u_m(x))\, dx + \int_\Omega L(x, u(x), \nabla u_m(x))\, dx -$$

$$- \int_\Omega L(x, u_m(x), \nabla u_m(x))\, dx\Big)\colon\ \{u_m\} \subset W^{1,1}(\Omega),\ \{u_m\} \text{ is a sequence,}$$

$$(u_m, \gamma(u_m)) \xrightarrow{w_q^* \times w_i^*} (u, f)\Big\}.$$

We consider the sequence

$$a_m = \left| \int_\Omega \big(L(x, u(x), \nabla u_m(x)) - L(x, u_m(x), \nabla u_m(x)\big)\, dx \right|;$$

let $\{a_{m_r}\}$ be a subsequence of $\{a_m\}$ and $\{a_{m_s}\}$ a subsequence of $\{a_{m_r}\}$ such that $\lim_{s\to+\infty} u_{m_s} = u$ a.e. in Ω. By (2.4) (i) we may use Fatou's lemma and obtain

$$\limsup_{s\to+\infty} a_{m_s} \leqslant \int_\Omega \limsup_{s\to+\infty} |L(x, u(x), \nabla u_{m_s}(x)) - - L(x, u_{m_s}(x), \nabla u_{m_s}(x))|\, dx = 0$$

by (2.4) (v).

So by a standard argument we have

$$\lim_{m\to+\infty} a_m = 0$$

and

$$(2.12)\quad F^{**}(\nabla(u_f)) \leqslant$$

$$\leqslant \min\Big\{\liminf_{m\to+\infty} \int_\Omega L(x, u_m(x), \nabla u_m(x))\, dx\colon \{u_m\} \text{ as in } (2.11)\Big\} = J_i(u, f),$$

where the last equality follows by (2.4) (vi) (see [6, Lemma 4.1]).

A comparison between (2.8) and (2.12) completes the proof. ∎

REMARK 2.1. If there exist $K > 0$ and $\theta \in L^1(\Omega)$ such that

$$L(x, u, v) \geq K(|u| + |v|) - \theta(x)$$

for every $(x, u, v) \in \Omega \times \mathbf{R} \times \mathbf{R}^n$ then (2.4) (vi) holds. If $L = L(x, v)$ then (2.4) (ii) implies (2.4) (vi). ■

The following lemma is similar to [12, Lemma 2] we used also in [5] and [6]. However out statement needs a completely different starting method in the proof.

LEMMA 2.3. Let λ and ζ be non-negative, continuous functions defined on $[0, +\infty)$ such that $\lambda(0) = \zeta(0) = 0$; moreover we suppose that there exists a constant c such that $\zeta(t) \leq ct$ for large t. Let

$$L\colon \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$$

satisfy

$$(2.13) \qquad \begin{cases} L \in C(\Omega \times \mathbf{R} \times \mathbf{R}^n), \\ L(x, u, \cdot) \text{ is convex for every } (x, u) \in \Omega \times \mathbf{R}, \\ L(x, u, v) \geq -\varphi(x), \text{ for every } (x, u, v) \in \Omega \times \mathbf{R} \times \mathbf{R}^n, \end{cases}$$

where $\varphi \geq 0$ and $|\varphi(x) - \varphi(x_1)| \leq \lambda(|x - x_1|)$ if $x, x_1 \in \Omega$,

$$(2.14) \qquad |L(x, u, v) - L(x_1, u_1, v)| \leq \lambda(|x - x_1|)[1 + L^+(x, u, v)] + \zeta(|u - u_1|),$$

for every $x, x_1 \in \Omega$, $u, u_1 \in \mathbf{R}$ and $v \in \mathbf{R}^n$.

Let $\bar{u} \in L^\infty_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$, $(u, \varphi) \in BV_b(\Omega) \oplus L^1(\partial\Omega)$ and $\{v_m\} \subset (L^1(\Omega))^n$ be a sequence such that

$$(2.15) \qquad \lim_{m \to +\infty} \int_\Omega G v_m = \int_{\bar{\Omega}} G \, \nabla(u_\varphi), \qquad \text{for every } G \in (C(\bar{\Omega}))^n.$$

Then

$$(2.16) \qquad \limsup_{h \to 0} \int_{\Omega_{(h)}} L(x, \bar{u}(x), \nabla u_h(x)) \, dx \leq \liminf_{m \to +\infty} \int_\Omega L(x, \bar{u}(x), v_m(x)) \, dx,$$

where u_h are the integral averages of u_φ (e.g. see [7]) and

$$\Omega_{(h)} = \{x \in \Omega : d(x, \partial\Omega) > h\} .$$

PROOF. Let K_h be mollifier functions as in [7] and $u_h(x) = \int_{\mathbb{R}^n} K_h \cdot (x - \xi)\, u_\varphi(\xi)\, d\xi$, $x \in \mathbb{R}^n$; (indeed we need only that u_h be the integral averages of any extension of u).

Let v_{m_h} be the integral averages of any extension of v_m and $x \in \overline{\Omega}_{(h)}$; we have

$$(2.17) \qquad |v_{m_h}(x) - \nabla u_h(x)| = \int_\Omega K_h(x-\xi)\, v_m(\xi)\, d\xi - \\ - \int_\Omega \nabla_x K_h(x-\xi)\, u(\xi)\, d\xi| = \delta'(m, h, x) ,$$

where by (2.15)

$$(2.18) \qquad \lim_{m\to+\infty} \delta'(m, h, x) = 0 \qquad \text{for every } h > 0 \text{ and } x \in \overline{\Omega}_{(h)} .$$

By (2.15) there exists $c > 0$ such that

$$(2.19) \qquad \|v_m\|_{(L^1(\Omega))^n} \leqslant c , \qquad \text{for every } m .$$

Fixed h, by (2.17) and (2.19) it follows that $\delta'(m, h, x)$ is a sequence of uniformly equicontinuous functions in $\overline{\Omega}_{(h)}$; moreover we have

$$|\delta'(m. h, x)| \leqslant \|v_m\|_{(L^1(\Omega))^n} + c(h)\|u\|_{L^1(\Omega)} \leqslant c + c(h)\|u\|_{L^1(\Omega)}$$

for a suitable $c(h) > 0$.

Then we may apply the Ascoli-Arzelà theorem and by (2.18) we obtain for each fixed h

$$(2.20) \qquad \lim_{m\to+\infty} \delta(m, h) = 0 ,$$

where $\delta(m, h) = \sup\{\delta'(m, h, x) : x \in \Omega_{(h)}\}$.

Now we put $f = L + \varphi \geqslant 0$; if $x \in \overline{\Omega}_{(h)}$ we have, by (2.17), (2.20) and the uniform continuity of f on the compact subset of $\Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$(2.21) \qquad |f(x, \overline{u}(x), \nabla u_h(x)) - f(x, \overline{u}(x), v_{m_h}(x))| \leqslant \varepsilon(m, h) ,$$

where $\lim_{m\to+\infty} \varepsilon(m, h) = 0$ for every h.

We remark that it is essential to derive (2.21) the hypothesis $\bar{u} \in L^\infty_{\text{loc}}(\Omega)$. By (2.21) and Jensen's inequality (which may be applied by (2.13)) we obtain

$$
\begin{aligned}
f(x, \bar{u}(x), \nabla u_h(x)) &\leqslant f(x, \bar{u}(x), v_{m_h}(x)) + \varepsilon(m, h) = \\
&= f\Big(x, \bar{u}(x), \int_\Omega K_h(x-\xi) v_m(\xi)\, d\xi\Big) + \varepsilon(m, h) \leqslant \\
&\leqslant \int_\Omega f(x, \bar{u}(x), v_m(\xi)) K_h(x-\xi)\, d\xi + \varepsilon(m, h) = \\
&= \int_\Omega K_h(x-\xi) f(\xi, \bar{u}(\xi), v_m(\xi))\, d\xi + \int_\Omega K_h(x-\xi) \cdot \\
&\qquad \cdot [f(x, \bar{u}(x), v_m(\xi)) - f(\xi, \bar{u}(\xi), v_m(\xi))]\, d\xi + \varepsilon(m, h)
\end{aligned}
$$

Integrating this inequality and by the use of Fubini's theorem we have

$$
\begin{aligned}
\int_{\Omega(h)} f(x, \bar{u}(x), \nabla u_h(x))\, dx &\leqslant \int_{\Omega(h)} \Big(\int_\Omega K_h(x-\xi) f(\xi, \bar{u}(\xi), v_m(\xi))\, d\xi \Big)\, dx + \\
&\quad + \int_{\Omega(h)} \varrho(m, x)\, dx + \varepsilon(m, h) \operatorname{mis} \Omega \leqslant \\
&\leqslant \int_\Omega f(\xi, \bar{u}(\xi), v_m(\xi))\, d\xi + \int_{\Omega(h)} \varrho(m, x)\, dx + \varepsilon(m, h) \operatorname{mis} \Omega\,,
\end{aligned}
$$

where

$$
\varrho(m, x) = \int_\Omega (fx, \bar{u}(x), v_m(\xi)) - f(\xi, \bar{u}(\xi), v_m(\xi))) K_h(x-\xi)\, d\xi\,.
$$

By the hypothesis on L and φ we have $L^+ \leqslant L + \varphi = f$ and

$$
(2.22) \quad |f(x, u, v) - f(x_1, u_1, v)| \leqslant 2\lambda(|x - x_1|)(1 + f(x, u, v)) + \zeta(|u - u_1|),
$$

for every $x, x_1 \in \Omega$ and $u, u_1 \in \mathbb{R}$.

Now we use (2.22) to evaluate $\varrho(m, x)$ and $\int_{\Omega(h)} \varrho(m, x)\, dx$.

If $x \in \bar{\Omega}_{(h)}$ we have

$$|\varrho(m, x)| \leqslant 2\int_{\Omega} K_h(x-\xi) \cdot$$
$$\cdot [\lambda(|x-\xi|)(1+f(\xi, \bar{u}(\xi), v_m(\xi))) + \zeta(|\bar{u}(x)-\bar{u}(\xi)|)]\, d\xi \leqslant$$
$$\leqslant 2\lambda(h)\Big[1+\int_{\Omega} K_h(x-\xi) f(\xi, \bar{u}(\xi), v_m(\xi))\, d\xi\Big] +$$
$$+\int_{\Omega} K_h(x-\xi)\zeta(|\bar{u}(x)-\bar{u}(\xi)|)\, d\xi\,.$$

Integrating and by the use of Fubini's theorem we obtain

$$\int_{\Omega_{(h)}} |\varrho(m, x)|\, dx \leqslant 2\lambda(h) \cdot$$
$$\cdot \Big[\operatorname{mis}\Omega + \int_{\Omega_{(h)}} \Big(\int_{\Omega} K_h(x-\xi) f(\xi, \bar{u}(\xi), v_m(\xi))\, d\xi\Big)\, dx\Big] +$$
$$+\int_{\Omega_{(h)}} \Big(\int_{\Omega} K_h(x-\xi)\zeta(|\bar{u}(x)-\bar{u}(\xi)|)\, d\xi\Big)\, dx \leqslant$$
$$\leqslant 2\lambda(h)\Big[\operatorname{mis}\Omega + \int_{\Omega} f(\xi, u(\xi), v_m(\xi))\, d\xi\Big] +$$
$$+\int_{\Omega} \Big(\int_{\Omega_{(h)}} K_h(x-\xi)\zeta(|\bar{u}(\xi)-\bar{u}(x)|)\, dx\Big)\, d\xi\,.$$

Without loss of generality we may suppose that ζ is concave; moreover we remark that

$$\int_{\Omega} \Big(\int_{\Omega_{(h)}} K_h(x-\xi)\, dx\Big)\, d\xi = \int_{\Omega_{(h)}} \Big(\int_{\Omega} K_h(x-\xi)\, d\xi\Big)\, dx = \operatorname{mis}\Omega_{(h)}\,.$$

Now we use Jensens inequality:

$$\int_{\Omega} \Big(\int_{\Omega_{(h)}} K_h(x-\xi)\zeta(|\bar{u}(\xi)-\bar{u}(x)|)\, dx\Big)\, d\xi \leqslant$$
$$\leqslant \zeta\left(\frac{\int_{\Omega}\Big(\int_{\Omega_{(h)}} K_h(x-\xi)|\bar{u}(\xi)-\bar{u}(x)|\, dx\Big)\, d\xi}{\operatorname{mis}\Omega_{(h)}}\right) \operatorname{mis}\Omega_{(h)}\,.$$

We have also

$$\int_{\Omega}\Big(\int_{\Omega_{(h)}} K_h(x-\xi)|\overline{u}(\xi)-\overline{u}(x)|\,dx\Big)\,d\xi =$$

$$=\int_{|z|<h}\Big(\int_{\Omega} K_h(z)|\overline{u}(\xi)-\overline{u}(\xi+z)|\,d\xi\Big)\,dz=\varepsilon_1(h)\,,$$

where $\lim_{h\to 0}\varepsilon_1(h)=0$.

Finally we may write

$$\int_{\Omega_{(h)}} f(x,\overline{u}(x),\nabla u_h(x))\,dx \leqslant \int_{\Omega} f(\xi,\overline{u}(\xi),v_m(\xi))\,d\xi +$$

$$+\,2\lambda(h)\Big[\operatorname{mis}\Omega+\int_{\Omega} f(\xi,\overline{u}(\xi),v_m(\xi)\,d\xi\Big]+\zeta\Big(\frac{\varepsilon_1(h)}{\operatorname{mis}\Omega_{(h)}}\Big)\operatorname{mis}\Omega_{(h)}+$$

$$+\,\varepsilon(m,h)\operatorname{mis}\Omega\,.$$

Letting $m\to+\infty$ we obtain

$$\int_{\Omega_{(h)}} f(x,\overline{u}(x),\nabla u_h(x))\,dx \leqslant \liminf_{m\to+\infty}\int_{\Omega} f(\xi,\overline{u}(\xi),v_m(\xi))\,d\xi +$$

$$+\,2\lambda(h)\Big[\operatorname{mis}\Omega+\liminf_{m\to+\infty}\int_{\Omega} f(\xi,\overline{u}(\xi),v_m(\xi))\,d\xi\Big]+\zeta\Big(\frac{\varepsilon_1(h)}{\operatorname{mis}\Omega_{(h)}}\Big)\operatorname{mis}\Omega\,.$$

If $\liminf_{m\to+\infty}\int_{\Omega} f(\xi,\overline{u}(\xi),v_m(\xi))\,d\xi=+\infty$ (2.16) holds; otherwise we obtain (2.16) letting $h\to 0$. ■

The following theorem is proved in [6]:

THEOREM 2.2. Let (2.4) (vi) and the hypothesis of Lemma 2.3 hold; moreover we suppose that there exist $A>0$ and $g\in L^1(\Omega)$ such that

(2.23) $$L(x,u,v)\leqslant A\big(g(x)+|u|+|v|\big)\,,\qquad (x,u,v)\in\Omega\times\mathbf{R}\times\mathbf{R}^n\,.$$

Then

(2.24) $$J_i(u,\gamma^-(u))=\lim_{h\to 0}\mathrm{I}_L(u_h')\,,\qquad u\in BV_b(\Omega)\,,$$

where u'_h are the integral averages of u' which is defined as follows: $u' = u$ in Ω, $u' \in BV_b(\mathbb{R}^n)$, $\gamma^+(u') = \gamma^-(u)$.

We have also

$$J_i(u, \gamma^-(u)) \leqslant J_i(u, f) \leqslant J_i(u, \gamma^-(u)) + A\int_{\partial\Omega} |f - \gamma^-(u)|\, dH_{n-1},$$

for every $(u, f) \in BV_b(\Omega) \oplus L^1(\partial\Omega)$. ■

REMARK 2.2. Condition (2.14) implies condition (2.4) (v). ■

Now we may prove our most important results.

THEOREM 2.3. Let (2.4), (2.23) and the hypothesis of Lemma 2.3 hold. Then

$$J_i(u, \gamma^-(u)) = \int_\Omega L(x, u(x), \nabla_p u(x))\, dx + \int_{\bar\Omega} r_L\left(x, \frac{d\nabla_s u'}{dq}(x)\right) q(dx),$$

for every $u \in BV_b(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$.

PROOF. We have

$$(2.25) \qquad \int_\Omega L(x, \bar u(x), \nabla u'_h(x))\, dx = \\ = \int_{\Omega_{(h)}} L(x, \bar u(x), \nabla u'_h(x))\, dx + \int_{\Omega - \Omega_{(h)}} L(x, u(x), \nabla u'_h(x))\, dx$$

and by (2.23) and the properties of integral averages

$$(2.26) \qquad \left|\int_{\Omega - \Omega_{(h)}} L(x, \bar u(x), \nabla u'_h(x))\, dx\right| \leqslant \\ \leqslant A\left(\int_{\Omega - \Omega_{(h)}} |g(x)|\, dx + \int_{\Omega - \Omega_{(h)}} |\bar u(x)|\, dx + \int_{\Omega^{(h)} - \Omega_{(2h)}} |\nabla u'|(dx)\right),$$

where $\Omega^{(h)} = \{x : d(x, \bar\Omega) < h\}$.

The limit of the right hand side of (2.26) is 0 since

$$\lim_{h\to 0} \int_{\Omega^{(h)} - \Omega_{(2h)}} |\nabla u'|(dx) = \int_{\partial\Omega} |\nabla u'|(dx) = \int_{\partial\Omega} |\gamma^+(u') - \gamma^-(u')|\, dH_{n-1} = 0\,.$$

Then, if $\bar{u} \in L^{\infty}_{\text{loc}}(\Omega) \cap L^1(\Omega)$, by (2.25) and (2.16) we obtain

$$\limsup_{h\to 0} \int_\Omega L(x, \bar{u}(x), \nabla u'_h(x))\, dx \leqslant \inf \left\{ \liminf_{m\to+\infty} \int_\Omega L(x, \bar{u}(x), v_m(x))\, dx \colon \{v_m\} \subset \right.$$

$$\left. \subset (L^1(\Omega))^n, \int_\Omega G v_m \to \int_\Omega G \nabla u \text{ for every } G \in (C(\bar{\Omega}))^n \right\},$$

and so, by (2.8) and (2.10),

$$(2.27) \qquad \limsup_{h\to 0} \int_\Omega L(x, \bar{u}(x), \nabla u'_h(x))\, dx \leqslant \int_\Omega L(x, \bar{u}(x), \nabla_p u(x))\, dx + \int_{\bar{\Omega}} r_L\left(x, \frac{d\nabla_s u'}{dq}(x)\right) q(dx),$$

for every $\bar{u} \in L^{\infty}_{\text{loc}}(\Omega) \cap L^1(\Omega)$ and $u \in BV_b(\Omega)$.

Now we take $u \in BV_b(\Omega) \cap L^{\infty}_{\text{loc}}(\Omega)$ and $\bar{u} = u$; then

$$(2.28) \qquad \limsup_{h\to 0} \int_\Omega L(x, u(x), \nabla u'_h(x))\, dx \leqslant \int_\Omega L(x, u(x), \nabla_p u(x))\, dx + \int_{\bar{\Omega}} r_L\left(x, \frac{d\nabla_s u'}{dq}(x)\right) q(dx).$$

By (2.4) (i) and (2.4) (v) as in the final part of the proof of Theorem 2.1, it is easily proved that

$$\lim_{h\to 0} \int_\Omega (L(x, u(x), \nabla u'_h(x)) - L(x, u'_h(x), \nabla u'_h(x)))\, dx = 0\,;$$

then by (2.28) and (2.24)

$$(2.29) \qquad J_i(u, \gamma^-(u)) \leqslant \int_\Omega L(x, u(x), \nabla_p u(x))\, dx + \int_{\bar{\Omega}} r_L\left(x, \frac{d\nabla_s u'}{dq}(x)\right) q(dx).$$

The proof is complete by a comparison between (2.29) and Theorem 2.1. ■

THEOREM 2.4. Let (2.4), (2.23) and the hypothesis of Lemma 2.3 hold. If $L = L(x, v)$, then

$$J_i(u, \gamma^-(u)) = \int_{\Omega} L(x, \nabla_p u(x))\, dx + \int_{\bar{\Omega}} r_L\left(x, \frac{d\nabla_s u'}{dq}(x)\right) q(dx)\,,$$

for every $u \in BV_b(\Omega)$.

PROOF. If $L = L(x, v)$, $\bar{u}$ does not appear in (2.27) and there is no restriction about u. ■

THEOREM 2.5. Let (2.4), (2.23) and the hypothesis of Lemma 2.3 hold; if in addition there exists a non decreasing continuous function $\eta\colon [0, +\infty) \to \mathbb{R}$ such that $\eta(0) = 0$ and

$$|L(x, u, v) - L(x, u, v_1)| \leqslant \eta(|v - v_1|) \tag{2.30}$$

then

$$J_i(u, \gamma^-(u)) = \int_{\Omega} L(x, u(x), \nabla_p u(x))\, dx + \int_{\bar{\Omega}} r_L\left(x, \frac{d\nabla_s u'}{dq}(x)\right) q(dx)\,,$$

for every $u \in BV_b(\Omega)$.

PROOF. It suffices to remark that in the proof of Lemma 2.3 we may derive, if (2.30) holds, inequality (2.21) for every $\bar{u} \in L^1(\Omega)$; then in (2.28) we may take $u \in BV_b(\Omega)$ instead of

$$u \in BV_b(\Omega) \cap L^\infty_{\text{loc}}(\Omega)\,. \qquad ■$$

REFERENCES

[1] O. CALIGARIS - F. FERRO - P. OLIVA, *Sull'esistenza del minimo per problemi di calcolo delle variazioni relativi ad archi di variazione limitata*, Boll. Un. Mat. Ital., (5), **14**-B (1977), pp. 340-369.

[2] O. CALIGARIS - P. OLIVA, *Problemi di Bolza per archi di variazione limitata ed estensione di funzionali variazionali*, Boll. Un. Mat. Ital., (5), **14**-B (1977), pp. 772-785.

[3] O. CALIGARIS - P. OLIVA, *Sulla caratterizzazione di problemi del calcolo delle variazioni per funzioni di variazione limitata*, Boll. Un. Mat. Ital., to appear.

[4] F. FERRO, *Sul minimo di funzionali definiti sullo spazio delle funzioni di variazione limitata in n dimensioni*, Ann. Mat. Pura Appl., IV, **117** (1978), pp. 153-171.

[5] F. FERRO, *Functionals defined on functions of bounded variation in* $\mathbb{R}^n$ *and the Lebesgue area*, SIAM J. Control Optimization, **16**, no. 5 (Sept. 1978).

[6] F. FERRO, *Variational functionals defined on spaces of BV functions and their dependence on boundary data*, Ann. Mat. Pura Appl., to appear.

[7] J. NECAS, *Les méthodes directes en théorie des equations elliptiques*, Masson et Cie, Paris; Academia, Editeurs, Prague, 1967.

[8] R. T. ROCKAFELLAR, *Convex analysis*, Princeton Univ. Press, 1970.

[9] R. T. ROCKAFELLAR, *Integrals which are convex functionals II*, Pacific J. Math., **39** (1971), pp. 439-469.

[10] R. T. ROCKAFELLAR, *Dual problem of Lagrange for arcs of bounded variation*, in *Calculus of variations and control theory*, D. L. Russel ed., Academic Press, New York, 1976; Proc. of a Conference of the Mathematical Research Center, University of Wisconsin, Madison, September 1975.

[11] R. T. ROCKEFELLAR, *Integral functionals, normal integrands and measurable selections*, in *Non linear operators and the Calculus of Variations*, Lecture Notes in Mathematics, no. 543, Springer-Verlag, Berlin - Heidelberg - New York, 1976.

[12] J. SERRIN, *On the definition and properties of certain variational integrals*, Trans. Amer. Math. Soc., **101** (1961), pp. 139-167.

Manoscritto pervenuto in redazione il 28 agosto 1978.