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Periodic Solutions of a Differential Delay Equation of Rayleigh Type.

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1. Introduction.

It is well-known that the ordinary differential equation of Rayleigh type

$$x''(t) + f(x'(t)) + g(x(t)) = h(t) \tag{R}$$

is physically significant. For instance, in the problem of vibrations of a suspended wire subjected to disturbances as wind (like an electrical transmission line), the periodic solutions of

$$x'' + |x'|x' + qx' + x - P^2x^3 = r \sin \omega t$$

are of interest (see Cecconi [1]). This suggests to study the existence of p-periodic solutions of the differential delay equation

$$x''(t) + f(x'(t + \sigma(t))) + g(x(t + \tau(t))) = h(t, x(t + r(t)), x'(t + s(t))) \tag{D}$$

where the deviations σ, τ, r, s are p-periodic, and h is a bounded function, p-periodic in t. We assume that g is differentiable and we allow g' to change sign: hence we need some « Lyapunov-Schmidt »

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technique. In particular, we shall use a theorem from the coincidence degree theory (see Mawhin [3]). A particular feature of our existence result for (D) (Theorem 1) is that we require only the continuity of f, according to the fact that the differentiability of a damping term is not a reasonable physical requirement (see Utz [6]).

As a corollary of Theorem 1, we have an existence theorem of periodic solutions of ordinary differential equations (Corollary 1), which contains a result due to Reissig (see [5]).

At the end of the paper, we get an existence-uniqueness theorem (Theorem 2) for periodic solutions of (R) under a monotonicity condition for g and a regularity condition for f.

2. Preliminaries.

We call $x\colon R \to R$ a p-periodic function ($p > 0$) if, for every $t \in R$, $x(t + p) = x(t)$. We denote $C^i(p, R)$ ($i = 0, 1, 2$) the Banach space of all p-periodic functions $x\colon R \to R$ of class C^i, with the norm $x \to \sum_{k=0}^{i} |x^{(k)}|_\infty$, where $|\cdot|_\infty$ denotes the supremum norm. Moreover, if $x \in C^0(p, R)$, the symbol $|x|_2$ denotes the $L^2(0, p)$-norm of x, i.e. $|x|_2 = \left(\int_0^p |x(t)|^2\, dt\right)^{\frac{1}{2}}$, and the symbol $\delta(x)$ denotes the diameter of the set $x(R) \cup \{0\}$. Observe that δ is an equivalent norm for $C^0(p, R)$.

In [2] the following technical lemma is proved:

LEMMA 1. *Let $\tau \in C^0(p, R)$. Then the formula*

$$x(\cdot) \to \int_0^{\tau(\cdot)} x(\cdot + s)\, ds$$

defines a linear operator $G(\tau)\colon C^0(p, R) \to C^0(p, R)$ such that for every x

$$|G(\tau)x|_2 \leqslant \delta(\tau)|x|_2 .$$

3. Main results.

We denote

$$L_f = \sup_{\xi, \eta \in R;\, \xi \neq \eta} \left| \frac{f(\xi) - f(\eta)}{\xi - \eta} \right| \qquad \text{(possibly } L_f = +\infty\text{)},$$

and we define similarly L_g. We assume the convenction that

$$0 \cdot (+\infty) = 0 .$$

THEOREM 1. *Let us consider the following equation*

$$x''(t) + f(x'(t+\sigma(t))) + g(x(t+\tau(t))) = \\ = h(t, x(t+r(t)), x'(t+s(t))) \tag{1}$$

where $f \in C^0(R, R)$, $g \in C^1(R, R)$, $h \in C^0(R^3, R)$ and it is p-periodic in the first variable, and the delays $\sigma, \tau, r, s \in C^0(p, R)$. Assume that

(i) *h is bounded, $|h(t, x, x')| \leqslant M$,*

(ii) *the derivative g' is bounded above, and the frequency $\omega = 2\pi/p$ satisfies $g'(\cdot) \leqslant K < \omega^2$ for some $K \in R$.*

If the norms $\delta(\sigma)$ and $\delta(\tau)$ are so small that

(iii) $\omega^2 L_f \delta(\sigma) + \omega L_g \delta(\tau) + K < \omega^2$,

and if

(iv) $\lim\limits_{|x| \to +\infty} g(x) \operatorname{sign} x = +\infty$ (*or* $-\infty$)

then (1) *has a least one p-periodic solution.*

REMARK 1. In the ordinary case, i.e. when $\sigma = \tau = 0$, we do not require any Lipschitz condition on f or on g, since in this case the hypothesis (iii) means simply $K < \omega^2$. For instance, if $\sigma = \tau = 0$, we can assume $g(x) =$ a polynomial in x of odd order with negative leading coefficient, as in the classical Rayleigh equation where $g(x) = x - P^2 x^3$. In fact for a polynomial of this kind, the hypothesis (ii) and the hypothesis (iv) with the limit equal to $-\infty$, are always satisfied, for suitable p.

COROLLARY 1. *If $g \in C^1(R, R)$ has its derivative bounded above by a constant $K < \omega^2$ ($\omega = 2\pi/p$), if $h \in C^0(R^3, R)$ is a bounded function, p-periodic in the first variable, and if $\lim\limits_{|x| \to +\infty} g(x) \operatorname{sign} x = +\infty$ (or $-\infty$), then the ordinary equation*

$$x'' + f(x') + g(x) = h(t, x, x')$$

has at least one p-periodic solution, whatever the function $f \in C^0(R, R)$ may be.

PROOF. Put $\sigma = \tau = r = s = 0$ in Theorem 1, and use the convenction $0 \cdot (+\infty) = 0$.

COROLLARY 2 (Reissig [5], Theorem 5). *The ordinary equation*

$$x'' + f(x') + Kx + \gamma(x) = e(t) ,$$

where f, γ, e are continuous and e is p-periodic, has at least one p-periodic solution when $0 < K < \omega^2$, $|\gamma(x)| \leqslant P$.

PROOF. Put $Kx = g(x)$, $e(t) - \gamma(x) = h(t, x)$, and use Corollary 1.

PROOF OF THEOREM 1. We use a result of coincidence degree theory. Let X_i $(i = 0, 1, 2)$ be Banach spaces, $X_2 \subseteq X_1 \subseteq X_0$ with completely continuous embeddings. Let $L\colon X_2 \to X_0$ be a continuous linear Fredholm map of index zero. This means that $\operatorname{im} L$ is closed and $\dim\ker L = \dim\operatorname{coker} L < \infty$. As a consequence, we can find two continuous projections $P\colon X_1 \to \ker L$, $(I - Q)\colon X_0 \to \operatorname{im} L$. The restriction $L\colon X_2 \cap \ker P \to \operatorname{im} L$ is bijective: we call K its inverse. Let $N\colon X_1 \to X_0$ be an L-completely continuous map: this means that $QN\colon X_1 \to X_0$ is continuous and maps bounded sets into bounded sets, and that $K(I - Q)N\colon X_1 \to X_1$ is completely continuous. Actually the map $A\colon X_1 \to X_0$, $Ax = Px$, is L-completely continuous. In fact, $QA\colon X_1 \to X_0$ and $K(I - Q)A\colon X_1 \to X_2$ are linear bounded (and the embedding $X_2 \to X_1$ is completely continuous). Moreover,

$$\ker(L - A) = \{0\} .$$

Then it follows directly from a theorem by Mawhin (see [3]) that if there exists $\varrho > 0$ such that $|x|_{X_1} < \varrho$ whenever $(\lambda, x) \in]0,1[\times X_2$ satisfies

$$Lx = (1 - \lambda) Ax + \lambda Nx ,$$

then the equation $Lx = Nx$ has at least one solution $x \in X_2$.

We shall apply this result with $X_i = C^i(p, R)$ $(i = 0, 1, 2)$. We define $L\colon C^2(p, R) \to C^0(p, R)$, $(Lx)(t) = -x''(t)$. It is well known that L is a continuous linear Fredholm map of index zero. Moreover the projections

$$P\colon C^1(p, R) \to \ker L = \{\text{constants maps } R \to R\}$$

and

$$Q\colon C^0(p, R) \to \{\text{constants maps } R \to R\}$$

can be choosen as follows:

$$(Px)(t) = (1/p)\int_0^p x(\xi)\, d\xi\,, \qquad (Qx)(t) = (1/p)\int_0^p x(\xi)\, d\xi\,.$$

We define $N\colon C^1(p, R) \to C^0(p, R)$

$$(Nx)(t) = f\big(x'(t+\sigma(t))\big) + g\big(x(t+\tau(t))\big) - h\big(t, x(t+r(t)), x'(t+s(t))\big)\,.$$

Since f, g, h are continuous, and Q is linear bounded, we have easily that the composite map $QN\colon C^1(p, R) \to C^0(p, R)$ is continuous and maps bounded sets into bounded sets. Moreover $K(I-Q)\colon C^0(p, R) \to \to C^2(p, R)$ is linear bounded; hence $K(I-Q)N\colon C^1(p, R) \to C^1(p, R)$ is completely continuous. It follows that N is L-completely continuous.

Now equation (1) has a p-periodic solution x if and only if the coincidence equation $Lx = Nx$ has a solution $x \in C^2(p, R)$. So, to prove the existence of a p-periodic solution of (1), in virtue of the Mawhin's theorem, we need only to show that there exists a constant $\varrho > 0$ such that, if $\lambda \in]0, 1[$ and $x \in C^2(p, R)$ verify

$$Lx = (1-\lambda)Ax + \lambda Nx \tag{2}$$

(where $Ax = (1/p)\int_0^p x(\xi)\, d\xi$), then we have $|x'|_\infty + |x|_\infty < \varrho$.

First we prove the existence of a bound for $|x'|^\infty$. If we multiply (2) by $-x''$ and we integrate on $[0, p]$, we have easily

$$|x''|_2^2 = -\lambda\int_0^p (Nx)\, x''\, dt\,.$$

We shall use now the definition of N, the boundedness of h (condition (i)), the upper bound of g' (condition (ii)), and, possibly, the

Lipschitz constants of f and g:

$$-\int_0^p (Nx)x''\,dt = -\int_0^p f(x'(t))x''(t)\,dt - \int_0^p \big(f(x'(t+\sigma(t))) - f(x'(t))\big)x''(t)\,dt -$$

$$-\int_0^p g(x(t))x''(t)\,dt - \int_0^p \big(g(x(t+\tau(t))) - g(x(t))\big)x''(t)\,dt +$$

$$+\int_0^p h(t, x(t+r(t)), x'(t+s(t)))x''(t)\,dt \leqslant$$

$$\leqslant 0 + L_f|x'(\cdot+\sigma) - x'|_2\,|x''|_2 + K|x'|_2^2 + L_g|x(\cdot+\tau) - x|_2\,|x''|_2 + Mp^{\frac{1}{2}}|x''|_2\,.$$

It follows from Lemma 1 that

$$|x'(\cdot+\sigma) - x'|_2 = \Big|\int_0^{\sigma(t)} x''(t+\xi)\,d\xi\Big|_2 \leqslant \delta(\sigma)|x''|_2\,,$$

$$|x(\cdot+\tau) - x|_2 = \Big|\int_0^{\tau(t)} x'(t+\xi)\,d\xi\Big|_2 \leqslant \delta(\tau)|x'|_2\,.$$

Using the Wirtinger inequality $\omega|x'|_2 \leqslant |x''|_2$ we obtain, since $0 < \lambda < 1$,

$$|x''|_2^2 \leqslant -\int_0^p (Nx)x''\,dt \leqslant \left(L_f\delta(\sigma) + \frac{1}{\omega}L_g\delta(\tau) + \frac{1}{\omega^2}K\right)|x''|_2^2 + Mp^{\frac{1}{2}}|x''|_2\,.$$

It follows from condition (iii) that $|x''|_2 \leqslant$ const, and this implies, by an elementary argument, that there exists a constant $\alpha > 0$ such that

$$|x'|_\infty \leqslant \alpha\,.$$

In order to show the existence of a bound for $|x|_\infty$, we shall use the condition (iv). There is no loss of generality if we assume that $g(x)\operatorname{sign} x \to +\infty$ (as $|x| \to +\infty$). In fact, if $g(x)\operatorname{sign} x \to -\infty$, we have only to define the map $A\colon X_1 \to X_0$ in the « abstract » part by $Ax = -Px$ instead of $Ax = Px$, that is, for the « concrete » case, $(Ax)(t) = -(1/p)\int_0^p x(\xi)\,d\xi$. It is easy to see that, with this sign modification, the a priori bound $|x'|_\infty \leqslant \alpha$ is still true, and that the a priori bound for $|x|_\infty$ we shall prove for the case $g(x)\,\mathrm{sign}\,x \to +\infty$ can be obtained, in the case $g(x)\,\mathrm{sign}\,x \to -\infty$, with the same argument.

We compute the average for both terms of (2): we have $-Qx'' = (1-\lambda)QAx + \lambda QNx$, that is

$$0 = (1-\lambda)Ax + \lambda QNx\,. \tag{3}$$

Claim. There exists $\beta > 0$ such that, for any $x \in C^2(p, R)$ which satisfies (3) with some $\lambda \in]0, 1[$,

$$|Ax| \leqslant \beta\,.$$

This statement guarantees the existence of a bound for $|x|_\infty$. In fact, for each $x \in C^1(p, R)$, for every $t \in [0, p]$, there exist two points ξ, η such that $x(t) = Ax + x'(\xi)(t-\eta)$. It follows that if x is a solution of (2) then $|x - Ax|_\infty \leqslant \alpha p$, and so, if the claim is true, we obtain $|x|_\infty \leqslant \alpha p + \beta$.

Let us assume our claim is false. We can find a suitable sequence of pairs $(\lambda_n, x_n) \in]0, 1[\times C^2(p, R)$ such that

(j) for every n, $0 = (1-\lambda_n)Ax_n + \lambda_n QNx_n$,

(jj) the sequence λ_n is convergent to some point of the closed interval $[0, 1]$,

(jjj) $Ax_n \to +\infty$ or $Ax_n \to -\infty$.

By definition, QNx_n is equal to the sum of the sequence

$$a_n = (1/p)\int_0^p g(x_n(t + \tau(t)))\,dt$$

and of another sequence of the form

$$b_n = (1/p)\int_0^p (f(x'(\dots)) - h(\dots))\,dt\,.$$

Clearly b_n is bounded (by $\sup_{|x'| \leqslant \alpha} |f(x')| + M$). Let us consider a_n. We assume that the function g reaches its minimum, on the interval

$[Ax_n - \alpha p, Ax_n + \alpha p]$, at the point u_n, and its maximum on the same interval at the point v_n. Since

$$a_n = (1/p)\int_0^p g\big(x_n(t+\tau(t)) - Ax_n + Ax_n\big)\,dt\,,$$

and since

$$\sup_{t\in[0,p]} |x_n(t+\tau(t)) - Ax_n| \leqslant \sup_{t\in[0,p]} |x_n(t) - Ax_n| \leqslant \alpha p\,,$$

we obtain easily that $g(u_n) \leqslant a_n \leqslant g(v_n)$. Thus, if $Ax_n \to +\infty$, we must have $u_n \to +\infty$. It follows from condition (iv) that $g(u_n) \to +\infty$ and hence $a_n \to +\infty$. This is a contradiction with (j), since we have simultaneously $Ax_n \to +\infty$ and $QNx_n \to +\infty$. On the other hand, if $Ax_n \to -\infty$, we obtain $g(v_n) \to -\infty$ and $a_n \to -\infty$, which is again a contradiction with (j).

This proves our claim and completes the proof of the theorem.

As a consequence of Corollary 1 we obtain the result that the non-linear ordinary differential equation

$$x'' + f(x') + g(x) = h(t)\,, \tag{4}$$

where $f \in C^0(R, R)$, $g \in C^1(R, R)$, $h \in C^0(p, R)$ has at least one p-periodic solution if $g'(\cdot) \leqslant K < 0$. In fact this condition implies that (ii) and (iv) hold.

A natural question arises: do the monotonicity condition $g'(\cdot) \leqslant \leqslant K < 0$ imply the uniqueness of the periodic solution of (4)? We are able to give an affermative answer provided that f satisfies only a regularity condition: f is of class C^1. For instance, all the viscous dampings $f(x') = \beta|x'|^\varrho \operatorname{sign}(x')$, with $\beta > 0$, $\varrho \geqslant 1$, can be considered.

THEOREM 2. *The ordinary differential equation*

$$x'' + f(x') + g(x) = h(t) \tag{5}$$

where h is continuous and p-periodic, $g \in C^1(R, R)$, and $g'(\cdot) \leqslant K < 0$, has exactly one p-periodic solution whatever $f \in C^1(R, R)$ may be.

PROOF. The existence follows from Corollary 1. Let us assume that x, y are p-periodic solutions of (5). Then the difference $z = x - y$ is a p-periodic function which satisfies the linear homogeneous equation

$$z''(t) + a(t)z'(t) + b(t)z(t) = 0 ,$$

where

$$a(t) = \int_0^1 f'\big(sx'(t) + (1-s)y'(t)\big)\,ds , \quad b(t) = \int_0^1 g'\big(sx(t) + (1-s)y(t)\big)\,ds ,$$

are continuous coefficients with $b(\cdot) < 0$. Let us define the auxiliary function $w = e^A(z^2)'$, where $A(t) = \int_0^t a(s)\,ds$. We have

$$w' = 2e^A\big(z'^2 + z(z'' + az')\big) = 2e^A(z'^2 - bz^2) \geqslant 0 ,$$

hence w is increasing. We consider the set $N = \{t \in R\colon z'(t) = 0\}$. Since z is periodic, N is not empty, and $\inf N = -\infty$, $\sup N = +\infty$. But clearly $w(N) = \{0\}$, and thus the monotonicity of w implies that $w(t) = 0$ for every t. From the definition of w, it follows that $z =$ a constant. Now the condition $b < 0$ implies that if z is a constant solution of $z'' + az' + bz = 0$, then we must have $z = 0$.

Remark 2. Theorem 2 can be proved using the Caccioppoli global inversion method (see [4]). In fact we can define a map

$$T\colon x \in C^2(p, R) \to x'' + f(x') + g(x) \in C^0(p, R)$$

and we need only to prove that T is proper and that at each point x the differential $DT(x)$ is bijective. The differential $DT(x)$ is a linear map defined by $DT(x)[v] = v'' + f(x')v' + g'(x)v$. Since $g'(x) < 0$, the argument of Theorem 2 shows that $DT(x)$ is one-to-one, hence it is onto by the Fredholm Alternative. To prove the properness of T, we take the L^2-inner product of $Tx = h$ with x'': we have

$$|x''|_2^2 - \int_0^p g'\big(x(t)\big)x'^2(t)\,dt = \int_0^p h(t)x''(t)\,dt .$$

It follows $|x''|_2^2 \leqslant K|x'|_2^2 + |h|_2|x''| \leqslant p^{\frac{1}{2}}|h|_\infty|x''|_2$. The usual technique yelds that $|x'|_\infty$ and consequently $|f(x')|_\infty$ is bounded in terms of $|h|_\infty$. using $Tx = h$, we deduce that $|g(x)|_2$ is bounded. Now it is easy to see that $|x|_2$ is bounded: in fact, for $s \neq 0$, we have $(g(s) - g(0))/s \leqslant K < 0$, and so $(g(s) - g(0))^2/s^2 \geqslant K^2 > 0$, that is $s^2 \leqslant (1/K)^2 \cdot (g(s) - g(0))^2$, or $s^2 \leqslant c_1|g(s)|^2 + c_2|g(s)| + c_3$, with $c_1 > 0$, $c_2, c_3 \geqslant 0$. This last inequality holds for every s. In particular, for $s = x(t)$, we can deduce that $|x|_2$ is bounded. An elementary argument shows that $|x|_\infty$ is bounded in terms of $|h|_\infty$. This implies that T is a proper map.

In this way we obtain the further result that *the unique p-periodic solution x of the equation* (5) *C^1-depends upon the forcing term h.*

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