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On Wave Functions in Quantum Mechanics. I.

A. BRESSAN (*)

SUMMARY - Present experiments are compatible with the possibility P of determining quantistic states by the expectations of fundamental observables—i.e. observables that are measurable with arbitrary precision. More than P is often assumed to prove Theor 2.1, the fundamental proportionality property of the wave functions of a same state; for this aim some axioms not only unsatisfactorily supported but even disproved by to-day experiments were, and for simplicity reasons are still assumed. In Part 1 Theor 2.1 is deduced from Post 4.1, a postulate much weaker than P on the state ψ^+ immediately after a measurement. Furthermore the observables used in its proof as fundamental, are surely so in that an ideal apparatus to measure them has been exhibited. In addition some well known postulates related with Post 4.1 are discussed and Post 4.1 is justified (on the basis of widely used postulates). In Part 2 the denial of P is supported and in Part 3 an axiomatic theory, $\mathcal{T}_1$, of quantum mechanics is introduced, in which wave functions are defined and Born's rule need not be postulated. This involves a deep change of the above postulate on ψ^+. The construction of Part 3 has two main aims. One is non-formal: the reduction of primitive notions, *e.g.* in order not to use as (primitive) fundamental observables (many) observables that are not surely fundamental. The other is formal: to state the quantistic axioms rigorously and completely (on the basis of a theory of modal logic)—see Summary of Part 3. For the sake of simplicity only systems formed with spinless and pairwise distinguishable particles are dealt with; and by stronger reason the state space is assumed at the outset to be the one considered by von Neumann.

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Part 1

On a Fundamental Property of Wave Functions. Its Deduction from Postulates with a Good Operative Character and Experimental Support.

1. Introduction.

Let us consider the following problem concerning foundations of quantum mechanics:

(*a*) *to introduce the wave functions of a pure state, s, for a quantal system,* $\mathfrak{S}$, *by means of suitable physical properties, and*

(*b*) *to prove Theor* 2.1, i.e. *that those wave functions are mutually proportional* (*in that so are their corresponding values*), *by*

(*c*) *using only primitive notions of a surely operative character, and by*

(*d*) *accepting only postulates that are satisfactorily supported by experiments* (*possibly in an indirect way*).

Let us specify requirement (*c*) into the following

(*c'*) *use as fundamental,* i.e.—cf. e.g. [6, p. 270]—*as ideally measurable with arbitrary precision, only observables* (*or physical magnitudes or else variables*) *that are surely so* ([1]).

In order to specify requirement (*d*), let us consider the usual Hilbert space $\mathcal{H}$ determined (or generated) by $\mathfrak{S}$'s wave functions—i.e. the wave functions in which $\mathfrak{S}$ can effectively be—and let us remember that in the early years after the birth of quantum mechanics every self-adjoint operator in $\mathcal{H}$ was assumed to represent—or briefly to be—a fundamental observable—cf. Post 2.4. Among other things, this allowed Von Neumann to write a very simple proof of the proportionality theorem 2.1—cf. [8](*a*) or (*b*). Later the discovery of superselection rules showed the non-acceptability of that assumption. One

([1]) Surely fundamental are (in our terminology) those observables—such as position, momentum, and spin—that can be measured directly by actually exhibited ideal apparata.

can formulate a weak version of it, e.g. Post 3.1, that unlike Post 2.4, is compatible with those rules and all known experiments, and that incidentally can be the basis of an analogous brief proof of Theor 2.1. However Post 3.1, as well as Post 2.4, has a scanty experimental support (2).

In Parts 1 and 3 requirement (*d*) is complied with in connection with Post 2.4 by substantially replacing it with Post. 7.2 on the state ψ^+ of $\mathfrak{S}$ immediately after a measurement. This postulate is a particular version of some assumptions widely used in connection with ideal position measurements of the first kind—cf. e.g. [3]. By the aforementioned wide use, Post 7.2 and hence the theories developed in Parts 1 and 3 appear satisfactory to a certain good degree (3).

Since Post 7.2 is related to some controversial postulates, to which other postulates are now often preferred, it is natural to discuss the question, the more so in that we can afford a serious motive, apparently yet unpublished, to discard some of the postulates above—cf. footnote 5 below—, which contributes to justify the afore-mentioned preference.

* * *

One often admits the possibility of identifying a (quantistic) state s with—or of characterizing it by—the function $\omega \rightarrow \mathcal{E}_s(\omega)$ that, for every fundamental observable ω, gives the expectation value of ω in s (in the well known sense).

In compliance with requirement (c') and the fact that very few observables are surely fundamental, we decide not to accept the above possibility (considering it as at least unsure). This point of view is strengthened in Part 2, N. 11, where the following thesis is supported:

(*e*) *a* (*pure*) *state*, s, *cannot be determined by the expectations* $\mathcal{E}_s(\omega)$ *for all fundamental observables* ω.

The point of view above contributes to push us to change the usual notion of state, and more precisely to characterize a pure state s not by simple measurements performed on $\mathfrak{S}$ (when it is) in s, but by a

(2) This may be the reason why I have never seen it in the literature.

(3) However before discovering superselection rules Post 2.2, which is now refused, appeared satisfactory; and actual experiments, though compatible with Post 7.2 (on ψ^+) are far from giving us a practical certainty of its validity. Hence the theory constructed in Part 1 and based on Post 7.2 will be improved and requirement (*d*) will be complied with at a higher degree when the use of Post 7.2 is avoided.

more complex system of measurements, possibly performed under varied external forces. In Parts 1 and 3 we comply with thesis (*e*) by considering, for almost all states, an ideal position measurement of the first kind, followed by a moment measurement. Incidentally, even if (*e*) were disproved, our theory would still be interesting, especially because it has less primitive notions than previous theories on the same subject.

Of course a state s of $\mathfrak{S}$ is a short for a preparation of $\mathfrak{S}$ according to this paper as well as according to [2] or [3], but in harmony with the considerations above we accept a different criterium (based on a measurement system) to judge whether or not any two of these preparations are equivalent—cf. footnote 2 in Part. 3.

In Part 1 the notion of wave functions is considered as primitive, following ordinary textbooks such as [5]. However it is strictly connected with the notion of pure states. Hence one has a deeper insight in the subject being considered by constructing a theory where wave functions are defined (and Born's rule need not be postulated). Such is the theory $\mathcal{T}_1$ constructed in Part 3. Of course the task of proving Theor 2.1 in $\mathcal{T}_1$ is more complex, because Post 7.2 on ψ^+, as well as the postulates on ψ^+ that are generally considered, speak of orthogonality relations just by means of the notion of wave functions which, in $\mathcal{T}_1$, is not available at the outset. Hence the counterpart in $\mathcal{T}_1$ of these postulates on ψ^+ must be deeply different. In fact it is tightly connected with the evolution properties of $\mathfrak{S}$.

* * *

Now let us describe the content of Part 1 in more detail. As preliminaries, in nn. 2, 3 we remember some well known postulates of quantum mechanics to be used later, and in particular Post 2.4, which is incompatible with superselection rules. We also write a natural analogue of it, Post 3.1, that complies with those rules and allows a quick proof of the proportionality theorem 2.1 for wave functions. In n. 4 we prove Theor 4.1, a weak version of Theor 2.1, without using Post 2.4 or Post 3.1 (4). In this proof we use Post 4.1, which in n. 4 may appear ad hoc, but in nn. 6, 7 is justified as a consequence of widely used postulates on ψ^+; furthermore Post 4.1 is weaker than the latter ones.

(4) On answering a bibliography question of mine G. Sartori kindly sketched a proof of Theor 2.1 that by some changes has become the one of Theor 4.1

If the pure state s is *connex*, i.e. its wave functions have connex supports, then the validities of Theor 4.1 and Theor 2.1 for s are equivalent. After showing that it is unsatisfactory to postulate that every pure state is connex, we fill the gap from Theor 4.1 to Theor 2.1 on the basis of the additional Post 5.1, which substantially asserts the following surely acceptable regularity property: every pure state is joined with a connex state by a physically possible process (possibly under varied external forces).

Post 6.1 and Post 6.2 on ψ^+ were substantially stated by Von Neumann in 1932 and Lüders in 1952 respectively—cf. [8](*a*) or (*b*) and [4]. Lüders criticized the non-compliance of Post 6.1 with a certain requirement of his own, on which he based Post 6.2. Since these postulates are related with Post 4.1, on which the main theorem 2.1 is based, and similar postulates are controversial and e.g. in [3] they are replaced by other postulates, in n. 6 we show—as was hinted at above—that there is a strong motive not to accept Post 6.1 or 6.2: they, and Lüders' requirement itself, are incompatible with the irrefutable Post 6.3 (5). This incompatibility does not hold for Post 7.1 to Post 7.3. These three postulates are similar to widely used postulates on ψ^+ and have increasing strengths. The last two of them (unlike the first) imply Post 4.1, which is thus justified.

Theorem 8.1 generalizes Theor 4.1, a weak version of Theor 2.1. Furthermore it can be considered as a step for a proof of Theor 2.1, that is not based on any postulate—such as Post 5.1—involving the evolution of $\mathfrak{S}$. Indeed to prove Theor 8.1, besides Post 4.1 we use (instead of Post 5.1) only, so to say, a certain dual of Post 4.1 itself. The problem of such an alternative proof of Theor 2.1 is open.

2. On the well known Theor. 2.1 concerning the indetermination of a wave function and on the postulates that allow a simple proof of it.

In many textbooks of quantum mechanics, e.g. in connection with systems having classical analogues, it is substantially said that

(5) From a recent kind letter of D. Dombrowski it appears that his brief criticism of Lüders' postulate in [1, p. 179]—is based on an unpublished example which is simpler than (and independent of) mine. However unlike the latter it cannot be applied to discuss Von Neumann's postulate 6.1 or 6.2.

(i) the (pure) state s of any $\mathfrak{S}$ of these systems can be represented by means of a normalized wave function ψ belonging to—or characterizing a vector of—a certain Hilbert space $\mathcal{H}$ (6); and that

(ii) every physical magnitude ω, whose values for $\mathfrak{S}$ can be observed, is representable by means of a self-adjoint (linear) operator A ($= A_{(\omega)}$) in $\mathcal{H}$ in that, for every bounded measurable real function $f(x)$, the expected value $\mathcal{E}_s[f(\omega)]$ of $f(\omega)$ (for $\mathfrak{S}$ when $\mathfrak{S}$ is) in s equals the scalar product $\langle\psi|f(A)\psi\rangle$ in $\mathcal{H}$.

Furthermore one postulates that

(iii) ψ satisfies a Schrödinger equation for $\mathfrak{S}$:

$$(2.1) \qquad i\hbar\frac{\partial\psi}{\partial t} = H\psi \qquad (2\pi\hbar = h = \text{Plank constant}).$$

Let $R[C]$ be the class of real [complex] numbers. Then $\mathbb{R}^n[C^n]$ expresses the cartesian space of the n-tuples of real [complex] numbers.

Since $\langle\psi|A\psi\rangle = \langle c\psi|Ac\psi\rangle$ for $\psi\in\mathcal{H}$ and $c\in C$ with $|c| = 1$, at this point it is clear that for any such c the function $\psi' = c\psi$—which we say to be *proportional* to ψ—can also represent the state s (7). Generally the following converse theorem is not considered in the afore-mentioned textbooks:

THEOR 2.1. *Wave functions representing the same state are proportional.*

Let $\mathcal{B}_N[\mathcal{B}'_N]$ be the class of the Borel [bounded Borel] subsets of $\mathbb{R}^N$. For $B\in\mathcal{B}_N$ let χ_B be the characteristic function of B: $\chi_B(x) = 1$ [$=0$] for x in B [in $\mathbb{R}^N - B$].

Let us incidentally remark that, equivalently, in assertion (i) only functions f of the form $\chi_{a\mapsto b}$ could be considered, where $a\mapsto b = \{x\in\mathbb{R}: a\leqslant x\leqslant b\}$.

Let $\|\psi\|$ be $\sqrt{\langle\psi|\psi\rangle}$ for $\psi\in\mathcal{H}$; and for any self-adjoint operator A in $\mathcal{H}$ let E^A be its *spectral measure* defined as that projector-valued

(6) Let $[\psi]$ be the class of the functions $\varphi\in(\mathbb{R}^N\to C)$ such that the integral of $|\varphi-\psi|$ over $\mathbb{R}^N$ is zero. For the sake of simplicity we shall often write ψ instead of $[\psi]$.

(7) In [6] ψ and $c\psi$ are postulated to represent the same state for every nonzero $c\in C$—cf. Ax 4 on p. 292 there.

function of domain $\mathcal{B}_1$ such that $E^A(B) = \chi_B(A)$ for all $B \in \mathcal{B}_1$. As is well known,

(*a*) $[E^A(B), E^A(B')] = 0$ *for* $B, B' \in \mathcal{B}_1$ $([C, D] = CD - DC)$, *and*

(*b*) E^A *is strongly* σ*-additive*, i.e. *if* $f \in \mathcal{H}$, $B_i \in \mathcal{B}_1$, *and* $B_i \cap B_j = \emptyset$ *for* $i \neq j$ *and* $i, j = 1, 2, \ldots,$ *then*

$$(2.2) \qquad \lim_{n\to\infty} \left\| E^A\Big(\bigcup_{r=1}^{n} B_r\Big) f - E^A\Big(\bigcup_{r=1}^{\infty} B_r\Big) f \right\| = 0 .$$

Let A_i represent the observable ω_i, so that if no confusion may arise A_i will also be called an observable ($i = 1, 2, \ldots, N$); and let

$$(2.3) \qquad [E^{A_i}(B), E^{A_j}(B')] = 0 \qquad \text{for } B, B' \in \mathcal{B}_1 \ (i, j = 1, \ldots, N) .$$

Then $A_1, \ldots, A_N[\omega_1, \ldots, \omega_N]$ will be said to *commute* [to be *compatible*]. For all $\varphi, \psi \in \mathcal{H}$ there is exactly one $\boldsymbol{C}$-valued measure on $\mathbf{R}^N$, say $\mu^{\{A_i\}}$, that (is defined on $\mathcal{B}_N$), fulfils the condition

$$(2.4) \qquad \mu^{\{A_i\}}_{\varphi,\psi}(B_1 \times \ldots \times B_N) = \langle \varphi | E^{A_1}(B_1) \ldots E^{A_N}(B_N) \psi \rangle \qquad (\forall B_1, \ldots, B_N \in \mathcal{B}_1) ,$$

and is such that, for every normalized $\psi \in \mathcal{H}$, $\bar{\mu} = \mu^{\{A_i\}}_{\psi,\psi}$ is a probability measure: $\bar{\mu}(\mathcal{H}) = 1$ and $\bar{\mu}(B) \geq 0$ $(\forall B \in \mathcal{B}_N)$.

If f is a measurable complex function of domain $\mathbf{R}^N$, the set $\mathcal{D}_{\{A_i\}}$ of the vectors ψ in $\mathcal{H}$ for which f is $\bar{\mu}$-integrable, is dense (in $\mathcal{H}$), as is well known ($\mathcal{D}_{\{A_i\}} = \mathcal{H}$ iff f is bounded, for $A_1 \neq 0, \ldots, A_N \neq 0$).

The observable $\omega = f(\omega_1, \ldots, \omega_N)$ is represented by the operator $A = f(A_1, \ldots, A_N)$ defined by the condition

$$(2.5) \qquad \langle \varphi | A \psi \rangle = \int_{\mathbf{R}^N} f \, d\mu^{\{A_i\}}_{\varphi,\psi} \qquad \text{for all } \varphi \in \mathcal{H} \text{ and all } \psi \in \mathcal{D}_{\{A_i\}} .$$

By definition the *spectral measure* of the system $\{A_i\}$ of compatible observables (operators) is the projector-valued function $E^{\{A_i\}}$ on $\mathcal{B}_N$ defined by $E^{\{A_i\}}(B) = \chi_B(A_1, \ldots, A_N)$ for all $B \in \mathcal{B}_N$. Then the analogues for $E^{\{A_i\}}$ and $\mathcal{B}_N$ of assertions (*a*) and (*b*) on E^A and $\mathcal{B}_1$ hold, as is well known.

We shall say that the observables ω_1 to ω_N—as well as the corresponding operators $A_i = A_{(\omega_i)}$ ($i = 1, \ldots, N$)—are *fundamental com-*

patible observables if for all $B \in \mathfrak{B}_N$, $\chi_B(\omega_1, \ldots, \omega_N)$—or $\chi_B(A_1, \ldots, A_N)$—is a *fundamental observable* according to [6], i.e. it can be measured (directly) with arbitrary precision.

* * *

In order to explain how we mean the notion of state—involved by Theor 2.1—it is useful to remark that some axiomatic theories—where maybe wave functions are not considered, cf. [1] and [2]—include a postulate such as

POST 2.1. *If for every bounded observable ω its expected values $\mathcal{E}_s(\omega)$ and $\mathcal{E}_{s'}(\omega)$ in the states s and s' coincide, then $s = s'$* (8).

In addition—see e.g. (5.3.1) in [1, p. 195]—the state s of $\mathfrak{S}$ is identified with the function $\omega \to \mathcal{E}_s(\omega)$ that carries every bounded observable ω into its expected value in s. Thus, if one consider only (bounded) observables that are fundamental, then s is identified with $\mathfrak{S}$'s statistical reactions to (precise) measurements of fundamental observables, performed (when $\mathfrak{S}$ is) in s. If we do so, we substantially mean the expectation value $\mathcal{E}_s(\omega)$ in s in the usual *direct* way and we identify Post 2.1 with the following

ASSUMPTION 2.1. *If s and s' are pure states and $\mathcal{E}_s(\omega) = \mathcal{E}_{s'}(\omega)$ for every fundamental observable ω, then $s = s'$.*

Since the state s of $\mathfrak{S}$ can be meant as a short for a preparation $\mathfrak{P}$ of $\mathfrak{S}$, if we accept the assumption above, it works as a direct operative criterium for distinguishing states (or for stating whether or not two preparations are equivalent). Unfortunately presently known experiments are compatible with the assumption above, but are far from assuring its truth. More, its acceptance contrasts with our thesis (e) in n. 1, supported in Part 2. In Part 3 we consider another operative criterium to distinguish pure states, compatible with thesis (e) and based mainly on $\mathfrak{S}$'s statistical behaviour under certain systems of measurements. We prove Theor 2.1, so that if s is a pure state and A a bounded (linear) operator, then $\langle\psi|A\psi\rangle$ is independent of the normalized wave function ψ of s. Hence we can regard

$$\langle A\rangle_s = \langle A\rangle_\psi = \langle\psi|A\psi\rangle \tag{2.6}$$

(8) For instance postulate (3.3.1) in [1, p. 192] substantially coincides with Post 2.1, under the obvious assumption that if f is a Borel mapping of $\mathbb{R}$ into $\mathbb{R}$ and ω is an observable, then such is also $f(\omega)$.

as an *expectation* of A in an *indirect* (but physical) sense. If A represents a fundamental observable ω, then $\langle A\rangle_s = \langle A_{(\omega)}\rangle = \mathcal{E}_s(\omega)$.

* * *

It is convenient to recall Born's interpretative postulate in the following form—somewhat similar with Def 1.3 in [6, p. 262]:

POST 2.2 (*Born's correspondence rule for determinative measurements*). *Let the self-adjoint operators A_1 to A_N represent the fundamental compatible observables ω_1 to ω_N. Then, for $B \in \mathcal{B}_N$ the theorically predicted probability that a simultaneous measurement of A_1 to A_N on $\mathfrak{S}$ in the pure state s of wave function ψ, gives a result $\beta = (\lambda_1, \ldots, \lambda_N)$ in B is*

$$\mathcal{E}_s(\omega) = \mu^{\{A_i\}}_{\psi,\psi}(B) = \langle\psi|E^{\{A_i\}}(B)\psi\rangle \tag{2.7}$$

$$\text{for } \omega = \chi_B(\omega_1, \ldots, \omega_N) \text{ and } \|\psi\| = 1\,.$$

REMARK. Obviously the observable ω_1 is fundamental if and only if $\{\omega_i\} = \{\omega_1\}$ is a set of fundamental compatible observables.

In 1932 Von Neumann substantially wrote the following (9).

POST 2.3 (*Von Neumann*). *Let a precise simultaneous preparatory measurement of the first kind be performed on $\mathfrak{S}$ in the pure state ψ, to know the values of the fundamental compatible observables ω_1 to ω_N represented by the bounded operators A_1 to A_N with purely point spectra. Let the result be $\beta = (\lambda_1, \ldots, \lambda_N)$ with β_i eigenvalue of A_i ($i = 1, \ldots, N$). Then the state ψ^+ of $\mathfrak{S}$ immediately after the measurement belongs to the eigenspace Π_{λ_i} of A_i relative to λ_i ($i = 1, \ldots, N$).*

In the early days of quantum mechanics—cf. e.g. [6, fnt. on p. 258]—there was the tendency to assuming the following postulate—see [8]:

POST 2.4. *Every self-adjoint operator in $\mathcal{H}$ represents a fundamental observable.*

(9) In Von Neumann's version of Post 2.3 arbitrary measurements are referred to. Margenau criticized it by remarking that preparatory measurements on $\mathfrak{S}$ may destroy $\mathfrak{S}$. Furthermore in [3], on p. 165, a measurement is called *of the first kind* if it would give the same value when immediately repeated; moreover, there one can read an example of a measurement of a different kind (called of the *second kind*). This explains the hypothesis on the measurement considered in Post 2.3.

However—cf. e.g. the same footnote—the discovery of superselection rules, which admittedly hold for systems with a variable number of particles, indicated for the first time that Post 2.4 lacks experimental support. This happens in that, since experiments support Post 2.3 very well, Post 2.4 turns out to be even logically incompatible with these rules:

SUPERSELECTION RULES. *The Hilbert space $\mathcal{H}$ where the (possible) pure states of $\mathfrak{S}$ are represented, is the closure of the direct sum of some linear manifolds $\mathcal{M}_1, \mathcal{M}_2, \ldots$ formed with the vectors that are parallel to those representing all aforementioned states.*

3. A substitute for Post. 2.4 that substantially keeps a simple proof of Theor. 2.1. Usefulness of reducing postulates.

Since Post 2.4 is incompatible with superselection rules, we now replace it with Post 3.1 below. Then we remember a simple proof of the afore-mentioned incompatibility to show that it does not keep holding after the replacement. Thus this replacement appears natural. It is also useful in that it keeps the validity of a simple proof of Theor 2.1. This will be shown explicitly, also in order to put in evidence a certain aspect of that proof, useful in the sequel.

POST 3.1. *Assume that* (i) *the Hilbert space $\mathcal{H}$ is the closure of the direct sum of the* (*possibly non-closed*) *linear manifolds $\mathcal{M}_1, \mathcal{M}_2, \ldots$,* (ii) *$\mathcal{M}_1 \cup \mathcal{M}_2 \cup \ldots$ is the class of the vectors in $\mathcal{H}$ parallel with those representing pure states of $\mathfrak{S}$, and* (iii) *the observable ω of $\mathfrak{S}$ is represented by the self-adjoint operator A in $\mathcal{H}$, of spectral measure E^A. Then ω is fundamental if and only if, for every $B \in \mathcal{B}_1$, the subspace $E^A(B)(\mathcal{H})$ on which $E^A(B)$ projects is generated by vectors belonging to $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \ldots$*

Let φ be a unit vector in $\mathcal{H} - \cup_n \mathcal{M}_n$, so that the bounded operator $P = |\varphi\rangle\langle\varphi|$ has the spectrum $\{0, 1\}$. Let us first accept Post 2.4, so that the observable ω represented by P is fundamental. Then by Post 2.2 a measurement of ω on $\mathfrak{S}$ in the (pure) state ψ gives the result 1 with the probability $\langle\omega\rangle_\psi = |\langle\varphi|\psi\rangle|$. Furthermore we can choose ψ in some $\mathcal{M}_i$ in such a way that $\langle\varphi|\psi\rangle \neq 0$. Hence, by Post 2.3, the state ψ^+ of $\mathfrak{S}$ immediately after this measurement can be outside every $\mathcal{M}_i$ (with a positive probability) in contrast to superselection rules.

Let us now replace Post 2.4 with Post 3.1. For $B = \{1\}$ the subspace on which $E^P(B)$ projects is $E^P(B)(\mathcal{H}) = \{\lambda\varphi : \lambda \in \boldsymbol{C}\}$; hence it is not generated by vectors in $\cup_n \mathcal{M}_n$, so that by Post 3.1 the observable ω is not fundamental and the preceding incompatibility reasoning does not hold.

THEOR 3.1. *The expected value* $\mathcal{E}_s(\omega)$ *of the (real) Borel function* $\omega = f(\omega_1, \ldots, \omega_N)$ *of the fundamental compatible observables* $\omega_1, \ldots, \omega_N$ *in a measurement of* ω_1 *to* ω_N *made on* $\mathfrak{S}$ *in the pure state* ψ ($\|\psi\| = 1$), *is given by*—cf. (2.7)

$$\mathcal{E}_\psi(\omega) = \langle A\rangle_\psi = \langle\psi|A\psi\rangle = \int_{\mathbf{R}^N} f\, d\mu_{\psi,\psi}^{\{A_i\}} \tag{3.1}$$

where $A = f(A_1, \ldots, A_N)$ *and* $A_i = A_{(\omega_i)}$ $(i = 1, \ldots, N)$.

DEDUCTION *of Theor* 2.1 *from postulates* 2.1, 2.2, *and* 3.1. Let ψ and ψ' be unit vectors in $\mathcal{M}_1 \cup \mathcal{M}_2 \cup \ldots$, so that they represent two pure states s and s', by the definition of the $\mathcal{M}_i$'s—cf. Post 3.1. Then, by the mathematical definition $(2.6)_2$,

$$\langle A\rangle_\psi = 1 \quad \text{and} \quad \langle A\rangle_{\psi'} = |\langle\psi|\psi'\rangle|^2 \qquad \text{for } A = |\psi\rangle\langle\psi|\,; \tag{3.2}$$

furthermore by Post 3.1 A represents a fundamental observable ω. Hence by the remark below (2.6) and by Theor 3.1, $\mathcal{E}_\psi(\omega) = \langle A\rangle_\psi$ and $\mathcal{E}_{\psi'}(\omega) = \langle A\rangle_{\psi'}$—cf. $(3.1)_1$. Now assume $s = s'$, hence $\mathcal{E}_\psi(\omega) = \mathcal{E}_{\psi'}(\omega)$ which yields $\langle A\rangle_\psi = \langle A\rangle_{\psi'}$. Then, by (3.2) $|\langle\psi|\psi'\rangle| = 1$, so that by the Schwartz inequality $\psi' = c\psi$ for some $c \in \boldsymbol{C}$ with $|c| = 1$. q.e.d.

Remark that the above use of Post 3.1 is essential because if A is not fundamental, the identity $s = s'$ does not imply $\langle A\rangle_\psi = \langle A\rangle_{\psi'}$.

* * *

Post 3.1, which replaces Post 2.4, has on it the advantage not to be surely false, but shares with Post 2.4 the lack of a satisfactory experimental support. More in particular the observables asserted by Post 3.1 to be fundamental are probably still too many. Indeed in ordinary textbooks—cf. [5], [6]—the position, momentum, and spin of a particle are explicitly considered as fundamental observables. To them energy is usually added. However the considerations in n. 11

(Part 2) push us to think that only a few other functionally independent observables can be fundamental, and by no means all of those that are so according to Post 3.1. As a consequence it is natural to search for a proof of Theor 2.1 where no fundamental observables other than position, momentum, and spin are used.

4. A proof of a weak version of Theor. 2.1 independent of certain postulates.

For the sake of simplicity we assume $\mathfrak{S}$ to have a classical analogue, and more precisely to consist of n particles M_1 to M_n that are pairwise distinguishable and spinless. We prove Theor 4.1 below, which is a weak version of Theor 2.1, on the basis of Post 4.1 below (to be justified in n. 7), but without using either Post 2.4 or Post 3.1.

Let $q = (q_1, \ldots, q_N)$, with $N = 3n$, be a system of Lagrangian coordinates for M_1 to M_n, so that any wave function for $\mathfrak{S}$ has the form $\psi(q)$. For every $B \in \mathfrak{B}_N$ we consider the projector θ_B:

$$(4.1) \qquad (\theta_B \psi)(q) = \chi_B(q)\psi(q) = \begin{cases} \psi(q) & \text{for } q \in B\,, \\ 0 & \text{for } q \notin B\,. \end{cases}$$

Post 4.1. *If the vectors ψ_1 and ψ_2 in $\mathcal{H}$ represent—i.e. are possibly non-normalized wave functions of—the same state s of $\mathfrak{S}$, $B \in \mathfrak{B}_N$, and $\|\theta_B\psi_1\| \neq 0 \neq \|\theta_B\psi_2\|$, then also $\theta_B\psi_1$ and $\theta_B\psi_2$ represent the same state s_B.*

This postulate, which may seem to be *ad hoc*, follows from Post 7.2 below, a weak version of the reduction of the wave packet, in that according to this reduction postulate $\theta_B\psi$ represents the state s_B of $\mathfrak{S}$ immediately after an ideal measurement of $\chi_B(q)$, which is of the first kind and is performed on $\mathfrak{S}$ in the state ψ.

Let us remember that the support Supp (ψ) of $\psi(\in \mathcal{H})$ is the set of the points $q \in \mathbb{R}^N$ for every neighborhood B of which $\int_B |\psi|^2 dq > 0$. Hence Supp (ψ) is closed.

Theor 4.1. *If the functions ψ_1 and ψ_2 are continuous, represent the same state s of $\mathfrak{S}$, and their first partial derivatives exist* a.e. *and are square integrable, then the restrictions of ψ_1 and ψ_2 to any* (*connex*) *component of* Supp (ψ_1) *are proportional.*

PROOF. By (4.1), for $B \in \mathfrak{B}_N$ we have

$$E^{\{Q_i\}}(B) = \theta_B \quad \text{where } (Q_h\psi)(q) \equiv q_h\psi(q) \quad \text{for } h = 1, \ldots, N\,. \tag{4.2}$$

By Post 2.2, (2.7) holds for $A_i = Q_i$ and $\omega_i = q_i$ $(i = 1, \ldots, N)$; furthermore we assume $\|\psi_1\| = \|\psi_2\| = 1$, which is not restrictive. Then

$$\mathcal{E}_s[\chi_B(q)] = \int_{\mathbf{R}^N} \psi_r^* \theta_B \psi_r \, dq = \int_B |\psi|^2 dq_i \qquad (B \in \mathfrak{B}_N;\ r = 1, 2)\,. \tag{4.3}$$

Since both ψ_1 and ψ_2 represent s, the validity of (4.3) for all $B \in \mathfrak{B}_N$ yields

$$|\psi_1| = |\psi_2| \text{ a.e.}, \qquad \text{hence} \qquad \operatorname{Supp}(\psi_1) = \operatorname{Supp}(\psi_2)\,.$$

Then we have

$$\psi_2(q) = \exp[i\varphi(q)]\psi_1(q) \qquad (q \in \mathbf{R}^N) \tag{4.5}$$

for some function φ that is real, is differentiable a.e., and is continuous outside a (possibly empty) hypersurface Σ on which it has the discontinuity $2k\pi$ (k integer).

For $h = 1, \ldots, N$ and $r = 1, 2$ we define the sets Δ_h^+ and Δ_h^-, and the scalar $\mathcal{N}_{r,B}$ by

$$\Delta_h^{\pm} = \left\{ q \in \mathbf{R}_N \colon \pm \frac{\partial\varphi}{\partial q_h} > 0 \right\}, \quad \mathcal{N}_{r,B} = \|\theta_B \psi_r\| \qquad (B \in \mathfrak{B}_N)\,. \tag{4.6}$$

Then, by (4.5)

$$\mathcal{N}_B = \mathcal{N}_{1,B} = \mathcal{N}_{2,B}\,, \qquad \mathcal{N}_{\Delta_h^+} + \mathcal{N}_{\Delta_h^-} = 1\,. \tag{4.7}$$

Now let P_h be the Fourier transform of the operator Q_h, so that P_h represents the momentum p_h conjugate to q_h.

First assume $\mathcal{N}_B \neq 0$, so that by $(4.7)_{1,2}$, $(4.6)_2$, and Post 4.1 $\theta_B\psi_1$ and $\theta_B\psi_2$ represent the same state s_B. Then by Post 2.2,

$$\langle P_h \rangle_{\theta_B\psi_1} = \mathcal{N}_B \mathcal{E}_{s_B}(p_h) = \langle P_h \rangle_{\theta_B\psi_2}\,.$$

Multiplication of this by $i\hbar^{-1}$ gives

$$\int_B \psi_1^* \frac{\partial \psi_1}{\partial q_h} dq = \int_B \psi_2^* \frac{\partial \psi_2}{\partial q_h} dq = \int_B \psi_1^* \frac{\partial \psi_1}{\partial q_h} dq + i \int_B |\psi_1(q)|^2 \frac{\partial \varphi}{\partial q} dq \,. \tag{4.8}$$

Hence, if $B = \Delta_h^+$ or $B = \Delta_h^-$ we have

$$\int_B |\psi_1(q)|^2 \left| \frac{\partial \varphi}{\partial q_h} \right| dq = 0 \tag{4.9}$$

in the case $\mathcal{N}_B \neq 0$. In the opposite case (4.9) holds trivially. Hence for $h = 1, \ldots, N$ (4.9) holds for $B = \Delta_h^+ \cup \Delta_h^-$, hence for $B = \operatorname{Supp}(\psi_1)$. Then by (4.5) and the continuity of ψ_1 and ψ_2 we conclude that the function φ—that fulfils condition (4.5) and is not completely determined by it—can be chosen to be constant in every component of Sup (ψ_1), so that Σ turns out to be empty. q.e.d.

5. Accomplishment of the proof of Theor. 2.1 on the basis of an additional postulate.

We want to complete the proof of Theor 2.1, on which Theor 4.1 constitutes the first step. It would become the last step if we postulated that the *support* of every pure state s, defined as the common support of the wave functions of $\mathfrak{S}$, is connex. This connection assertion is compatible with actual experiments but contrasts to the following important requirement:

REQUIREMENT 5.1. *Every regularity assumption on wave functions must be time invariant in the solutions of the Schrödinger equations for reasonably regular problems.*

Indeed, if $\mathfrak{S}$ is a spinless free particle, the dispersion of its wave packet ψ_t is well known. Then, if Supp (ψ_t) has (only) n components for $t = 0$, these reduce to one at some $t > 0$. Since now $\mathfrak{S}$ is invariant under time inversion, some connex pure state appears to evolve into a non-connex one.

Before replacing the postulate hinted at above by Post 5.1 below, which complies with requirement 5.1, it is useful remarking that the system $\mathfrak{S}$ considered in n. 4 consists of the particles M_1 to M_n and has an internal structure, which determines internal forces. Furthermore the external forces acting on $\mathfrak{S}$ are given. Hence, when a system of canonical variables p, q is fixed, the total (classical or quantistic) Hamiltonian H of $\mathfrak{S}$ and the contribution $H^{(e)}$ of external forces to it are determined up to the addition of a function $c(t)$ of time; and so is $H^{(i)} = H - H^{(e)}$.

Let $\mathfrak{S}^{(i)}$ be the system $\mathfrak{S}$ considered disregarding its external forces, so that it is natural to write $\mathfrak{S} = (\mathfrak{S}^{(i)}, H^{(e)})$. We can keep $\mathfrak{S}^{(i)}$ fixed and can vary $H^{(e)}$; *i.e.* we can associate $\mathfrak{S}$ with $\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$ in connection with an arbitrary change of external forces. A *state* s' *for* $\mathfrak{S}'$ *at* t'—i.e. a state s' possible for $\mathfrak{S}'$ at the instant t'—can be called a *state for* $\mathfrak{S}^{(i)}$ *at* t'. Furthermore, if $H'^{(e)} = H^{(e)}$ up to some instant τ, and if $\tau \leqslant t'$, then we may regard s' as a *state for* $\mathfrak{S}$ *at* t' *under varied external forces*. In addition, for some process $t \to s_t$ possible for $\mathfrak{S}'$ in $\tau \vdash\!\dashv t' (= \{t : \tau \leqslant t \leqslant t'\})$ assume $s_\tau = s$ and $s_{t'} = s'$; then we say that s *is joinable with* s' (*under varied external forces*).

Post 5.1. *Every pure state* s *of the quantal system* $\mathfrak{S}$ *is joinable with a connex* (*pure*) *state* s' (*under varied external forces*).

This postulate appears to be reasonably both true and in agreement with requirement 5.1. It seems to be only reasonably true, because e.g. one may suspect that one can define a wave function ψ —whose support has infinitely many components—that represents a state of $\mathfrak{S}$ that cannot be joined with any connex state. Thus Post 5.1 can be imputed to recognize as (possible) states only a part of them. However this limitation (if true) is much lighter than e.g. the acceptance of a classical rather than a relativistic quantum theory. So Post 5.1 seems quite acceptable. (At most it rules out physically uninteresting states.)

Proof of Theor 2.1. Let $\psi^{(1)}$ and $\psi^{(2)}$ be two wave functions of the pure state s for $\mathfrak{S}$ at τ. By Post 5.1, s is joinable with a connex state s'. Hence $\mathfrak{S}' = (\mathfrak{S}^{(i)}, H'^{(e)})$ can be chosen in such a way that it can undergo a process $t \to s_t$ in some interval $\tau \vdash\!\dashv t'$, for which $s_\tau = s$ and $s_{t'} = s'$. Now, by some well known postulates not written here explicitly, $\psi^{(1)}$ and $\psi^{(2)}$ are so regular that for a suitable choice of the Hamiltonian H' of $\mathfrak{S}'$, (i) the Schrödinger equation

$$(5.1) \qquad i\hbar \frac{\partial \psi}{\partial t} = H' \psi \qquad \text{with } H' = H^{(i)} + H'^{(e)}$$

is solved by a representation $t \to \psi_t^{(r)}$ of the process $t \to s_t$ that fulfils the initial condition $\psi_\tau^{(r)} = \psi^{(r)}$, so that $\psi_t^{(r)}$ represents s_t for $t \in \tau \vdash\!\dashv t'$ and $r = 1, 2$, (ii) the uniqueness theorem holds for the above solutions of $(5.1)_1$, and (iii) the function $(t, q) \to \psi_t^{(r)}(q)$ is continuous and $\partial\psi_t^{(r)}(q)/\partial q_h$ exists a.e. and is square integrable ($h = 1, \ldots, N$; $r = 1,2$).

Then $\psi_{t'}^{(1)}$ and $\psi_{t'}^{(2)}$ represent the connex pure state s', so that by Theor 4.1 they are proportional, i.e. $\psi_{t'}^{(2)} = c\psi_{t'}^{(1)}$ for some $c \in \boldsymbol{C}$. As a consequence, by the definition $\psi_t = \psi_t^{(2)} - c\psi_{t'}^{(1)}$ $(\tau \leqslant t \leqslant t')$, the function $t \to \psi_t$ solves equation $(5.1)_1$ in $\tau \vdash\!\dashv t'$ and vanishes for $t = t'$. Then by the afore-mentioned uniqueness theorem, $\psi_t = 0$ for $\tau \leqslant t \leqslant t'$, so that $\psi^{(2)} = \psi_\tau^{(2)} = c\psi_\tau^{(1)} = c\psi^{(1)}$ and Theor 2.1 is proved.

6. On two postulates of Von Neumann and Lüders respectively.

Here we discuss some postulates related with Post 4.1 on which the proofs of theorems 4.1 and 2.1 are based. We begin with the following one, substantially stated by Von Neumann in 1932 —cf. [4, p. 325].

POST 6.1 *If $\mathcal{A}$ is an apparatus for measuring the (fundamental) observable ω represented by A and capable of only the values $\lambda_1, \lambda_2, \ldots$, then for every $k \in \{1, 2, \ldots\}$ there is a subspace Σ_k of the eigenspace $\Pi_{\lambda_k}^A$ of A for the proper value λ_k, such that if a measurement of ω is performed on $\mathfrak{S}$ in the state ψ by using $\mathcal{A}$ and with the result λ_k, then the orthogonal projection ψ^+ of ψ on Σ_k represents the state of $\mathfrak{S}$ immediately after the measurement.*

In 1952 Lüders made two criticizms—cf. [4, p. 325]—to Post 6.1, the second of which complains that Post 6.1 fails to fulfil the following

REQUIREMENT (Lüders). *The state s^+ of $\mathfrak{S}$ immediately after a measurement of the observable ω is determined by ω, the state s of $\mathfrak{S}$ immediately before the measurement, and the result λ_k of this measurement.*

In armony with this requirement, in [4] one asserts substantially the following

POST 6.2 *Von Neumann's Post 6.1 always holds for $\Sigma_k = \Pi_{\lambda_k}^A$.*
The requirement above fails to be fulfilled by the assumptions of various authors—see e.g. [3], p. 166—and in particular by our postulates 7.1-3

on one of which the justification of Post 4.1 will be based. Therefore it is useful to emphasize that the disagreement with the requirement above is not at all a defect, by showing that

(a) Von Neumann's postulate 6.1 is unsatisfactory, because its disagreement with Lüders's requirement is in some sense too little, and

(b) Lüders's postulate 6.2—which has been criticized by various authors, see e.g. [1], p. 179 and our footnote 5—is not compatible with the following irrefutable

POST 6.3. *If f is a measurable mapping of $\mathbb{R}^N$ into $\mathbb{R}$, every measurement of the compatible observables ω_1 to ω_N is a measurement of $\omega = f(\omega_1, \ldots, \omega_N)$.*

With a view to proving assertions (a) and (b), we remark that $\mathfrak{S}$ —see n. 4—has some observable ω represented by a (bounded) operator A with the spectrum $\{-1, 0, 1\}$—e.g. $A = f(Q_1)$ with $f(q) = \chi_{1\vdash\dashv 2}(q) - \chi_{3\vdash\dashv 4}(q)$. Then ω^2 is represented by A^2, whose spectrum is $\{0, 1\}$. The eigenspace $\Pi_1^{A^2}$ (of A^2 for the eigenvalue 1) is the space generated by Π_1^A and Π_{-1}^A, i.e. $\Pi_1^A + \Pi_{-1}^A$.

To prove assertion (a) let $s[s^+]$ be the state of $\mathfrak{S}$ immediately before [after] a measurement of ω on $\mathfrak{S}$ with the apparatus $\mathcal{A}$; let λ_k be the result. Post 6.1 implies that s^+ is determined by ω, $\mathcal{A}$, s, and λ_k. This is unacceptable by the following motive.

Let Σ_k ($\subseteq \Pi_{\lambda_k}^A$) be the space corresponding to λ_k according to Post 6.1 ($|k| \leqslant 1$). These spaces are mutually orthogonal and s can be assumed to be represented by a unit vector ψ forming with Σ_1 and Σ_{-1}, two angles equal to $\pi/4$, so that

$$(6.1) \qquad \psi \in \Sigma_1 + \Sigma_{-1} - (\Pi_1^A \cup \Pi_{1-}^A) \subseteq \Pi_1^{A^2} - (\Pi_1^A \cup \Pi_{-1}^A)\,.$$

The result of a measurement of ω with $\mathcal{A}$ (on $\mathfrak{S}$ in s) must be 1 or -1, and both have a positive probability. If the result is $1[-1]$, then by Post 6.1 s^+ is represented by the orthogonal projection $\psi^+[\psi^-]$ of ψ on Σ_1 $[\Sigma_{-1}]$ (hence $\psi^+ \in \Pi_1^A \cup \Pi_{-1}^A$).

By Post 6.3 ω^2 has also been measured, and the considerations above show that the result 1 does not determine the state s^+ in that it can belong to Σ_1 and can belong to Σ_{-1}. Hence assertion (a) holds.

Incidentally a measurement of ω^2 on $\mathfrak{S}$ in the eigen-state ψ for ω^2 has been shown to be able to turn the state of $\mathfrak{S}$ into a mixture (formed with ψ^+ and ψ^-).

To prove assertion (*b*) we keep the above assumptions on ψ, ψ^+, s, and s'. By Post 6.2 $\Sigma_k = \Pi_k^A$ for $|k| \leqslant 1$, so that by $(6.1)_1$ $\psi \in \Pi_1^A + \Pi_{-1}^A - (\Pi_1^A \cup \Pi_{-1}^A)$ and by the considerations above on the measurement of ω with the apparatus $\mathcal{A}$, $\psi^+ \in \Pi_1^A \cup \Pi_{-1}^A$.

Now we consider again the measurement above as one of ω^2, so that by Post 6.2, ψ^+ can be identified with the projection of ψ on $\Pi_1^{A^2}$; hence $\psi^+ = \psi$ by (6.1). This implies $\psi^+ \notin \Pi_1^A \cup \Pi_{-1}^A$ in contrast to a preceding deduction. Hence assertion (*b*) holds.

We conclude that if f is not injective, a measurement of ω cannot be an ideal measurement of $f(\omega)$.

7. Some postulates to justify the rather « ad hoc » Post. 4.1.

Since there are strong motives not to accept Lüders's Post 6.2 and especially Lüders's requirement n. 6, in harmony with Von Neumann [Post 6.1] and especially with [3, p. 166] we admit that measurements of the same observable ω on $\mathfrak{S}$ in the state s, made with different apparata (and of the first kind) may result in different states after the interaction, and that, at least for some choices of ω, ideal measurements of ω, of the first kind, are possible. More in particular we consider Posts 7.1-3 below, of increasing strengths, about the state s^+ of $\mathfrak{S}$ immediately after a measurement of the first kind. The second (or third) of them will be used to justify Post 4.1, hence as a basis of the proof of Theor 2.1 given in n. 5. To this end Post 7.1 is too weak. However its consideration helps to make the (whole) situation clearer (also in connection with Parts 2-3).

A self-adjoint operator A is called *discrete* if it has a purely point spectrum. The observable ω is called *discrete* [*bounded*] if so can be called the corresponding operator $A_{(\omega)}$.

DEF 7.1 [7.2]. *Assume that* (*a*) ω_1 *to* ω_n *are discrete fundamental observables*, (*b*) $\psi(\in \mathcal{H})$ *represents* [s *is*] *a pure state of* $\mathfrak{S}$, *and* (*c*) *if, by using the apparatus* $\mathcal{A}$, *we measure* ω_1 *to* ω_n *simultaneously on* $\mathfrak{S}$ *in the state* $\psi[s]$ *with the result* $\lambda = (\lambda_1, \ldots, \lambda_n)$, *then the state* s^+ *of* $\mathfrak{S}$ *immediately after the measurement is represented by the orthogonal projection* $\psi^+[\varphi^+]$ *on* $\Pi_{\lambda_1}^{A_1} \cap \ldots \cap \Pi_{\lambda_n}^{A_n}$ *of* ψ [*of the arbitrary wave function* φ *for* s]. *Then we say that the apparatus* $\mathcal{A}$ *to measure* ω_1 *to* ω_n, *is orthogonal for the wave function* ψ [*the state* s].

The orthogonality of $\mathcal{A}$ for the wave function ψ appears to be equivalent to the one for the state s represented by ψ only when the validity of Theor 2.1 is known. However, even disregarding this validity, since we now admit to know what a wave function is, we may assert that

$\mathcal{A}$ is orthogonal for every wave function of $\mathfrak{S}$ if and only if it is orthogonal for every state s of $\mathfrak{S}$.

If the last orthogonality condition holds, $\mathcal{A}$ will be simply said to be *orthogonal.* The distinction of the two orthogonality notions introduced by Defs 7.1-2 is interesting especially in connection with the theory $\mathcal{T}_1$, developed in Part 3, where wave functions are defined.

POST 7.1 [7.2]. *If ω_1 to ω_n are fundamental compatible observables of $\mathfrak{S}$ and $\psi[s]$ represents [is] a pure state of $\mathfrak{S}$, then there is an apparatus $\mathcal{A}$ to measure ω_1 to ω_n, orthogonal for $\psi[s]$.*

POST 7.3. *If ω_1 to ω_n are as in Post 7.1, some apparatus to measure them is orthogonal.*

The paradoxical assertions (a) and (b) in n. 6 have no (true) analogue for any of Posts 7.1-3. We now turn to the main aim of this section.

THEOR 7.1. *Post 7.2 implies Post 4.1 on which our proof of Theor 4.1 is based.*

Indeed, for $B \in \mathcal{B}_N$ the observable $\omega = \chi_B(q_1, \dots, q_N)$ is represented by $\chi_B(Q_1, \dots, Q_N) = \theta_B$—cf. (4.1)—which is the orthogonal projector on Π_1^A, the space of the vectors in $L^2(\mathbb{R}^N)$ represented by functions vanishing a.e. outside B.

Now let ψ_1 and ψ_2 represent the state s of $\mathfrak{S}$ immediately before a measurement of ω on $\mathfrak{S}$ with the result 1; and suppose that this measurement is performed by means of an apparatus $\mathcal{A}$ orthogonal for the state s. Then by Post 7.2, both $\theta_B\psi_1$ and $\theta_B\psi_2$ represent the same state, the one immediately after the measurement. Hence Post 4.1 holds. q.e.d.

Obviously Post 7.3 can be substituted for Post 7.2 in the preceding proof. Instead Post 7.1 cannot, because it simply tells us that $\theta_B\psi_1$ and $\theta_B\psi_2$ represent the states of $\mathfrak{S}$ immediately after two measurements similar to the aforementioned one, but performed with two apparata. These apparata may be different, hence the same holds for the states $\theta_B\psi_1$ and $\theta_B\psi_2$.

Incidentally, in stating Post 6.1 [6.2] Von Neumann [Lüders] considered Theor 2.1 as independent of that postulate, while in stating our analogues for it, i.e. Posts 7.1-3, we aim at proving Theor 2.1 by some of them. This contributes to explain why our approach is more complex.

8. An attempt at a second way of completing the proof of Theor. 2.1.

It is natural to invert the roles of the observables q and p in n. 4. Thus Post 4.1 and Theor 4.1 [our proof of Theor 4.1] are easily turned into Post 8.1 and Theor 8.1 below [a proof of Theor 8.1], where by $\hat{\psi}$ we denote the Fourier transform of any $\psi \in \mathcal{H}$:

$$h^{N/2}\hat{\psi}(p) = \int_{\mathbf{R}^N} \left(\exp\frac{i}{\hbar}\right) q\cdot p\ \psi^*(q)\,dq$$

$$\text{where } q\cdot p = \sum_{h=1}^{N} q_h p_h \text{ and } dq = dq_1 \dots dq_N\,.$$

Post 8.1. *If the vectors ψ_1 and ψ_2 in $\mathcal{H}$ represent the same state s of $\mathfrak{S}$, $B \in \mathfrak{B}_N$, and $\|\theta_B\hat{\psi}_1\| \neq 0 \neq \|\theta_B\hat{\psi}_2\|$*—cf. (4.1)—, *then $\theta_B\hat{\psi}_1$ and $\theta_B\hat{\psi}_2$ are the Fourier transforms of functions representing the same state.*

Theor 8.1. *If the wave functions ψ_1 and ψ_2 of the state s (of $\mathfrak{S}$), $\hat{\psi}_1$, and $\hat{\psi}_2$ are continuous and have square integrable first partial derivatives existing a.e., then the restrictions of $\hat{\psi}_1$ and $\hat{\psi}_2$ to any component of* Supp $(\hat{\psi}_1)$ *are proportional.*

At this point it is natural to consider again the problem of proving Theor 2.1 also for wave functions with non-connex supports, a problem left open by Theor 4.1 and solved in n. 5 by taking the evolution of $\mathfrak{S}$ into account; more precisely it is natural to ask whether this problem can be solved disregarding that evolution, by accepting both Post 4.1 and Post 8.1, and by assuming that wave functions fulfil all regularity conditions mentioned in either Theor 4.1 or Theor 8.1.

Under the aforementioned assumptions let B_j $[B'_r]$ be the j-th $[r$-th$]$ component of the common support of φ and ψ $[\hat{\varphi}$ and $\hat{\psi}]$, so that there are functions ψ_j and ψ'_r with the regularity properties above, for which

$$(8.1)\qquad \psi = \sum_j \psi_j\,, \qquad \hat{\varphi} = \sum_r \hat{\varphi}'_r\,, \qquad \psi_j = \psi\cdot\chi_{B_j}\,, \qquad \hat{\varphi}'_r = \hat{\varphi}\cdot\chi_{B'_r}\,;$$

furthermore, by Theors 4.1 and 8.1, for some nonzero c_i, $c'_r \in \boldsymbol{C}$

$$(8.2)\qquad \varphi = \sum_j c_j \psi_j\,, \qquad \hat{\varphi} = \sum_r c'_r \hat{\varphi}'_r \qquad (c_i \neq 0 \neq c'_r)\,.$$

Let

$$(8.3)\qquad \Gamma_j = \{r\colon B_j \cap \mathrm{Supp}\,(\psi'_r) \neq \emptyset\}\,;$$

hence, by (8.1) and (8.2), in B_j we have

$$(8.4)\qquad \sum_{r\in\Gamma_j} c'_r \psi'_r = \varphi = c_j \psi_j = c_j \psi = c_j \sum_{r\in\Gamma_j} \psi'_r \qquad (\text{in } B_j)\,.$$

Intuitively, at least for Γ_j finite, this implies, in general $c'_r = c_j$ for $r \in \Gamma_j$. In this connection we can prove the following particular but precise theorem.

THEOR 8.2. *Accept Post* 4.1 *and Post* 8.1; *furthermore let* ψ *and* φ *fulfil the conditions on* ψ_1 *and* ψ_2 *assumed in Theors* 4.1 *and* 8.1, *and the condition*

(*a*) *The components of* Supp ($\hat{\varphi}$) *are bounded and their number* ν *is finite.*

Then φ *and* ψ *are proportional.*

Indeed formulas (8.1-4) above hold. Moreover by (*a*) $\hat{\varphi}'_r$ has a compact support, so that ψ'_r is analytic—cf. Assertion (*b*) on p. 368 in [7]. Since ψ'_r is square-integrable and $\neq 0$, Supp $(\psi'_r) = \mathbb{R}^N$. Hence $\Gamma_j = \{1, \ldots, \nu\}$ for every j. Thus (8.4) asserts the equality, in B_j, of the non-zero functions $c'_1\psi'_1 + \ldots + c'_\nu\psi'_\nu$ and $c_j(\psi'_1 + \ldots + \psi'_\nu)$. Since ψ'_1 to ψ'_ν are analytic, this yields $(c'_1 - c_j)\psi'_1 + \ldots + (c'_\nu - c_j)\psi_\nu = 0$ in $\mathbb{R}^N$. In addition Supp $(\hat{\varphi}_r) \cap$ Supp $(\hat{\varphi}_s) = \emptyset$ for $r \neq s$, so that ψ'_1 to ψ'_ν are linearly independent. Hence $c'_1 = \ldots = c'_\nu = c_j$ for every j. By $(8.1)_1$ and $(8.2)_1$ this implies $\varphi = c_1\psi$. q.e.d.

Of course, the validity of Theor 8.2 is invariant under the replacement of assumption (*a*) on $\hat{\varphi}$ with its analogue for ψ.

REFERENCES

[1] H. D. DOMBROWSKI, *On simultaneous measurements of incompatible observables*, Arch. Rat. Mech. Anal. (1969), pp. 178-210.
[2] G. EMCH, *Algebraic methods in statistical mechanics and quantum field theory*, Wiley-Interscience (1972), 333 pp.
[3] J. M. JAUCH, *Foundations of quantum mechanics*, Addison Wesley Publ. (1968), 299 pp.
[4] G. LÜDERS, *Über die Zustandsänderung durch den Messprozess*, Ann. Phys., **8** (1952), pp. 322-328.
[5] A. MESSIAH, *Quantum mechanics*, North Holland (1970), 1136 pp.
[6] E. PRUGOVEČKI, *Quantum mechanics in Hilbert space*, Academic Press, New York (1971), 648 pp.
[7] W. RUDIN, *Real and complex analysis*, McGraw-Hill (1966), 412 pp.
[8] J. VON NEUMANN, *a*) *Mathematische Grundlagen der Quantenmechanik*, Berlin (1932); *b*) English translation, Princeton Un. Press (1955), 445 pp.

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