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Abelian Groups with Anti-Isomorphic Endomorphism Rings.

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All groups considered in this paper are abelian. We say that a group G is *E-dual* if there exists a group H such that the endomorphism rings $E(G)$ and $E(H)$ are anti-isomorphic; G is said to be *E-self-dual* if $E(G)$ has an anti-automorphism. In this note we investigate some properties of E-dual and E-self-dual groups. In section 1, we examine some closure properties of the classes of E-dual and E-self-dual groups. In fact, we prove that direct summands of E-self-dual groups are not necessarily E-self-dual, and direct sums of E-self-dual groups are not necessarily E-dual. In section 2, we show that a torsion group G is E-dual if and only if, for every prime p, its p-component $t_p(G)$ is either a p-group of finite rank or a torsion-complete p-group with finite Ulm invariants. In section 3, we describe some classes of E-dual cotorsion groups. As we shall see, a reduced cotorsion group G is E-dual if and only if, for every prime p, the p-adic component of G is either a J_p-module of finite rank or the p-adic completion of an E-dual reduced p-group. We also prove that a divisible group G is E-dual if and only if G is either a torsion E-dual group or a torsion-free group of finite rank. In section 4, we show that plenty of reduced torsion-free groups are E-dual. In fact, every controlled group G such that $E(G)$ is of cardinality $< \aleph_i$, the first strongly inaccessible cardinal, is an E-dual group. In the torsion-free case some pathologies of the class of E-dual groups appear. For instance, by Corner's realization

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theorems, completely different reduced torsion-free groups have anti-isomorphic endomorphism rings. Finally, we remark that there exist torsion, mixed and torsion-free E-dual groups which are not E-self-dual.

For all unexplained terminology and notation we refer to ([5]); in particular $\mathbb{N}$ is the set of natural numbers, P the set of prime numbers; $\mathbb{Z}$, $\mathbb{Q}$, J_p are respectively the groups (or rings) of integers, rational numbers, p-adic integers; $\mathbb{Z}(p)$ is the group (or field) with p elements. If G is a group and $g \in G$, then $o(g)$ is the order of g and, if G is a p-group, $o(g) = p^{e(g)}$, where $e(g)$ is the exponent of g. If G' is a pure subgroup of G, we write $G' \leqslant_* G$. If G is torsion-free and S is a subset of G, then $\langle S \rangle_*$ is the pure subgroup of G generated by S. For every set X, $G^{(X)}$ is the group of all functions from X to G with finite support. If R is a ring, then R^0 is its opposite ring and, for every $n \in \mathbb{N}$, $M_n(R)$ is the ring of all $n \times n$ matrices with entries in R. For every p-group G and every ordinal σ, $f_\sigma(G)$ is the σ-th Ulm invariant of G. When we shall say that $B = \bigoplus_{n \in \mathbb{N}} B_n$ is a basic subgroup of the p-group G, we always adopt the convention that B_n is a direct sum of cyclic groups of order p^n. If G is a reduced cotorsion group, then we write $G = \prod_{p \in P} G_p$, where each G_p is the p-adic component of G.

§ **1.** Let G and H be groups and assume there is an anti-isomorphism between $E(G)$ and $E(H)$. Since idempotents of $E(G)$ are mapped onto idempotents of $E(H)$, the following lemma is obvious.

LEMMA 1.1. *Direct summands of E-dual groups are E-dual.*

The situation is different in the class of E-self-dual groups.

LEMMA 1.2. *Direct summands of E-self-dual groups are not necessarily E-self-dual.*

PROOF. We shall prove that if $G = \bigoplus_{i=1}^{4} G_i$, where $G_1 = G_2 = \prod_{p \in P} \mathbb{Z}(p)$; $G_3 = G_4 = \bigoplus_{p \in P} \mathbb{Z}(p)$, then G is E-self-dual, but there exists a direct summand of G which is not E-self-dual. In the following $\mathbf{\Pi}$ denotes the group (or ring) $\prod_{p \in P} \mathbb{Z}(p)$ and $\mathbf{\Sigma}$ denotes the group $\bigoplus_{p \in P} \mathbb{Z}(p)$. Thus $\mathrm{Hom}(\mathbf{\Sigma}, \mathbf{\Pi}) \simeq \mathbf{\Pi}$; $\mathrm{Hom}(\mathbf{\Pi}, \mathbf{\Sigma}) \simeq \mathbf{\Sigma}$; $E(G_i) \simeq \mathbf{\Pi}$ $(1 \leqslant i \leqslant 4)$. Let A and A^* be the following subrings of $M_4(\mathbf{\Pi})$:

$$A = \{a = [a_{ij}] \in M_4(\mathbf{\Pi}) \colon a_{ij} \in \mathbf{\Sigma};\ 3 \leqslant i \leqslant 4;\ 1 \leqslant j \leqslant 2\}$$

$$A^* = \{a = [a_{ij}] \in M_4(\mathbf{\Pi}) \colon a_{ij} \in \mathbf{\Sigma};\ 1 \leqslant i \leqslant 2;\ 3 \leqslant j \leqslant 4\}\ .$$

Then, by ([5] Theorem 106.1), A is isomorphic to $E(G)$ and, using the decomposition $G = G_3 \oplus G_4 \oplus G_1 \oplus G_2$, the same applies to A^*. Since the transposition of $M_4(\mathbf{\Pi})$ induces an anti-isomorphism between A and A^*, we conclude that G is E-self-dual. To complete the proof, we now show that $G' = \bigoplus_{i=1}^{3} G_i$ is not an E-self-dual group. Otherwise, suppose $E(G')$ has an anti-automorphism which takes e_i, the projection of G' onto G_i, to a suitable $\varepsilon_i \in E(G')$ $(1 \leqslant i \leqslant 3)$. Then $G' = \bigoplus_{i=1}^{3} H_i$, where $H_i = \varepsilon_i(G')$ for every i. Evidently $E(H_i) \simeq \mathbf{\Pi}$ and $\bigcap_{p\in P} p^\omega H_i = 0$; hence, by ([15] Theorem 2), $\mathbf{\Sigma} \leqslant H_i \leqslant_* \mathbf{\Pi}$ $(1 \leqslant i \leqslant 3)$. Also note that $\mathrm{Hom}\,(H_3, H_i) \simeq \mathrm{Hom}\,(G_i, G_3) \simeq \mathbf{\Sigma}$ $(1 \leqslant i \leqslant 2)$. For every prime p, let 1_p be the unit of $\mathbb{Z}(p)$ and let $x = (1_p)_{p\in P} \in G_1$; $y = (1_p)_{p\in P} \in G_2$. On the other hand $x = \sum_{i=1}^{3} x_i$; $y = \sum_{i=1}^{3} y_i$ for some $x_i, y_i \in H_i$ $(1 \leqslant i \leqslant 3)$. To find a contradiction, we first prove that $x_i, y_i \in \mathbf{\Sigma}$ $(1 \leqslant i \leqslant 2)$. For instance, we show that $x_1 \in \mathbf{\Sigma}$. Since $\varepsilon_1(ax) = a\varepsilon_1(x) = ax_1 \in H_1$ $(a \in G_1 = \mathbf{\Pi})$ and $H_3 \leqslant \mathbf{\Pi}$, there exists a homomorphism φ: $H_3 \to H_1$ such that $\varphi(z) = zx_1$ for every $z \in H_3$; therefore $n\varphi = 0$ for some $n \in \mathbf{N}$. Since $nx_1\mathbf{\Sigma} = n\varphi(\mathbf{\Sigma}) \leqslant n\varphi(H_3) = 0$, we must have $x_1 \in \mathbf{\Sigma}$, as claimed. Consequently $P^* = \{p \in P / t_p(G_1) \oplus t_p(G_2) \leqslant \mathrm{Ker}\,(\varepsilon_1 + \varepsilon_2)\}$ contains all but finitely many primes. Fix $p \in P^*$ and let $j\colon \bigoplus_{i=1}^{3} G_i \to \bigoplus_{i=1}^{3} H_i$ denote the identity map of G'. Then the choice of p implies $j(t_p(G_1) \oplus t_p(G_2)) \leqslant H_3$; on the other hand $t_p(G_1)$, $t_p(G_2)$ and $t_p(H_3)$ are all isomorphic to $\mathbb{Z}(p)$. This contradiction establishes that G' is not E-self-dual, and the lemma follows. Another application of ([5] Theorem 106.1) shows that $E(G')$ is anti-isomorphic to $E(G'')$, where $G'' = G_2 \oplus G_3 \oplus G_4$. Hence G'' is another direct summand of G which is not E-self-dual. □

LEMMA 1.3. *Finite direct sums of E-self-dual groups are not necessarily E-dual.*

PROOF. It is enough to observe that $\mathbf{Q}$, $\mathbb{Z}(p^\infty)$, $\mathbb{Z}$ are clearly E-self-dual; however, as we shall see in sections 3 and 4, the groups $\mathbf{Q} \oplus \mathbb{Z}(p^\infty)$, $\mathbf{Q} \oplus \mathbb{Z}$, $\mathbb{Z}(p^\infty) \oplus \mathbb{Z}$ are not E-dual. □

Before classifying all E-dual and E-self-dual torsion groups by means of a suitable realization of their endomorphism ring, we summarize the results previously obtained about this kind of problem.

Liebert has shown ([10] Lemma A) that the endomorphism ring of a finite p-group has an anti-automorphism. By a result of Faltings ([4] Lemma 2.10), the same property holds for every torsion-complete p-group with finite Ulm invariants. A new theorem of Liebert ([12] Theorem 8.1) states that if G is a torsion-complete p-group, then $E(G)$ has an anti-automorphism if and only if G has finite Ulm invariants.

§ **2.** In the first part of this section we prove that if G is an E-dual reduced p-group, then G must be a torsion-complete E-self-dual p-group. We begin with two lemmas.

LEMMA 2.1. *Let G be a reduced p-group. If G is E-dual, then $f_\sigma(G)$ is finite, for every $\sigma < \omega$.*

PROOF. Let G be as in the hypotheses and assume $B = \bigoplus_{n\in\mathbb{N}} B_n$ is basic in G. We now prove that B_1 is finite. By 1.1, there exists a group H and an anti-isomorphism $f\colon E(B_1) \to E(H)$. Since $pH = 0$, an application of ([1] General Existence Theorem, p. 193) shows that B_1 is finite. An elementary proof of this fact is the following. Assume $|B_1| \geqslant \aleph_0$. To find a contradition, it is enough to prove that B_1 cannot be of cardinality $\aleph_0$. Suppose the contrary. Then H is not finite and $E(B_1)$ has only one proper two-sided ideal consisting of all endomorphisms of finite rank ([8], Chapter 4; also see [1], p. 198). Since the endomorphism ring of an uncountable vector space has at least two proper two-sided ideals, i.e. the ideals of all endomorphisms of finite or countable rank, we conclude that $|H| = \aleph_0$. Let π be a minimal idempotent of $E(B_1)$ and let $\pi' = f(\pi)$. Then $|E(B_1)\pi| = |\mathrm{Hom}\,(\pi(B_1), B_1)| = \aleph_0$, while $|\pi' E(H)| = |\mathrm{Hom}\,(H, \pi'(H))| = 2^{\aleph_0}$. This contradiction establishes that B_1 is finite. To complete the proof, it is enough to check that B_{n+1} is finite ($n \in \mathbb{N}$). Fix $n \in \mathbb{N}$, and let $f\colon E(B_{n+1}) \to E(H)$ be an anti-isomorphism. Remark that H is a direct sum of cyclic groups of order p^{n+1}, because $p^{n+1}H = 0$ and $E(H)$ has no idempotent of order $< p^{n+1}$. Let $\sigma_1\colon E(B_{n+1}) \to E(B_{n+1})/p^n E(B_{n+1})$ and $\sigma_2\colon E(H) \to E(H)/p^n E(H)$ be the natural homomorphisms. Since $\mathrm{Ker}\,\sigma_1 = \mathrm{Ker}\,\sigma_2 f$ and f is an anti-isomorphism, there exists an anti-isomorphism $\bar{f}\colon \sigma_1(E(B_{n+1})) \to \sigma_2(E(H))$ such that $\bar{f}\sigma_1 = \sigma_2 f$. Evidently $\sigma_1(E(B_{n+1})) \cong E(p^n B_{n+1})$ and $\sigma_2(E(H)) \cong E(p^n H)$. Therefore B_{n+1} is finite, and the lemma follows. □

LEMMA 2.2. *Let G be a reduced p-group. If G is E-dual, then $p^\omega G = 0$.*

PROOF. As before, let $B = \bigoplus_{n\in \mathbb{N}} B_n$ be a basic subgroup of G. Let π_n denote the projection of G onto B_n with $\operatorname{Ker}\pi_n = \bigoplus_{m\neq n} B_m + p^n G$ $(n \in \mathbb{N})$. Assume G is E-dual; then, there is an anti-isomorphism $f\colon E(G)\to E(H)$ for some H. To see that G is separable, we shall use the following properties of H:

(1) $t(H) = t_p(H)$. Let $q\in P$; $q\neq p$. Since G has no summand whose endomorphism ring is isomorphic to J_q or to $\mathbb{Z}(q^n)$ for some $n\in\mathbb{N}$, the same applies to H. Therefore $t_q(H) = 0$.

(2) H is reduced. By the previous remark, it is enough to observe that $\mathbb{Q}$ and $\mathbb{Z}(p^\infty)$ cannot be subgroups of H.

(3) H is a J_p-module. This follows from the fact that the center of $E(H)$ is isomorphic to the center of $E(G)$, and the center of $E(G)$ is isomorphic to J_p or to $\mathbb{Z}(p^n)$ for some $n\in\mathbb{N}$ ([5] Theorem 108.3).

(4) If $\pi_n' = f(\pi_n)$ and $\pi_n'(H) = B_n'$ $(n\in\mathbb{N})$, then $B' = \bigoplus_{n\in\mathbb{N}} B_n'$ is a basic subgroup of $t_p(H)$. Since B' is a direct sum of cyclic groups and $B' \leqslant_* t_p(H)$, there is a basic subgroup $B'' = \bigoplus_{n\in\mathbb{N}} B_n''$ such that $B'\leqslant B''$. Our claim is that $B' = B''$. Assume this is not true. Choose $m\in\mathbb{N}$ such that $B_m' < B_m''$. Let η' be a projection of H onto B_m'', and let $\eta' = f(\eta)$. Therefore $\eta(G)$ is a direct sum of cyclic groups of order p^m and clearly $B_m < \eta(G)$. This contradiction proves that $B' = B''$.

(5) $p^\omega t_p(H) = 0$. To see this, suppose the contrary. Then, there is $\varphi\in E(G)$ such that $0\neq f(\varphi) = \varphi'\in E(H)[p]$ and $\varphi'(H)\leqslant p^\omega t_p(H)$. Since $\pi_n'\varphi' = 0$, we get $\varphi\pi_n = 0$ $(n\in\mathbb{N})$. It follows that $\varphi = 0$, and this contradicts the hypothesis that $\varphi'\neq 0$; consequently $p^\omega t_p(H) = 0$.

The last remark tells us that, if H is a p-group, G is separable. To end the proof, assume that $p^\omega G\neq 0$. Then there is a suitable $\varphi\in E(G)[p]$ such that $\varphi\neq 0$; $\pi_n\varphi = 0$ $(n\in\mathbb{N})$. Let $\varphi' = f(\varphi)$; since $\varphi'\pi_n' = 0$ $(n\in\mathbb{N})$, φ' is 0 on B'. Using (1), (2) and (4), we conclude that $\varphi'(t(H)) = 0$, and therefore $\operatorname{Hom}(H/t(H), H[p])\neq 0$. This implies that $H/t(H)$ is not p-divisible; hence, by (3), there exists $x\in H$, $x\notin t(H)$ such that $J_p x$ is a direct summand of H. But this is impossible, because G has no summand isomorphic to J_p or $\mathbb{Z}(p^\infty)$. This contradiction proves that $p^\omega G = 0$. □

REMARK. In G is an infinite reduced E-dual p-group, $|G| = 2^{\aleph_0}$. In fact, with the notations of 2.2, $|G|\leqslant \left|\prod_{n\in\mathbb{N}} B_n\right| = 2^{\aleph_0}$. To prove the reverse inequality, take $m \in \mathbb{N}$ such that $B_m \neq 0$. Since $|H/p^m H| \geqslant |B'/p^m B'| = \aleph_0$, we clearly have

$$|G| \geqslant |G[p^m]| = |\mathrm{Hom}\,(B_m, G)| = |E(G)\pi_m| = |\pi'_m E(H)| = \\ = |\mathrm{Hom}\,(H, B'_m)| = |\mathrm{Hom}\,(H/p^m H, B'_m)| \geqslant 2^{\aleph_0}.$$

Following ([12], p. 350), we say that a p-group G is torsion-compact if G is torsion-complete and every Ulm invariant of G is finite. We now give a realization of the endomorphism ring of a torsion-compact p-group G. If G is finite, an application of ([5] Theorem 106.1) shows that there exist $r, n \in \mathbb{N}$ such that $E(G)$ is isomorphic to a subring of $M_r(\mathbb{Z}(p^n))$ fully invariant under the transposition of $M_r(\mathbb{Z}(p^n))$. This is an elementary proof of a result ([10] Lemma A) mentioned in section 1. Suppose now that G is not finite. Fix a basic subgroup $B = \bigoplus_{n \in \mathbb{N}} \langle x_n \rangle$ of G such that $o(x_r) \leqslant o(x_s)$ if $r \leqslant s$. Regarding G as embedded in $\prod_{n \in \mathbb{N}} \langle x_n \rangle$, let π_n denote the projection of G onto $\langle x_n \rangle$ $(n \in \mathbb{N})$; for every $x \in G$ we may write $x = (\alpha_n x_n)_{n \in \mathbb{N}}$, where $\alpha_n x_n = \pi_n(x)$ and α_n is a suitable p-adic integer $(n \in \mathbb{N})$. Let h be the p-adic valuation of J_p and let A be additive group of all $\aleph_0 \times \aleph_0$ matrices over J_p of the form $a = [\alpha_{rs}]$, where $h(\alpha_{rs}) \geqslant \lambda_{rs} = \max\big(0, e(x_r) - e(x_s)\big)$ for every $r, s \in \mathbb{N}$. Then A is a ring with the usual rows by columns product and the subset I of all $a = [\alpha_{rs}] \in A$ such that $h(\alpha_{rs}) \geqslant e(x_r)$ $(r, s \in \mathbb{N})$ is a two-sided ideal of A. Remark that every $\varphi \in E(G)$ is completely determined by the elements $\{\varphi(x_s) = (\alpha_{rs} x_r)_{r \in \mathbb{N}}\ (s \in \mathbb{N})\}$. With these notations, let $\varrho\colon E(G) \to A/I$ be the map defined by $\varrho(\varphi) = [\alpha_{rs}] + I$ for all $\varphi \in E(G)$. Evidently ϱ is a group isomorphism. We claim that ϱ is a ring isomorphism. To see this, choose $\varphi, \psi \in E(G)$; then there are suitable $\alpha_{rs}, \beta_{rs}, \delta_{rs} \in J_p$ $(r, s \in \mathbb{N})$ such that $\varrho(\varphi) = [\alpha_{rs}] + I$; $\varrho(\psi) = [\beta_{rs}] + I$; $\varrho(\psi\varphi) = [\delta_{rs}] + I$. Fix $r, s \in \mathbb{N}$; then there exists some $k \in \mathbb{N}$ such that $e(x_k) \geqslant e(x_r) + e(x_s)$. Consequently $\varphi(x_s) \equiv \sum_{i=1}^{k} \alpha_{is} x_i \bmod p^{e(x_r)} G$. Since

$$\delta_{rs} x_r = \pi_r\big(\psi\varphi(x_s)\big) = \pi_r\Big(\sum_{i=1}^{k} \alpha_{is} \psi(x_i)\Big) = \Big(\sum_{i=1}^{k} \beta_{ri} \alpha_{is}\Big) x_r,$$

we obtain $\delta_{rs} \equiv \sum_{i=1}^{k} \beta_{ri} \alpha_{is} \bmod p^{e(x_r)} J_p$. Using the hypothesis that $h(\alpha_{ns}) \geqslant e(x_r)$ for all $n > k$, we conclude that $\delta_{rs} \equiv \sum_{n \in \mathbb{N}} \beta_{rn} \alpha_{ns} \bmod p^{e(x_r)} J_p$.

This proves that ϱ is a ring isomorphism, because $\varphi, \psi \in E(G)$ and $r, s \in \mathbf{N}$ are arbitrary elements.

The following theorem characterizes all E-dual reduced p-groups.

THEOREM 2.3. *Let G be a reduced p-group. The following are equivalent*:

(1) *G is torsion-compact.*

(2) *G is E-self-dual.*

(3) *G is E-dual.*

PROOF (1) $\Rightarrow$ (2). As already observed, finite p-groups are E-self-dual. Assume G is not finite and fix a basic subgroup $B = \bigoplus_{n\in\mathbf{N}} \langle x_n \rangle$ of G such that $e(x_r) \leqslant e(x_s)$ if $r \leqslant s$ $(r, s \in \mathbf{N})$. Let A be the ring of all matrices $a = [\alpha_{rs}]$, where $\alpha_{rs} = p^{\lambda_{rs}}\gamma_{rs}$ for some $\gamma_{rs} \in J_p$ and $\lambda_{rs} = \max\big(0, e(x_r) - e(x_s)\big)$ $(r, s \in \mathbf{N})$. Evidently the map $t\colon A \to A$ such that ${}^t a = {}^t[p^{\lambda_{rs}}\gamma_{rs}] = [p^{\lambda_{rs}}\gamma_{sr}]$ $(a \in A)$ is a group isomorphism. To show that t is a ring anti-automorphism, pick $a, a' \in A$. Let $a = [p^{\lambda_{rs}}\gamma_{rs}]$; $a' = [p^{\lambda_{rs}}\gamma'_{rs}]$; $a'a = [\beta_{rs}]$ and ${}^t a\, {}^t a' = [\delta_{rs}]$. For every $r, s \in \mathbf{N}$ we may write $\beta_{rs} = \sum_{n\in\mathbf{N}} p^{\sigma_n}\gamma'_{rn}\gamma_{ns}$; $\delta_{sr} = \sum_{n\in\mathbf{N}} p^{\tau_n}\gamma'_{rn}\gamma_{ns}$ where $\sigma_n = \lambda_{rn} + \lambda_{ns}$ and $\tau_n = \lambda_{sn} + \lambda_{nr}$ $(n \in \mathbf{N})$. Let $r \leqslant s$; it is easy to check that

$$\sigma_n = \begin{cases} e(x_r) - e(x_n) \\ 0 \\ e(x_n) - e(x_s) \end{cases} \qquad \tau_n = \begin{cases} e(x_s) - e(x_n) & e(x_n) \leqslant e(x_r) \\ e(x_s) - e(x_r) & e(x_r) \leqslant e(x_n) \leqslant e(x_s) \\ e(x_n) - e(x_r) & e(x_s) \leqslant e(x_n) \end{cases}$$

Since $\lambda_{sr} = e(x_s) - e(x_r)$, we clearly have $\tau_n = \sigma_n + \lambda_{sr}$ $(n \in \mathbf{N})$, and therefore $\delta_{sr} = p^{\lambda_{sr}}\beta_{rs}$. Let $r \geqslant s$; this hypothesis implies

$$\sigma_n = \begin{cases} e(x_r) - e(x_n) \\ e(x_r) - e(x_s) \\ e(x_n) - e(x_s) \end{cases} \qquad \tau_n = \begin{cases} e(x_s) - e(x_n) & e(x_n) \leqslant e(x_s) \\ 0 & e(x_s) \leqslant e(x_n) \leqslant e(x_r) \\ e(x_n) - e(x_r) & e(x_n) \leqslant e(x_r) \end{cases}$$

Since $\lambda_{rs} = e(x_r) - e(x_s)$, we get $\tau_n = \sigma_n - \lambda_{rs}$ $(n \in \mathbf{N})$; thus $\delta_{sr} = p^{-\lambda_{rs}}\beta_{rs}$. Consequently ${}^t[\beta_{rs}] = [\delta_{rs}]$, and t is a ring anti-automorphism, as required. Let $r, s \in \mathbf{N}$ and $\gamma_{rs} \in J_p$; then $h(p^{\lambda_{rs}}\gamma_{rs}) \geqslant e(x_r)$ if and only if $h(p^{\lambda_{sr}}\gamma_{rs}) \geqslant e(x_s)$. Hence t induces an anti-automorphism of A/I, that we still call t. Since A/I is isomorphic to $E(G)$, G is E-self-dual.

(2) $\Rightarrow$ (3): This is obvious.

(3) $\Rightarrow$ (1): Let G be an E-dual p-group. If G is finite, G is clearly torsion-compact. If G is not finite, Lemmas 2.1 and 2.2. enable us to assume that $G \leqslant \bar{B} = t\left(\prod_{n\in\mathrm{N}} \langle x_n\rangle\right)$, where $B = \bigoplus_{n\in\mathrm{N}} \langle x_n\rangle$ is a basic subgroup of G and $e(x_r) \leqslant e(x_s)$ if $r \leqslant s$ $(r, s \in \mathrm{N})$. To prove that $\bar{B} \leqslant G$, we introduce some endomorphisms of G similar to those used in ([9] Theorem 28). Let π_n be the projection of G onto $\langle x_n\rangle$ $(n \in \mathrm{N})$. Then, for all $r, s \in \mathrm{N}$ we define $e_{rs} \in E(G)$ as follows: $e_{rs}(1 - \pi_s) = 0$ and $e_{rs}(x_s) = p^{\lambda_{rs}}x_r$, where $\lambda_{rs} = \max\big(0, e(x_r) - e(x_s)\big)$. Since G is E-dual, there exists an anti-isomorphism $f\colon E(G) \to E(H)$ for some H. Write $f(e_{rs}) = e'_{sr}$ $(r, s \in \mathrm{N})$ and choose $y_n \in H$ such that $e'_{nn}(H) = \langle y_n\rangle$ $(n \in \mathrm{N})$. For every $r, s \in \mathrm{N}$, let ε_{rs} be the endomorphism of H uniquely determined by the following conditions $\varepsilon_{rs}(y_s) = p^{\lambda_{rs}}y_r$, $\varepsilon_{rs}(1 - e'_{ss}) = 0$. Remark that $e_{rs} = e_{rr}e_{rs}e_{ss}$ implies $e'_{sr} = e'_{ss}e'_{sr}e'_{rr}$; therefore $e'_{sr} = u_{sr}\varepsilon_{sr}$ for some $u_{sr} \in J_p \setminus pJ_p$ $(r, s \in \mathrm{N})$. Assume $G < \bar{B}$ and choose $x \in \bar{B} \setminus G$. Then there exist $\alpha_n \in J_p$ $(n \in \mathrm{N})$ and $m \in \mathrm{N}$ such that $x = (\alpha_n x_n)_{n\in\mathrm{N}}$ and $o(x_n) \geqslant o(x)$ for all $n \geqslant m$. Let $\alpha_n^* = 0$ if $n < m$, and let $\alpha_n^* = \alpha_n$ if $n \geqslant m$. We claim that $x^* = (\alpha_n^* x_n)_{n\in\mathrm{N}} \in G$. To see this, let φ' denote an endomorphism of H with the following properties: $\varphi' = e'_{mm}\varphi'$, $\varphi'(y_n) = \alpha_n^* u_{mn} y_m$ $(n \in \mathrm{N})$. Now consider the endomorphism φ of G such that $f(\varphi) = \varphi'$. By hypothesis, we clearly have $e'_{mm}\varphi' e'_{nn} = \alpha_n^* u_{mn}\varepsilon_{mn} = (\alpha_n^* u_{mn})(u_{mn}^{-1} e'_{mn}) = \alpha_n^* e'_{mn}$ $(n \in \mathrm{N})$. It follows that $\varphi(x_m) = (\alpha_n^* x_n)_{n\in\mathrm{N}} = x^* \in G$. Since $x \in x^* + B$, we obtain $x \in G$. This contradiction shows that $G = \bar{B}$ and the proof is complete. □

Remark. Let G be an infinite torsion-compact p-group. With the previous notations, the anti-automorphism t of $E(G)$ defined in the first part of the proof has the property that $e'_{rs} = \varepsilon_{rs}$; thus we may assume $u_{rs} = 1$ for every $r, s \in \mathrm{N}$. This follows from the fact that t is induced by the most obvious transposition of the matrix ring A.

A remark of ([12], p. 352) states that if G is a p-group, then G is E-self-dual if and only if G is either a torsion-compact group or a divisible group of finite rank. As we shall see, if a divisible p-group is E-dual then it is E-self-dual, but there exist E-dual p-groups which are not E-self-dual.

THEOREM 2.4. *Let G be a divisible p-group. Then G is E-dual if and only if it is of finite rank.*

PROOF. If G is a divisible p-group of rank n, then $E(G) \cong M_n(J_p)$ clearly has an anti-automorphism. By 1.1, to prove the theorem, it is enough to show that a divisible p-group of rank $\aleph_0$ cannot be E-dual. Assume this is not true. Write $G = \bigoplus_{n\in\mathbb{N}} G_n$, where $G_n \cong \mathbb{Z}(p^\infty)$ for every $n \in \mathbb{N}$, and choose an anti-isomorphism $f\colon E(G) \to E(H)$ for some H. Let π_n be the projection of G onto G_n ($n \in \mathbb{N}$), and let $\varphi' = f(\varphi)$ ($\varphi \in E(G)$). First note that the groups $\pi_n'(H)$ ($n \in \mathbb{N}$) are all isomorphic. Otherwise, there exist $m, n \in \mathbb{N}$ such that $\pi_m'(H) \cong \mathbb{Z}(p^\infty)$ and $\pi_n'(H) \cong J_p$. But this is impossible, because $\pi_n' E(H)\pi_m' = 0$, while $\pi_m E(G)\pi_n \neq 0$ (compare with ([13] Lemma 1.2)). Remark now the following properties of H:

(1) H is not a divisible p-group. Suppose (1) does not hold. Let $\sigma_1\colon E(G) \to E(G)/pE(G)$; $\sigma_2\colon E(H) \to E(H)/pE(H)$ be the natural homomorphisms. Then $\sigma_1(E(G))$, $\sigma_2(E(H))$ are isomorphic to the endomorphism rings of two infinite vector spaces over $\mathbb{Z}(p)$, and this is a contradiction. In fact, the existence of f implies that $\sigma_1(E(G))$ is anti-isomorphic to $\sigma_2(E(H))$. Hence (1) is true.

(2) H is torsion-free. Since G has no finite summand, the same holds for H. Consequently $t_q(H) = 0$ for every prime $q \neq p$, because G cannot have J_q or $\mathbb{Z}(q^\infty)$ for a summand. It remains to check that $t_p(H) = 0$. Assume the contrary. Then H is a mixed group and $t(H)$ is a divisible p-group. Thus $t(H)$ is a proper fully invariant direct summand of H. On the other hand G has no proper fully invariant direct summand. This contradiction proves that H is torsion-free.

(3) H is a reduced J_p-module. Since H is torsion-free and G is a p-group, it suffices to repeat the proof of (2) and (3) of Lemma 2.2.

It is now clear that we may assume $\pi_1'(H) = J_p$. As an immediate consequence $J_p^{\mathbb{N}} \cong \operatorname{Hom}(G, G_1) \cong \operatorname{Hom}(\pi_1'(H), H) \cong \operatorname{Hom}_{J_p}(\pi_1'(H), H)$. By (3), the map which takes φ' to $\varphi'(1)$ for all $\varphi' \in \operatorname{Hom}(\pi_1'(H), H)$ is an isomorphism between $\operatorname{Hom}(\pi_1'(H), H)$ and H; therefore $H \cong J_p^{\mathbb{N}}$. Since H properly contains the p-adic completion of $\bigoplus_{n\in\mathbb{N}} \pi_n'(H)$, the group $\bar{H} = H/\bigoplus_{n\in\mathbb{N}} \pi_n'(H)$ is not p-divisible. This implies that $\bar{H}$ has a pure subgroup, hence a direct summand, isomorphic to J_p. Since $\operatorname{Hom}(\bar{H}, H) \neq 0$, there exists a non-zero endomorphism φ' of H such that $\varphi'\left(\bigoplus_{n\in\mathbb{N}}\pi_n'(H)\right)=0$. But this means that if $\varphi\in E(G)$ and $f(\varphi)=\varphi'$, then $\varphi\neq 0$, while $\pi_n\varphi=0$ $(n\in\mathbb{N})$. This contradiction establishes that divisible p-groups of infinite rank are not E-dual. □

REMARK. Let G be a divisible p-group of infinite rank $\mathfrak{m}$. Then, as observed in ([5] vol. II, p. 220) $E(G)$ is isomorphic to the ring of all column-convergent $\mathfrak{m}\times\mathfrak{m}$ matrices with entries in J_p (i.e. in every column almost all entries are divisible by p^n for any $n\in\mathbb{N}$). In this case the asymmetry between rows and columns cannot be removed by means of a suitable transposition, as in the case of infinite torsion-compact p-groups.

THEOREM 2.5. *Let G be a p-group. Then G is E-dual if and only if either G is torsion-compact or G is of finite rank.*

PROOF. By the previous results, we assume G is neither reduced not divisible. Suppose first that G has finite rank. Then G has a decomposition $G=\bigoplus_{i=1}^{r}G_i$ with the following properties: G_i is a cyclic group, if $1\leqslant i\leqslant n$, and $G_i\cong\mathbb{Z}(p^\infty)$, if $n+1\leqslant i\leqslant r$. Using ([5] Theorem 106.1), we identify $E(G)$ with the ring of all $r\times r$ matrices $[\alpha_{ij}]$, where $\alpha_{ij}\in\operatorname{Hom}(G_j,G_i)$. Define H to be the group $H=\bigoplus_{i=1}^{r}H_i$, where $H_i=G_i$ if $1\leqslant i\leqslant n$, and $H_i=J_p$ if $n+1\leqslant i\leqslant r$. Another application of ([5] Theorem 106.1) shows that $E(H)$ is isomorphic to the ring of all $r\times r$ matrices $[\alpha_{ij}]$, where $\alpha_{ij}\in\operatorname{Hom}(H_j,H_i)$. Identifying the groups $\operatorname{Hom}(G_j,G_i)$ and $\operatorname{Hom}(H_i,H_j)$ $(1\leqslant i,j\leqslant n)$, let $t\colon E(G)\to E(H)$ be the map that sends $a=[\alpha_{ij}]$ to ${}^ta=[\alpha_{ji}]$ for all $a\in E(G)$. Since t is a ring anti-isomorphism, G is E-dual. Conversely, let $G=D\oplus R$ be an E-dual p-group, where $D\cong(\mathbb{Z}(p^\infty))^r$ for some $r\in\mathbb{N}$ and R is reduced. Our claim is that R is finite. Suppose R is not finite and fix an anti-isomorphism $f\colon E(G)\to E(H)$. Let π_1,π_2 be the projections of G onto D and R respectively, and let $\pi_i'=f(\pi_i)$, $H_i=\pi_i'(H)$ $(1\leqslant i\leqslant 2)$. Then $H=H_1\oplus H_2$. Since $\pi_1'E(H)\pi_2'=0$, H_1 is isomorphic to J_p^r. The proof of Lemma 2.2 enables us to regard H_2 as embedded in a group of the form $\prod_{n\in\mathbb{N}}B_n'$, where $B'=\bigoplus_{n\in\mathbb{N}}B_n'$ is isomorphic to a basic subgroup $B=\bigoplus_{n\in\mathbb{N}}B_n$ of R. Since $|R/B|=2^{\aleph_0}$, we get $|\operatorname{Hom}(R,D)|=\left|\operatorname{Hom}(B,\mathbb{Z}(p^\infty))\oplus J_p^{2^{\aleph_0}}\right|>2^{\aleph_0}$ ([5] Theorem 47.1). This contradicts the fact that $|\operatorname{Hom}(H_1,H_2)|\leqslant\left|\prod_{n\in\mathbb{N}}\operatorname{Hom}(J_p^r,B_n')\right|\leqslant 2^{\aleph_0}$. Therefore R is finite, and the proof is complete. □

COROLLARY 2.6. *Let G be a torsion group. Then G is E-dual if and only if $t_p(G)$ is E-dual for every prime p.*

PROOF. Necessity follows from 1.1. Assume that $t_p(G)$ is E-dual $(p \in P)$. Let $H_p = t_p(G)$ if $t_p(G)$ is reduced, and let $H_p = J_p^r \oplus R$ if $t_p(G) \cong (\mathbf{Z}(p^\infty))^r \oplus R$, where $r \in \mathbf{N}$ and R is reduced. If $H = \bigoplus_{p\in P} H_p$, then $E(H) \cong \prod_{p \in P} E(H_p)$. Since $E(t_p(G))$ is anti-isomorphic to $E(H_p)$ for all $p \in P$, $E(G)$ is anti-isomorphic to $E(H)$. Thus G is E-dual. □

COROLLARY 2.7. *If G and H are torsion groups with anti-isomorphic endomorphism rings, the following conditions hold for every prime p:*

(i) *$t_p(G)$ is either reduced or divisible.*

(ii) *$t_p(G)$ is isomorphic to $t_p(H)$.*

PROOF. (i) Assume $t_p(G)$ is neither reduced nor divisible. Then our hypotheses imply that $t_p(H)$ is a group of the form $t_p(H) = D \oplus R$, where D is divisible, R is reduced and Hom $(D, R) \neq 0$. Since this is clearly impossible, (i) holds.

(ii) If $t_p(G)$ is reduced, then the proof of 2.2 indicates that $t_p(G)$ and $t_p(H)$ must be torsion-compact p-groups with the same Ulm invariants. Consequently $t_p(G)$ is isomorphic to $t_p(H)$. If $t_p(G)$ is divisible, the statement is obvious, because $t_p(G)$ and $t_p(H)$ must have the same rank. □

REMARK. By 2.6 and 2.7, if G and H are torsion groups and there exists an anti-isomorphism $f\colon E(G) \to E(H)$, then G belongs to a restricted class of torsion groups and H is isomorphic to G. In particular, let G, H, f be as above; then the following conditions are equivalent:

(1) f is induced by a group isomorphism $\tau\colon G \to H$ (i.e. $f(\varphi) = \tau\varphi\tau^{-1}(\varphi \in E(G))$).

(2) $E(G)$ is commutative.

In fact, assume first that (1) is true. Since $\tau\psi\varphi\tau^{-1} = \tau\varphi\psi\tau^{-1}$ $(\varphi, \psi \in E(G))$, (2) clearly holds. This completes the proof, because the implication (2) ⇒ (1) follows from the Baer-Kaplansky theorem. Hence, by ([15] Theorem 1), condition (1) is not generally satisfied.

§ **3.** The characterization of all E-dual torsion groups enables us to prove the following

THEOREM 3.1. *Let* $G = \prod_{p \in P} G_p$ *be a reduced cotorsion group. Then* G *is* E*-dual if and only if, for every* $p \in P$*, its* p*-adic component* G_p *is either a* J_p*-module of finite rank or the* p*-adic completion of a torsion-compact* p*-group.*

PROOF. Necessity. Suppose G is E-dual and fix a prime p. To prove that G_p has the required properties, we distinguish three cases.

(i) G_p adjusted. Since $E(G_p)$ is isomorphic to $E(t_p(G_p))$ ([13] Theorem 3.3), from Theorem 2.3 we deduce that G_p is the p-adic completion of a torsion compact p-group.

(ii) G_p torsion-free. An application of ([11] Theorem 5.5) shows that $E(G_p) \cong E(\mathbb{Q}_p/J_p \otimes G_p)$, where $\mathbb{Q}_p$ is the field of p-adic numbers. Using 2.4, we conclude that G_p is isomorphic to J_p^r for some $r \in \mathbb{N}$.

(iii) G_p neither adjusted nor torsion-free. It is not restrictive to assume $G_p = J_p^r \oplus G'_p$, where $r \in \mathbb{N}$ and G'_p is adjusted. We claim that G'_p is finite. Suppose the contrary. Let $B = \bigoplus_{n \in \mathbb{N}} B_n$ be a basic subgroup of $t_p(G'_p)$ and assume $G'_p \leqslant \prod_{n \in \mathbb{N}} B_n$. Fix a group H and an anti-isomorphism between $E(G_p)$ and $E(H)$, which takes φ to φ' $(\varphi \in E(G_p))$. Let π_1, π_2 be the projections of G_p onto J_p^r and G'_p respectively, and let $H_i = \pi'_i(H)$ $(1 \leqslant i \leqslant 2)$. Evidently $H = H_1 \oplus H_2$ and $H_1 \cong (\mathbb{Z}(p^\infty))^r$ because H_1 is fully invariant in H. Since $E(G'_p)$ is isomorphic to $E(t_p(G'_p))$, the proof of Lemma 2.2 shows that H_2 is a reduced J_p-module. The hypothesis that G'_p is not finite and the fact that $H_1 \cong (\mathbb{Z}(p^\infty))^r$ guarantee that H_2 is not a p-group. Let x be a torsion-free element of H_2, and let $y = ux$, where u is a p-adic integer algebraically independent over $\mathbb{Z}_p$. Choose $\bar{x}, \bar{y} \in H_1$ such that $\bar{y} \neq u\bar{x}$. Then H has an endomorphism φ' that maps x and y onto $\bar{x}$ and $\bar{y}$ respectively. Since $\mathrm{Hom}(J_p^r, G'_p) \leqslant \mathrm{Hom}_{J_p}\left(J_p^r, \prod_{n \in \mathbb{N}} B_n\right)$, the center of $E(G)$ is isomorphic to J_p and the same applies to $E(H)$. But this is a contradiction, because $u\varphi' \neq \varphi' u$. Consequently G'_p is finite.

Sufficiency. Assume $G = \prod_{p \in P} G_p$, where each G_p is as in (i), (ii) or (iii). Then 2.5, 2.6 and the result used in (i) tell us that, for every $p \in P$, there exists a group H_p such that $E(G_p)$ is anti-isomorphic to $E(H_p)$

and H_p is fully invariant in $H = \bigoplus_{p \in P} H_p$. Since $E(H) \simeq \prod_{p \in P} E(H_p)$, G is E-dual. □

Corollary 3.2. *Let $G = \prod_{p \in P} G_p$ be a reduced cotorsion group. The following are equivalent*:

(1) *For every prime p, G_p is either a torsion-free J_p-module of finite rank or the p-adic completion of a torsion-compact p-group.*

(2) *G is E-self-dual.*

Proof. Since G is E-self-dual if and only if the p-adic component G_p of G is E-self-dual ($p \in P$), the result is an immediate consequence of the previous theorem. In fact, the first part of the proof of 3.1 shows that if G_p is E-dual, then G_p is E-self-dual if and only if it is either adjusted or torsion-free. □

Remark. In ([9], p. 73) Kaplansky asserts that there are reasons for believing that two modules with isomorphic (or anti-isomorphic) endomorphism rings are isomorphic or « dual ». This suggests that we translate 3.1 and 3.2 as follows: The correspondence given by Harrison ([7]) between torsion groups and reduced cotorsion groups induces a correspondence between E-dual (E-self-dual) torsion groups and E-dual (E-self-dual) reduced cotorsion groups. It is natural to compare this statement with a result of May and Tubassi ([13] Main Theorem) about groups with isomorphic endomorphism rings, i.e. the characterization of all groups G and H such that $E(G) \simeq E(H)$ and $t(G) \not\simeq t(H)$. Even in this case, a theory of duality, more precisely Harrison's duality, clarifies the situation.

Theorem 3.3. *Let G be a divisible group. Then G is E-dual if and only if either $G = \bigoplus_{p \in P} D_p$ with D_p a divisible p-group of finite rank or G is a torsion-free group of finite rank.*

Proof. Sufficiency immediately follows from 2.5, because if G is torsion-free of finite rank r, then $E(G) \simeq M_r(\mathbf{Q})$. Since arguments very similar to those used in the first part of 2.1 show that infinite dimensional vector spaces over $\mathbf{Q}$ cannot be E-dual, it remains to prove that the group $G = \mathbf{Q} \oplus \mathbb{Z}(p^\infty)$ is not E-dual ($p \in P$). Suppose this does not hold. Let π_1, π_2 be the projections of G onto $\mathbf{Q}$ and $\mathbb{Z}(p^\infty)$ respectively, and let π_1', π_2' be the corresponding elements under an anti-isomorphism between $E(G)$ and $E(H)$ for some H. Write

$H = H_1 \oplus H_2$, where $H_i = \pi_i'(H)$ $(1 \leqslant i \leqslant 2)$. Since $H_1 \simeq \mathbf{Q}$ and $\pi_2' E(H) \pi_1' = 0$, we must have $H_2 \simeq J_p$. Therefore $|E(H)| > 2^{\aleph_0} = |E(G)|$, and this contradiction proves that divisible mixed groups are not E-dual. □

The following result is an obvious consequence of Theorem 3.3.

COROLLARY 3.4. *Let G be a divisible group. Then G is E-dual if and only if G is E-self-dual.*

COROLLARY 3.5. *Let $G = D \oplus R$ and let D, the divisible part of G, be non-zero and torsion-free. Then G is E-dual if and only if D and R are E-dual and R is a torsion group.*

PROOF. Let G be an E-dual group as in the hypotheses. Fix a group H such that $E(G)$ and $E(H)$ are anti-isomorphic. Write $H = H_1 \oplus H_2$, where $\mathrm{Hom}(H_1, H_2) \simeq \mathrm{Hom}(R, D)$; $\mathrm{Hom}(H_2, H_1) = 0$ and $E(H_1)$, $E(H_2)$ are anti-isomorphic to $E(D)$ and $E(R)$ respectively. Then H_1 is isomorphic to D, while H_2 is a reduced torsion group. By symmetry, we conclude that R is a torsion group. The other assertions follow from Lemma 1.1 and the fact that D and R are fully invariant in G. □

COROLLARY 3.6. *Let $G = D \oplus R$; let D be a non-zero divisible torsion group and $R = \prod_{p \in P} R_p$ an adjusted cotorsion group. Then G is E-dual if and only if D is E-dual and R is finite.*

PROOF. By 1.1 and 2.6, we need only prove that if G is an E-dual group as in the hypotheses, then R is finite. To see this, fix a group H and an anti-isomorphism between $E(G)$ and $E(H)$ mapping φ onto φ' for every $\varphi \in E(G)$. Let π_1, π_2 be the projections of G onto D and R respectively, and let $H = H_1 \oplus H_2$, where $H_i = \pi_i'(H)$ $(1 \leqslant i \leqslant 2)$. Assume first that $R = R_p$ for some prime p. Our claim is that R is finite. Suppose the contrary. Then there exists $\varphi \in E(G)$ such that $\varphi \neq 0$, $\varphi(t(G)) = 0$. Since $G/t(G)$ is divisible and torsion-free, $\varphi \in p^\omega E(G)$ ([5] vol. I, p. 182) and, obviously, $\varphi = \varphi\pi_2$. Hence $\varphi'(H)$ is a non-zero subgroup of $p^\omega H_2$. On the other hand, by 3.1, $t(R)$ is a torsion-compact p-group. Since $E(R)$ is isomorphic to $E(t(R))$, the proof of Lemma 2.2 assures us that $p^\omega H_2 = 0$. This contradiction establishes that R is finite. To complete the proof, it remains to show that the hypothesis that G is E-dual always implies that R is finite. Assume this is not true. Then, as before, there exists $\varphi \in E(G)$ such that $\varphi \neq 0$ and $\varphi(t(G)) = 0$. For every prime p, let e_p denote the projection of G onto R_p. Remark that $\varphi'(H) \cap t_p(H_2) = 0$, because $t_p(H_2) = e'_p(H)$ and $\varphi'(H) \cap e'_p(H) = 0$ $(p \in P)$. Since $G/t(G)$ is divisible and torsion-free, it follows that $\varphi \in \bigcap_{p \in P} p^\omega E(G)$. Therefore $\varphi'(H)$ must be a torsion-free divisible subgroup of H_2, and this is clearly impossible. In fact, R has no subgroup isomorphic to $\mathbf{Q}$ and the same applies to H_2. This contradiction proves that R is finite, and the proof is complete. □

REMARK 1. Let G be as in 3.6. Then a necessary and sufficient condition for G to be E-dual is that D and R are E-dual with R a torsion group. In fact, by ([5] Corollary 54.4), reduced cotorsion torsion groups are bounded. The result now follows from 2.6.

REMARK 2. The hypotheses of 3.6 cannot be weakened, because there exist reduced E-dual groups G of the form $G = T \oplus R$, where T is a non-zero torsion group and R is an infinite adjusted cotorsion group. For instance, Lemma 1.2 tells us that the group $G = \bigoplus_{p \in P} \mathbf{Z}(p) \oplus \prod_{p \in P} \mathbf{Z}(p)$ is E-dual.

PROPOSITION 3.7. *If G is a mixed E-dual group, the following facts hold*:

(i) *$G/t(G)$ is not necessarily E-dual.*

(ii) *$t_p(G)/p^\omega t_p(G)$ is E-dual, for every prime p.*

PROOF. (i) Since $G = \prod_{p \in P} \mathbf{Z}(p)$ is E-dual and $G/t(G)$ is a divisible torsion-free group of rank $2^{\aleph_0}$, (i) follows from 3.3.

(ii) Let $B = \bigoplus_{n \in \mathbf{N}} B_n$ be a basic subgroup of $t_p(G)$. Since B_n is a summand of G, B_n is finite $(n \in \mathbf{N})$. If B is finite, then the statement clearly holds. Assume B is not finite. Then there exist suitable $x_n \in B$ $(n \in \mathbf{N})$ such that $B = \bigoplus_{n \in \mathbf{N}} \langle x_n \rangle$ and $o(x_r) \leqslant o(x_s)$ $(r, s \in \mathbf{N};\ r \leqslant s)$. Fix pairwise orthogonal projections $\pi_n\colon G \to \langle x_n \rangle$ $(n \in \mathbf{N})$ so that if $\eta\colon t_p(G) \to \prod_{n \in \mathbf{N}} \langle x_n \rangle$ is the product map, i.e. $\eta(x) = (\pi_n(x))_{n \in \mathbf{N}}$ $(x \in t_p(G))$, then $\operatorname{Ker} \eta = p^\omega\, t_p(G)$. It remains to show that $t\Big(\prod_{n \in \mathbf{N}} \langle x_n \rangle\Big) \leqslant \eta(t_p(G))$. Let e_{rs} denote the endomorphism of G uniquely defined by the following conditions: $e_{rs}(1 - \pi_s) = 0$ and $e_{rs}(x_s) = p^{\lambda_{rs}} x_r$, where $\lambda_{rs} =$

$= \max(0, e(x_r) - e(x_s))$ $(r, s \in \mathbb{N})$. As in the proof of 2.3, the existence of these elements implies $t\left(\prod_{n\in\mathbb{N}} \langle x_n \rangle\right) \leqslant \eta(t_p(G))$. □

REMARK. Condition (ii) indicates that only very particular torsion groups may be the torsion part of an E-dual group. We don't know examples of E-dual groups G such that, for some prime p, $p^\omega t_p(G)$ is not divisible. However, we can give a sufficient condition in order that $p^\omega t_p(G)$ is divisible. In fact, if G and H have anti-isomorphic endomorphism rings and $H/t(H)$ is p-divisible, then $t_p(G)$ is E-dual. To see this, assume the anti-isomorphism between $E(G)$ and $E(H)$ takes φ to φ' $(\varphi \in E(G))$. Write $t_p(G) = D \oplus R$; $t_p(H) = D' \oplus R'$ where D, D' are divisible and R, R' are reduced. We claim that $p^\omega R = 0$. Suppose this does not holds. Then there is an endomorphism φ of G such that $0 \neq \varphi(G) \leqslant p^\omega R[p]$. Let π_n $(n \in \mathbb{N})$ be as before. An argument similar to that used in 2.2 shows that $\bigoplus_{n\in\mathbb{N}} \pi'_n(H)$ is a basic subgroup of R'. Since $\pi_n \varphi = 0$ $(n \in \mathbb{N})$, φ' is 0 on $t(H)$ and clearly $0 \neq \varphi'(H) \leqslant H[p]$. But this is impossible, because $H/t(H)$ is p-divisible. This contradiction establishes that $p^\omega R = 0$. Consequently $p^\omega t_p(G)$ is divisible.

§ **4.** In this section we investigate some properties of torsion-free E-dual groups. Since Corollary 3.5 gives the structure of an E-dual group containing $\mathbb{Q}$, we can confine ourselves to the reduced case. First we recall some definitions.

If G is any group, the finite topology of $E(G)$ has the family of all $U_X = \{\varphi \in E(G) : \varphi(X) = 0\}$, with X a finite subset of G, as a basis of neighborhoods of 0. It is well known ([5] Theorem 107.1) that $E(G)$, with respect to the finite topology, is a complete Hausdorff topological ring. According to ([2], p. 63), reduced torsion-free groups of cardinality $< 2^{\aleph_0}$ are called control groups. If G is a group and, for some control group C, every subgroup of G of finite rank is isomorphic to a subgroup of C, then G is a controlled group. In the following, $\aleph_i$ denotes the first strongly inaccessible cardinal ([5] vol. II, p. 129).

THEOREM 4.1. *If G is a controlled group and $E(G)$ is of cardinality $< \aleph_i$, then G is E-dual.*

PROOF. It is enough to show that the ring $A = (E(G))^0$, equipped with the discrete topology, satisfies the hypotheses of ([2] Theorem 2.2). This clearly holds, if we only show that the group $E(G)$ is controlled. To this purpose, regard $E(G)$ as embedded in $\prod_X E(G)/U_X$, the product being extended over all finite subsets X of G, and let K be a subgroup of $E(G)$ of finite rank. Take linearly independent elements $\varphi_1, \ldots, \varphi_r \in E(G)$ such that $K \leqslant \langle \varphi_1, \ldots, \varphi_r \rangle_* \leqslant E(G)$. Then there exist a finite subset X' of G such that the natural projection $\pi\colon \prod_X E(G)/U_X \to$ $\to E(G)/U_{X'}$ maps $\varphi_1, \ldots, \varphi_r$ onto linearly independent elements. Since $K \cap \operatorname{Ker} \pi = 0$, K is isomorphic to a subgroup of $E(G)/U_{X'}$. The choice of K assures us that every subgroup of $E(G)$ of finite rank is is isomorphic to a subgroup of $\bigoplus_X E(G)/U_X$. Using ([2] Proposition 2.1), we conclude that $E(G)$ is controlled. This completes the proof. □

COROLLARY 4.2. *If G is a reduced torsion-free separable group and $E(G)$ is of cardinality $< \aleph_i$, then G is E-dual.*

PROOF. Let $C = \bigoplus_{p \in P} \mathbb{Z}_p^{(\mathbb{N})}$. Since C is a control group and every subgroup of G of finite rank may be embedded in C, the result follows from the previous theorem. □

REMARK. There exists an E-dual group G such that $G^{(\mathbb{N})}$ is E-dual. In fact, the group $\mathbb{Z}^{(\mathbb{N})}$ satisfies the hypotheses of 4.1. Observe that, by 2.5 and 3.3, this possibility cannot occur if G either a torsion or a divisible group.

Comparing 3.1 and 3.3 with 4.1, we see that the behaviour of torsion-free cotorsion groups is completely different from that of torsion-free non cotorsion groups. Also note that, by Corner's theorems, very complicated torsion-free groups have uncomplicated, even commutative, endomorphism rings ([14], p. 180; [15], p. 62). On the other hand, if G and H are arbitrary reduced torsion-free groups with anti-isomorphic endomorphism rings, then H does not generally inherit many properties of G. For instance, it has been proved ([6] Theorem 1.2) that if $G = \mathbb{Z}^{(\mathbb{N})}$, then there is no reduced torsion-free group H of the same type as $\mathbb{Z}$ such that $E(G)$ and $E(H)$ are anti-isomorphic. More generally, we have the following

PROPOSITION 4.3. *There exist a free group G and a non controlled group H such that $E(G)$ and $E(H)$ are anti-isomorphic.*

PROOF. Let $G = \mathbb{Z}^{(\mathbb{N})}$ and $A = (E(G))^0$. We shall show first that A, endowed with the discrete topology, satisfies the hypotheses of ([3] Theorem 1). Fix a prime p. Since $|A| = 2^{\aleph_0}$ and $p^\omega A = 0$, it suffices to prove that J_p is linearly disjoint from the group A, that is from the group $E(G)$, over $\mathbf{Z}_p$ $\left(\text{i.e. if } \sum_{i=1}^{n} \alpha_i \varphi_i = 0 \text{ in } \widehat{E(G)}\text{, the } p\text{-adic}\right.$ completion of $E(G)$, with $\varphi_1, \ldots, \varphi_n \in E(G)$; $\alpha_1, \ldots, \alpha_n \in J_p$ and linearly independent over $\mathbf{Z}_p$, then $\left.\varphi_1 = \ldots = \varphi_n = 0\right)$. Assume this is not true. Then we may write $\sum_{i=1}^{n} \alpha_i \varphi_i = 0$, where $\varphi_i \in E(G)$ $(1 \leqslant i \leqslant n)$, $\varphi_1 \neq 0$ and the α_i's are as before. Let $G = \bigoplus_{n \in \mathbf{N}} G_n$, where $G_n = \mathbf{Z}$, $x_n = 1 \in G_n$ and π_n is the projection of G onto G_n $(n \in \mathbf{N})$. By hypothesis, there exist $r, s \in \mathbf{N}$ such that $\pi_r\left(\sum_{i=1}^{n} \alpha_i \varphi_i(x_s)\right) = 0$ is a linear combination of the α_i's with coefficients in $\mathbf{Z}$ not all equal to 0. This contradiction establishes that A has the required property. We claim that there exists a non controlled group H whose endomorphism ring, with the finite topology, is the discrete ring A. In fact, for every $a \in A$ we can choose a p-adic integer $\alpha(a)$ with the following properties:

(i) The set $\{\alpha(a) \colon a \in A\}$ is algebraically independent over $\mathbf{Z}_p$.

(ii) J_p has trascendence degree $2^{\aleph_0}$ over the subring generated by the $\alpha(a)$'s. Let H be the following pure subgroup of the p-adic completion $\hat{A}$ of A

$$H = \langle A, A\alpha(a)(a \in A)\rangle_* \leqslant \hat{A} .$$

Since ([3] Theorem 1) assures that $E(H)$ is isomorphic to A, it remains to check that H is not controlled. To see this, let S denote the subset of all $\varphi \in E(G)$ such that $\pi_1 \varphi \pi_1 = \pi_1$; $\pi_r \varphi \pi_s = 0$ $(r, s \in \mathbf{N};\ r \neq s)$. Now consider the pure subgroups S' and S'', where

$$S' = \langle 1, \alpha(\varphi)\ (\varphi \in S)\rangle_* \leqslant J_p; \qquad S'' = \langle \pi_1, \pi_1 \varphi \alpha(\varphi)\ (\varphi \in S)\rangle_* \leqslant \widehat{E(G)} .$$

By ([3] Proposition 1), S' is not controlled. Since S'' is isomorphic to S' and $S'' \leqslant H$, we conclude that H is not controlled. □

PROPOSITION 4.4. *There exists a countable reduced torsion-free group G such that $E(G)$ is not anti-isomorphic to the endomorphism ring of a countable reduced torsion-free group.*

PROOF. Let $G = \mathbf{Z}^{(\mathbf{N})}$. With the same notations of 4.3, let e_{rs} be the endomorphism of G defined by $e_{rs}(x_s) = x_r$; $e_{rs}(1 - \pi_s) = 0$ $(r, s \in \mathbf{N})$. Let $f \colon E(G) \to E(H)$ be any anti-isomorphism. To end the proof, it is enough to show that H is not countable. Assume the contrary.

Then H is a countable reduced torsion-free group and f is continuous with respect to the finite topologies of $E(G)$ and $E(H)$. This is an immediate consequence of ([14] Lemma 4.3), because if U is a subgroup of $E(H)$, then

$$U \text{ open} \Leftrightarrow E(H)/U \cong E(G)/f^{-1}(U) \text{ countable}$$
$$\text{reduced torsion-free} \Leftrightarrow f^{-1}(U) \text{ open}.$$

Since $E(G)$ is not discrete, the same applies to $E(H)$. Therefore $E(H)$ has a proper open left ideal U and $f^{-1}(U)$ is a proper open right ideal of $E(G)$. It is now clear that there exists $m \in \mathbf{N}$ such that $\{\varphi \in E(G): \varphi\pi_r = 0 \ (1 \leqslant r \leqslant m)\} \leqslant V$, where V is an open two-sided ideal of $E(G)$ and $V \leqslant f^{-1}(U)$. Choose $\varphi \in E(G) \setminus V$ and define $\varphi', \varphi'' \in E(G)$ as follows: $\varphi'\pi_r = \varphi\pi_r$, $\varphi''\pi_r = 0$ $(1 \leqslant r \leqslant m)$; $\varphi'\pi_r = 0$, $\varphi''\pi_r = \varphi\pi_r$ $(r > m)$. Evidently $\varphi = \varphi' + \varphi''$ and $\varphi'' \in V$. Write $\varphi' = \sum_{r=1}^{m} \varphi_r$, where $\varphi_r\pi_r = \varphi'\pi_r$, $p_r(1 - \pi_x) = 0$ $(1 \leqslant r \leqslant m)$. Then there are suitable $n_{kr} \in \mathbf{Z}$ $(k \in \mathbf{N};\ 1 \leqslant r \leqslant m)$ almost all 0 such that $\varphi_r = \sum_{k \in \mathbf{N}} n_{kr} e_{kr}$. Since

$$e_{kr} = e_{k,m+1} e_{m+1,r} \in V \qquad (k \in \mathbf{N};\ 1 \leqslant r \leqslant m),$$

we conclude that $\varphi \in V$. This contradiction proves that H is not countable, and the proof is complete. □

REMARK 1. There exists a non commutative topological ring A such that A is the endomorphism ring of a countable reduced torsion-free group and the same applies to its opposite ring. In fact, let $G = \mathbf{Z}^{(\mathbf{N})}$ and, using the notations of 4.4, let A be the subring of $E(G)$ consisting of all φ such that $\pi_r\varphi\pi_s = 0$ $(r, s \in \mathbf{N};\ r > s)$, i.e. A is isomorphic to the subring of all upper triangular $\aleph_0 \times \aleph_0$ matrices with entries in $\mathbf{Z}$ ([5] Theorem 106.1). It is easy to see that A, with the topology induced by the finite topology of $E(G)$, has a family of two-sided ideals as a basis of neighborhoods of 0 and satisfies the hypotheses of ([2] Theorem 1.1).

REMARK 2. The direct sum of two reduced torsion-free E-self-dual groups is not necessarily E-dual. To prove this, fix a prime p and let $G = \mathbf{Z} \oplus J_p$. We claim that G is not E-dual. Otherwise, $E(G)$ is anti-isomorphic to $E(H)$ for some group H of the form $H = H' \oplus H''$, where $E(H') \cong \mathbf{Z}$; $E(H'') \cong J_p$; $\mathrm{Hom}(H', H'') = 0$ and

$\operatorname{Hom}(H'', H') \simeq J_p$. Since H' is reduced and torsion-free, we may assume $H'' = J_p$. Consequently $\operatorname{Hom}(H'', H') = \operatorname{Hom}_{J_p}(H'', H')$ Choose a non-zero homomorphism $\varphi: H'' \to H'$ and regard H' as a pure subgroup of its $\mathbb{Z}$-adic completion $\hat{H}' = \prod_{q \in P} \hat{H}'_q$. Since $\varphi(H'') = J_p\varphi(1) \leqslant \leqslant H' \cap \hat{H}'_p$, there exists $x \in H' \setminus pH'$ such that $\varphi(1) = p^n x$ for some $n \in \mathbb{N}$. These conditions imply that $J_p x$ is a pure subgroup, hence a direct summand, of H'. But this is clearly impossible, because $E(H')$ is isomorphic to $\mathbb{Z}$. This contradiction establishes that G' is not E-dual.

The previous example suggests that we determine some properties of all E-dual groups admitting a free summand.

Proposition 4.5. *If $G = \mathbb{Z} \oplus G'$ is E-dual, then the following conditions hold:*

(i) *G' is reduced and torsion-free.*

(ii) *G' is not cotorsion.*

(iii) *G' is not necessarily a controlled group.*

Proof. (i) We first prove that G' is torsion-free. Suppose $E(G)$ is anti-isomorphic to $E(H)$. Then H has a decomposition $H = H' \oplus H''$, where $E(H') \simeq \mathbb{Z}$ and $\operatorname{Hom}(H'', H') \simeq \operatorname{Hom}(\mathbb{Z}, G') \simeq G'$. Since H' is torsion-free, the same applies to G'. Using Corollary 3.5, we conclude that G' is reduced.

(ii) This immediately follows from (i) and Remark 2.

(iii) Fix a prime p. Let G' denote a pure subgroup of J_p with the following properties: $1 \in G'$; $|G'| = 2^{\aleph_0}$ and the trascendence degree of J_p over the subring generated by G' is $2^{\aleph_0}$. Let $G = \mathbb{Z} \oplus G'$; then, as in 4.3, one can show that the ring $(E(G))^0$, with the discrete topology, satisfies the hypotheses of ([3] Theorem 1). Thus G is E-dual and, by ([3] Proposition 1), G' is not controlled. □

Remark. More generally, if R is a rational group, p is a prime and $G = R \oplus G'$ is E-dual, then $pR \neq R$ implies $t_p(G') = 0$. In fact, we can find a group $H = H' \oplus H''$ such that $E(G)$ is anti-isomorphic to $E(H)$, $E(H') \simeq E(R)$ and $\operatorname{Hom}(H'', H') \simeq \operatorname{Hom}(R, G')$. Since H' is torsion-free and $R \neq pR$, G' has no element of order p. Finally note that if $R = \mathbb{Q}$, then the structure of G' is completely determined by 3.5.

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