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LEOPOLDO NACHBIN

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On the Closure of Modules of Continuously Differentiable Mappings.

Leopoldo Nachbin (*)

1. Introduction.

In 1948, Whitney proved a conjecture of Laurent Schwartz and characterized the closure of an ideal in the algebra $\mathcal{C}^m(U; \mathbb{R})$ of all continuously m-differentiable real functions on the nonvoid open subset U of $\mathbb{R}^n$; see § 2 below for details.

In 1949, we proved a result characterizing dense subalgebras of $\mathcal{C}^m(U; \mathbb{R})$ in the spirit of the Weierstrass-Stone theorem for continuous functions; see [8].

Subsequently, our density theorem was extended from $\mathbb{R}^n$ to arbitrary dimensions, and from scalar values to vector values; see the work of Lesmes [4], Llavona [5], [6], Prolla [11], [12] and Aron [1].

Next, the Whitney ideal theorem has been extended from $\mathbb{R}^n$ to arbitrary dimensions, and from scalar values to vector values; see the work of Guerreiro [2], [3].

In a previous paper, we have presented part of the work of Lesmes, Llavona and Prolla is an improved and simplified form; see [9]. The purpose of the present article is to offer likewise a better and straightforward version of part of the work of Guerreiro, in the style that we adopted in [9]. As we aimed only at illustrating a fashion of exposition, we avoided being exhaustive both in [9] as well as in here.

(*) Indirizzo dell'A.: Department of Mathematics, University of Rochester, Rochester, N.Y. 14627, U.S.A.; Departmento de Matemática Pura, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ ZC-32 Brasil.

2. The Whitney ideal theorem.

The standard references are Whitney [14], Malgrande [7], Tougeron [13] and Poenaru [10].

Let U be a nonvoid open subset of $\mathbb{R}^n$, where $n \in \mathbb{N}^*$. Consider the algebra $\mathcal{C}^m(U; \mathbb{R})$ of all continuously m-differentiable real functions on U, where $m \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, If $f \in \mathcal{C}^m(U; \mathbb{R})$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, then $D^\alpha f$ denotes the partial α-derivative of f of order $|\alpha| = \alpha_1 + \ldots + \alpha_n \leqslant m$. Endow $\mathcal{C}^m(U; \mathbb{R})$ with the natural topology $\mathcal{T}_m$ defined by the family (with parameter α, K) of seminorms

$$f \in \mathcal{C}^m(U; \mathbb{R}) \mapsto \sup\{|D^\alpha f(x)|;\, x \in K\} \in \mathbb{R} \qquad \text{for } \alpha \in \mathbb{N}^n,\ |\alpha| \leqslant m\,,$$

where K is a nonvoid compact subset of U. Let I be an ideal in $\mathcal{C}^m(U; \mathbb{R})$ and $f \in \mathcal{C}^m(U; \mathbb{R})$. Then f belongs to the closure of I in $\mathcal{C}^m(U; \mathbb{R})$ if and only if the following equivalent conditions are satisfied:

1) For every $x \in U$ and $k \in \mathbb{N}$, $k \leqslant m$, there is some $g \in I$ such that $D^\alpha f(x) = D^\alpha g(x)$ for all $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k$.

2) For every $x \in U$, $k \in \mathbb{N}$, $k \leqslant m$ and $\varepsilon > 0$ there is some $g \in I$ such that $|D^\alpha g(x) - D^\alpha f(x)| \leqslant \varepsilon$ for all $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k$.

We may restate the above results as follows. For every $x \in U$ and $k \in \mathbb{N}$, $k \leqslant m$, let $I_k(x)$ be the closed ideal of $\mathcal{C}^m(U; \mathbb{R})$ formed by the $f \in \mathcal{C}^m(U; \mathbb{R})$ such that $D^\alpha f(x) = 0$ for all $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k$. Moreover, for every $x \in U$, $k \in \mathbb{N}$, $k \leqslant m$ and $\varepsilon > 0$, let $I_{k\varepsilon}(x)$ be the set formed by the $f \in \mathcal{C}^m(U; \mathbb{R})$ such that $|D^\alpha f(x)| \leqslant \varepsilon$ for all $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k$. Then the closure $\bar{I}$ of I in $\mathcal{C}^m(U; \mathbb{R})$ is given by

$$\bar{I} = \bigcap_{x,k} \{I + I_k(x)\}\,, \tag{1}$$

$$\bar{I} = \bigcap_{x,k,\varepsilon} \{I + I_{k\varepsilon}(x)\}\,. \tag{2}$$

There is another way of stating the Whitney ideal theorem, as follows. The closure of an ideal I of $\mathcal{C}^m(U; \mathbb{R})$ for the topology $\mathcal{T}_m$ is equal to the closure of I for the topology $\mathcal{T}_{ms}$ on $\mathcal{C}^m(U; \mathbb{R})$ defined by the family (with parameters α and x) of seminorms

$$f \in \mathcal{C}^m(U; \mathbb{R}) \mapsto |D^\alpha f(x)| \in \mathbb{R}$$

for $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant m$, where $x \in U$.

3. Modules of continuously differentiable mappings.

Let E, F be Hausdorff real locally convex spaces $E \neq 0$, $F \neq 0$, U be a nonvoid open subset of E and $m \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We denote by $\mathcal{C}^m(U; F)$ the vector space of all mappings $f\colon U \to F$ that are continuously m-differentiable in the following sense:

1) f is finitely m-differentiable; that is, for every finite dimensional vector subspace S of E with $S \neq 0$, $U \cap S$ nonvoid, we assume that the restriction $f|(U \cap S)$ is m-differentiable in the classical sense. Hence we have the differentials

$$d^k f\colon\ U \mapsto \mathcal{L}_{as}({}^kE; F)$$

for $k \in \mathbb{N}$, $k \leqslant m$, with values in the vector space $\mathcal{L}_{as}({}^kE; F)$ of all symmetric k-linear mappings of E^k into F.

2) The mapping

$$(x, t) \in U \times E \mapsto d^k f(x)\, t^k \in F$$

is continuous, for every $k \in \mathbb{N}$, $k \leqslant m$; in particular each $d^k f(x)$ belongs to the vector subspace $\mathcal{L}_s({}^kE; F)$ of all continuous symmetric k-linear mappings of E^k into F.

We endow $\mathcal{C}^m(U; F)$ with the topology $\mathcal{T}_m$ defined by the family (with parameters k, β, K, L) of seminorms

$$f \in \mathcal{C}^m(U; F) \mapsto p_{KL}^{k\beta}(f) = \sup\,\{\beta[d^k f(x)\, t^k];\ x \in K,\ t \in L\} \in \mathbb{R}$$

for $k \in \mathbb{N}$, $k \leqslant m$, where β is a continuous seminorm on F and K, L are nonvoid compact subsets of U, E respectively. If $F = \mathbb{R}$, the topology $\mathcal{T}_m$ is defined by the family (with parameters k, K, L) of seminorms

$$f \in \mathcal{C}^m(U; \mathbb{R}) \mapsto p^k_{KL}(f) = \sup\{|d^k f(x) t^k|;\ x \in K,\ t \in L\} \in \mathbb{R}$$

with k, K, L as above.

We recall that E is said to have the Banach-Grothendiek approximation property if the identity mapping I_E belongs to the closure of the vector subspace $E' \otimes E$, formed by all continuous linear endomorphisms of E with finite dimensional images, for the compact-open topology on the vector space $\mathcal{L}(E; E)$ of all continuous linear endomorphisms of E.

THEOREM 1. Let W be a vector subspace of $\mathcal{C}^m(U; F)$ which is a module over the algebra $\mathcal{C}^m(U; \mathbb{R})$. Assume that there is a subset G of $E' \otimes E$ such that:

$E1$) The identity mapping I_E belongs to the closure of G for the compact-open topology on $\mathcal{L}(E; E)$.

$E2$) $W \circ G \subset \overline{W}$ in the following sense: for every $J \in G$, every nonvoid open subset V of U such that $J(V) \subset U$ and every $f \in W$, then the restriction $(f \circ J)|V = f \circ (J|V)$ belongs to the closure of the restriction $W|V$ in $\mathcal{C}^m(V; F)$ for $\mathcal{T}_m$.

Assume moreover that:

$F1$) F has the approximation property.

$F2$) $(F' \otimes F) \circ W \subset W$, that is, $(\psi \otimes b) \circ f = (\psi \circ f) \otimes b \in W$ if $\psi \in F'$, $b \in F$, $f \in W$.

Then $f \in \mathcal{C}^m(U; F)$ belongs to the closure of W in $\mathcal{C}^m(U; F)$ if and only if, for every $x \in U$, $k \in \mathbb{N}$, $k \leqslant m$, every neighborhood Y of O in F and every $t = (t_1, \ldots, t_n) \in E^n$ with $n \in \mathbb{N}^n$, there is some $g \in W$ such that

$$\frac{\partial^\alpha g}{\partial t^\alpha}(x) - \frac{\partial^\alpha f}{\partial t^\alpha}(x) \in Y$$

for all $a \in \mathbb{N}$, $|\alpha| \leqslant k$.

PROOF. Necessity is clear. It does not depend on the conditions $E1$), $E2$), $F1$), $F2$), and follows from a classical polarization formula expressing a symmetric k-linear mapping as a finite sum in terms of the corresponding k-homogeneous polynomial.

We will subdivide the proof of sufficiency into two parts. Fix $f \in \mathcal{C}^m(U; F)$ satisfying the assumed conditions with respect to W.

PART 1. Suppose that $F = \mathbb{R}$, so that W is an ideal in $\mathcal{C}^m(U; \mathbb{R})$. The case in which E is finite dimensional (when G is not utilized for we may take G reduced to I_E) is the classical Whitney ideal theorem (see the above § 2). Let E be arbitrary. Fix $K \subset U$, $L \subset E$ both compact nonvoid, $k \in \mathbb{N}$, $k \leqslant m$ and $\varepsilon > 0$. By condition $E1$) in the statement of the theorem, there are $J \in G$ and a nonvoid open subset $V \subset U$ such that $K \subset V$, $J(V) \subset U$ and

$$(1) \qquad p^i_{KL}[f \circ (J|V) - f|V[\leqslant \varepsilon$$

for $i = 0, \ldots, k$ (the proof of this assertion will not be repeated here and is given in full detail in Part 2 of the proof of the main theorem in [9]). Set $E_J = J(E)$; it is a finite dimensional vector subspace of E. We may assume that $E_J \neq 0$ (see the beginning of Part 3 in the proof of the main theorem in [9]). Set $U_J = U \cap E_J$; it is an open subset of E_J that is nonvoid since $U_J \supset U \cap J(V) = J(V)$ and V is nonvoid. Let W_J be the ideal of $\mathcal{C}^m(U_J; \mathbb{R})$ formed by the restrictions $g_J = g|U_J$ for $g \in W$. Then $f_J = f|U_J$ satisfies the Whitney conditions with respect to W_J because f satisfies the assumed conditions with respect to W. By Whitney's classical theorem there is $g \in W$ such that

$$p^i_{J(K)J(L)}(g_J - f_J) \leqslant \varepsilon$$

which means

$$|d^i g_J[J(x)[J(t)^i - d^i f_J[J(x)[J(t)^i| \leqslant \varepsilon$$

that is

$$|d^i[g \circ (J|V)](x) t^i - d^i[f \circ (J|V)](x) t^i| \leqslant \varepsilon$$

for $x \in K$, $t \in L$, hence

$$(2) \qquad p^i_{KL}[g \circ (J|V) - f \circ (J|V)] \leqslant \varepsilon$$

for $i = 0, \ldots, k$. According to condition $E2$) in the statement of the theorem, there is $h \in W$ such that

$$(3) \qquad p^i_{KL}[h|V - g \circ (J|V)] \leqslant \varepsilon$$

for $i = 0, \ldots, k$. Then (1), (2), (3) give us

$$p^i_{KL}(h|V - f|V) \leqslant 3\varepsilon$$

that is, once V is an open neighborhood of K in U,

$$p^i_{KL}(h - f) \leqslant 3\varepsilon$$

for $i = 0, \ldots, k$. Thus f belongs to the closure of W in $\mathcal{C}^m(U; \mathbb{R})$.

PART 2. Let now E, F be arbitrary. Fix any $\psi \in F'$. It is clear that G and the ideal $\psi \circ W$ of $\mathcal{C}^m(U; \mathbb{R})$ satisfy condition $E2$) of the statement of the theorem with $F = \mathbb{R}$, because G and the module W over $\mathcal{C}^m(U; \mathbb{R})$ satisfy that condition $E2$) with just F. Moreover, once f satisfies the assumed conditions with respect to W, it follows that $\psi \circ f$ satisfies the corresponding conditions with respect to $\psi \circ W$. By Part 1, we have

$$\psi \circ f \in \overline{\psi \circ W} \tag{4}$$

for any $\psi \in F'$, where closure is taken in $\mathcal{C}^m(U; \mathbb{R})$. Fix any $b \in F$. The linear mapping $g \in \mathcal{C}^m(U; \mathbb{R}) \mapsto g \otimes b \in \mathcal{C}^m(U; F)$ is continuous, and so

$$\left(\overline{\psi \circ W}\right) \otimes b \subset \overline{(\psi \circ W) \otimes b} \tag{5}$$

where closures in the left and right hand sides are taken in $\mathcal{C}^m(U; \mathbb{R})$ and $\mathcal{C}^m(U; F)$, respectively. By (4), (5) and condition $F2$) of the statement of the theorem, we have that

$$(\psi \otimes b) \circ f = (\psi \circ f) \otimes b \in \left(\overline{\psi \circ W}\right) \otimes b$$
$$\subset \overline{(\psi \circ W) \otimes b} = \overline{(\psi \otimes b) \circ W} \subset \overline{W}$$

for every $\psi \in F'$, $b \in F$, that is

$$(F' \otimes F) \circ f \subset \overline{W}, \tag{6}$$

where closure is taken in $\mathcal{C}^m(U; F)$. Now, we know that the linear mapping $T \in \mathcal{L}(F; F) \mapsto T \circ f \in {}^n\mathcal{C}^m(U; F)$ is continuous if $\mathcal{L}(F; F)$ is given the compact-open topology. Thus we have

$$f = I_F \circ f \in \overline{(F' \otimes F) \circ f}, \tag{7}$$

where closure is taken in $\mathcal{C}^m(U; F)$, because we assumed that

$$I_F \in \overline{F' \otimes F},$$

where closure is taken in $\mathcal{L}(F; F)$, by condition $F1$) of the statement of the theorem. Finally (6) and (7) imply that $f \in \overline{W}$. Q.E.D.

Let us also endow $\mathcal{C}^m(U; F)$ with the topology $\mathcal{T}_{ms}$ defined by the family (with parameters k, β, x, t) of seminorms

$$f \in \mathcal{C}^m(U; F) \mapsto \beta[d^k f(x) t^k] \in \mathbf{R}$$

for $k \in \mathbf{N}$, $k \leqslant m$, where β is a continuous seminorm on F and $x \in U$, $t \in E$.

Corollary 2. Let W, G be as in the statement of Theorem 1, and assume conditions $E1$), $E2$), $F1$), $F2$). Then the closure of W for $\mathcal{T}_m$ is equal to the closure of W for $\mathcal{T}_{ms}$.

Proof. It follows from the polarization formula that $\mathcal{T}_{ms}$ on $\mathcal{C}^m(U; F)$ is also defined by the family (with parameters $k, \beta, x, t_1, \ldots, t_k$) of seminorms

$$f \in \mathcal{C}^m(U; F) \mapsto \beta[d^k f(x)(t_1, \ldots, t_k)] \in \mathbf{R}$$

for $k \in \mathbf{N}$, $k \leqslant m$, where β is a continuous seminorm on F and $x \in U$, $t_1, \ldots, t_k \in E$. Thus the result follows. Q.E.D.

Remark 3. For every $x \in U$ and $k \in \mathbf{N}$, $k \leqslant m$, let $W_k(x)$ be the closed module over $\mathcal{C}^m(U; \mathbf{R})$ formed by the $f \in \mathcal{C}^m(U; F)$ such that

$$d^i f(x) = 0 \qquad \text{for all } i = 0, \ldots, k .$$

Moreover, for every $x \in U$, $k \in \mathbf{N}$, $k \leqslant m$, every neighborhood Y of 0 in F and $t = (t_1, \ldots, t_n) \in E^n$ with $n \in \mathbf{N}^*$, let $W_{kYt}(x)$ be the set formed by the $f \in \mathcal{C}^m(U; F)$ such that

$$\frac{\partial^\alpha f}{\partial t^\alpha}(x) \in Y$$

for all $\alpha \in \mathbf{N}^n$, $|\alpha| \leqslant k$. Then we may ask if the closure $\overline{W}$ of W in $\mathcal{C}^m(U; F)$ is given by

$$\overline{W} = \bigcap_{x,k} \{W + W_k(x)\}, \tag{1}$$

$$\overline{W} = \bigcap_{x,k,Y,t} \{W + W_{kYt}(x)\}, \tag{2}$$

by analogy with (1) and (2) of section 2, respectively. Theorem 1 means indeed that, under the assumptions in its statement, (2) does hold true. Moreover, since $W_{kYt}(x) \supset W_k(x)$, it then follows under the assumptions in the statement of Theorem 1, that the left hand side of (1) contains its right hand side. However, the following example shows that (1) may break down if E is infinite dimensional and $F = \mathbf{R}$.

EXAMPLE 4. Assume that E is a real normed space, $e_i \in E$ $(i \in \mathbf{N})$ and $\varphi_j \in E'$ $(j \in \mathbf{N})$ are such that $\varphi_j(e_i) = \delta_{ij}$ $(i, j \in \mathbf{N})$, and the φ_j $(j \in \mathbf{N})$ generate a vector subspace S which is dense in E'. We may assume that

$$\sum_j \|\varphi_j\| < +\infty \tag{1}$$

by adjusting. If $x \in E$, let $I_0(x)$ be the closed ideal of $\mathcal{C}^1(E; \mathbf{R})$ formed by the $f \in \mathcal{C}^1(E; \mathbf{R})$ such that $f(x) = 0$; and $I_1(x)$ be the closed ideal of $\mathcal{C}^1(E; \mathbf{R})$ formed by the $f \in I_0(x)$ such that $df(x) = 0$. Finally, let I be the ideal of $\mathcal{C}^1(E; \mathbf{R})$ formed by the $f \in I_0(0)$ such that

$$\frac{\partial f}{\partial e_i}(0) = 0$$

for all but finitely many $i \in I$. We have that $I_1(0) \subset I \subset I_0(0)$ and we claim that

$$I = \bigcap_x \{I + I_1(x)\}; \tag{2}$$

in fact, clearly $I \subset I + I_1(x)$ for every $x \in E$, and $I = I + I_1(0)$ once $I_1(0) \subset I$. To prove that

$$I \neq I_0(0)\,, \qquad \bar{I} = I_0(0)\,, \tag{3}$$

we argue as follows. By using (1), we introduce

$$f = \sum_i \varphi_i \in E' \subset \mathcal{C}^1(E; \mathbf{R})\,.$$

Then $f(0) = 0$ and $df(x) = f$ for every $x \in E$, which shows that

$$\frac{\partial f}{\partial e_i}(0) = df(0)(e_i) = f(e_i) = \varphi_i(e_i) = 1$$

for every $i \in I$. Thus $f \in I_0(0)$ and $f \notin I$. This proves the first half of (3). Next, consider any $f \in I_0(0)$. Put $g = df(0) \in E'$. There are $g_n \in S$ $(n \in \mathbf{N})$ such that $\|g_n - g\| \to 0$ as $n \to \infty$. Define $f_n = f - (g - g_n) \in \mathcal{C}^1(E; \mathbf{R})$, $(n \in \mathbf{N})$. Notice that $f_n(0) = 0$ and $df_n(x) = df(x) - (g - g_n)$ for every $x \in E$. In particular $df_n(0) = g_n$ and

$$\frac{\partial f_n}{\partial e_i}(0) = df_n(0)(e_i) = g_n(e_i) = 0$$

for all but finitely many $i \in I$. Thus $f_n \in I$ for every $n \in \mathbf{N}$. Moreover we have

$$|f_n(x) - f(x)| \leq \|g_n - g\| \cdot \|x\|\,,$$

$$\|df_n(x) - df(x)\| = \|g_n - g\|\,,$$

for every $n \in \mathbf{N}$, $x \in E$, showing that $f_n \to f$ as $n \to \infty$ in $\mathcal{C}^1(E; \mathbf{R})$. This proves the second half of (3). Thus I is not closed in $\mathcal{C}^1(E; \mathbf{R})$. Then (2) implies that

$$\bar{I} \neq \bigcap_x \{I + I_1(x)\}\,,$$

that is, (1) of Remark 3 breaks down. With regard to conditions $E1$), $E2$) of the statement of Theorem 1, once E has the approximation property, we take $G = E' \otimes E$ and notice that $I \circ G \subset I$. Of course, conditions $F1$), $F2$) of the same statement are to be disregarded once $F = \mathbf{R}$.

BIBLIOGRAPHY

[1] R. M. Aron - J. B. Prolla, *Polynomial approximation of differentiable functions on Banach spaces*, Journal für die Reine und Angewandte Mathematik, to appear.

[2] C. S. Guerreiro, *Ideias de funçoes diferenciáveis*, Anais da Academia Brasileira de Ciências, **49** (1977), pp. 47-70.

[3] C. S. Guerreiro, *Whitney's spectral synthesis theorem is infinite dimensions*, in *Approximation Theory and Functional Analysis* (Editor: J. B. Prolla), Notas de Matemática (1979), North-Holland, to appear.

[4] J. Lesmes, *On the approximation of continuously differentiable functions in Hilbert spaces*, Revista Colombiana de Matemática, **8** (1974), pp. 217-223.

[5] J. G. Llavona, *Aproximación de funciones diferentiables*, Universidad Complutense de Madrid, 1975.

[6] J. G. Llavona, *Approximation of differentiable functions*, Advances in Mathematics, to appear.

[7] B. Malgrange, *Ideals of differentiable functions*, Oxford University Press, 1966.

[8] L. Nachbin, *Sur les algèbres denses de fonctions différentiables sur une variété*, Comptes Rendus de l'Académie des Sciences de Paris, **228** (1949), pp. 1549-1551.

[9] L. Nachbin, *Sur la densité des sous-algèbres polynomiales d'applications continûment différentiables*, Séminaire Pierre Lelong - Henri Skoda (1976-1977), Lecture Notes in Mathematics 694 (1978), to appear.

[10] V. Poenaru, *Analyse différentielle*, Lecture Notes in Mathematics 371 (1974).

[11] J. B. Prolla, *On polynomial algebras of continuously differentiable functions*, Rendiconti della Accademia Nazionale dei Lincei, **57** (1974), pp. 481-486.

[12] J. B. Prolla - C. S. Guerreiro, *An extension of Nachbin's theorem to differentiable functions on Banach spaces with the approximation property*, Arkiv för Mathematik, **14** (1976), pp. 251-258.

[13] J. C. Tougeron, *Idéaux de fonctions différentiables*, Springer-Verlag, 1972.

[14] H. Witney, *On ideals of differentiable functions*, American Journal of Mathematics, **70** (1948), pp. 635-658.

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