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A. Barlotti
E. Schreiber
K. Strambach

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The Group of Projectivities in Free-Like Geometries.

A. BARLOTTI (*)

E. SCHREIBER - K. STRAMBACH (**)

Introduction.

Since the time of the masterly work of von Staudt the group of projectivities of a line onto itself has played an important role in geometry; from the ideas of von Staudt and F. Schur we know that if in a geometry every element ($\neq 1$) of this group has only a few fixed points the geometry has a classical representation. If, for instance, the geometry is a projective plane or a Benz plane and the group is sharply 3-transitive, then the plane is pappian, or miquelian respectively (see [5], [12] and [13]); if the geometry is an affine plane and if the group Π of affine projectivities is sharply 2-transitive, then the plane is desarguesian.

It is not the case that by progressive weakenings of the number of fixed points of projectivities different from the identity we get classes of non-classical « nice » geometries in which some characteristic configurational propositions hold. For example, Barlotti [2] showed on the one hand that in a free projective plane (which contains no confined configurations) the stabilizer in Π of any 6 distinct points consists only of the identity; Schleiermacher [14] proved on the other hand that if the stabilizer in Π of any 5 distinct points is always the identity, no element ($\neq 1$) can have more than two different fixed points, and the plane is pappian.

(*) Indirizzo dell'A.: Università di Bologna.

(**) Indirizzo degli A.A.: Universität Erlangen - Nürnberg, Rep. Fed. Tedesca.

With respect to the number of fixed points for elements ($\neq 1$) of Π it is interesting to note that the free projective planes are nearer to the pappian planes than the desarguesian (non pappian) projective planes are, since in the latter planes there are stabilizers of sets of infinite points which are different from the identity (Schleiermacher [15], lemma 2.8). This goes in the same direction as a result of Joussen (3) which states that a finitely generated free projective plane has an archimedean order whereas no desarguesian non pappian projective plane possesses an archimedean order (see [13] p. 243).

One of the purposes of the present paper is to initiate the study of the group of projectivities of a line (block) onto itself for geometries different from projective planes, such a study there being already underway. From this points of view it is interesting to determine the number n of points on a block in a free Benz plane (or free affine plane) for which the stabilizer in Π of every $(n+1)$-tuple of points consist of the identity, but there exist n-tuples whose stabilizer in Π is different from the identity. Funk discovered that for free Benz planes this number n is 6. In section 1 we study this problem for the affine planes, and we prove that in a free affine plane $n = 4$. Results (similar to Schleiermacher's above quoted result [14] for projective planes) for the gap between $n = 3$ for the miquelian Benz planes and the group of projectivities [$n = 2$ for the desarguesian affine planes and the group of affine projectivities] and $n = 6$ [$n = 4$] for the corresponding free geometries, are as yet unknown.

Another aim of the present paper is to show that in the classes of projective and Benz planes (or in the class of affine planes) we can find a geometry in which the stabilizer in Π of $(m+1)$-tuples consists only of the identity for every m with $6 \leqslant m \leqslant \aleph_0$ (or $4 \leqslant m \leqslant \aleph_0$ respectively), but there are stabilizers of m distinct points which are not the identity. The construction of these geometries takes place by extending a closed configuration which increases as m grows. This may give us a first insight into the fact that the group of projectivities in the classical geometries is further from the group in a geometry with closed configurations than from the group in a free geometry.

Another result, which is worthy of mention, is proved here: Contrary to the cases of both the classical and the free geometries, for every type of geometry, we construct examples in which the group of projectivities is transitive on (ordered) n-tuples for every finite n.

(3) Result not yet published.

This negative result indicates how large and unwieldy the group of projectivities can be.

Although our constructions in § 2 and § 3 are dependent on a line [circle] chosen in advance, the properties of the group of projectivities of a line [circle] onto itself are the same for every line [circle] since groups of projectivities on different lines [circles] are isomorphic as permutation groups (see [1]).

0. Basic notions.

In this paper we shall deal with projective and affine planes and with Benz planes. For the definitions of projective and affine planes see [11] or [13]; we work with the definition of a Benz plane given in [7], (12. 3). In a Minkowski plane we shall call two points non-parallel if they are neither plus nor minus parallel.

Geometries which are defined by free extension processes play a very important role in our considerations. For free extensions in the class of projective planes see, e.g. [13]; in the class of affine, Möbius and Laguerre planes see [17]; and for free extensions which lead to Minkowski planes see [9]. For the class of Benz planes, M. Funk [6] unifies the procedure of the preceding authors; we use his steps to unify the extension process.

We shall prove theorems about the group of projectivities of a line or a circle onto itself in all the above geometries. Every projectivity of a line (or a circle) onto a line (or a circle) is defined as a product of perspectivities.

In a projective plane a perspectivity $\alpha = [G, \mathfrak{X}, H]$ of a line G onto a line H is defined in a natural way by the pencil $\mathfrak{X}$ of lines incident with the point p, with $p \notin G, H$ ([13], p. 9). The point p is called the center of α.

In an affine plane an (affine) perspectivity $\alpha = [G, \mathfrak{X}, H]$ of a line G to a line H is defined naturally by the class $\mathfrak{X}$ of parallel lines with $G, H \notin \mathfrak{X}$ ([4] p. 161; [5]): in what follows an affine perspectivity of an affine plane will be simply called a perspectivity.

In a Benz plane we consider perspectivities of the following three types.

A proper perspectivity $\alpha = [K_1, \mathfrak{X}, K_2]$ of a circle K_1 onto a circle K_2 is defined in a natural way by the pencil $\mathfrak{X}$ of circles incident with different non-parallel points p_1 and p_2 (which are called the centers of α) for which $p_i \in K_i$, $p_i \notin K_j$ $(i \neq j)$(see [5] p. 2). The fixed points of α are exactly the points of $K_1 \cap K_2$. We note also that in the case of a Minkowski plane the point on K_1 which is plus parallel (minus parallel) to p_2 is mapped by α onto the point of K_2 which is minus parallel (plus parallel) to p_1 (see [12]).

An affine perspectivity $\alpha = [K_1, \mathfrak{X}, K_2]$ of a circle K_1 onto a circle K_2 with $K_1 \cap K_2 \neq \emptyset$ is defined in a natural way by a pencil $\mathfrak{X}$ of circles which are tangent at a point p (called the center of α) with $p \in K_1 \cap K_2$ and $K_i \notin \mathfrak{X}$ $(i = 1, 2)$. The fixed points of α are the points of $K_1 \cap K_2$.

A Funk perspectivity $\alpha = [K_1, \mathfrak{X}, K_2]$ of a circle K_1 onto a circle K_2 with $K_1 \cap K_2 \neq \emptyset$ is defined by a pencil $\mathfrak{X}$ of circles which are incident with the different non-parallel points p_f and p_s (called the free center and the intersection center) with $p_f \notin K_1, K_2$ and $p_s \in K_1 \cap K_2$, in the following way: If $x \in K_1$ and x is not parallel to p_f, then we consider the circle K incident with x, p_f and p_s, and intersect K with K_2. If $|K \cap K_2| = 2$ we define $\alpha(x)$ to be the point of $K \cap K_2$ which is different from p_s. If $|K \cap K_2| = 1$, then $\alpha(x) = p_s$. If the plane is a Möbius plane α is well defined. If the plane is a Laguerre plane and x is the point on K_1 which is parallel to p_f, then its image under α is the point on K_2 which is parallel to p_f. If the plane is a Minkowski plane and x the plus parallel (minus parallel) point on K_1 to p_f, then $\alpha(x)$ is the point on K_2 which is plus parallel (minus parallel) to p_f. A Funk perspectivity has only one fixed point, namely the intersection of K_1 and K_2 different from p_s if $|K_1 \cap K_2| > 1$, and p_s otherwise. Every Funk perspectivity is a product of two proper perspectivities with the same pencil $\mathfrak{X}$ (see [6]).

Given a line (circle) G in one of the above geometries we can consider all products:

$$\prod_{i=1}^{n} [G_{i-1}, \mathfrak{X}_i, G_i]$$

with $G_0 = G_n = G$. This set of mappings forms a group in a natural way: the group $\Pi = \Pi(G)$ of projectivities of G onto itself (4). This group is invariant for a given geometry since different lines (circles) give isomorphic groups Π (as permutation groups).

A projectivity π can have several representations. The most relevant among the representations of π are the irreducible rep-

(4) We remark explicitly that in the case of an affine plane a projectivity is a product of affine perspectivities.

resentations

$$\pi = \prod_{i=1}^{n} [G_{i-1}, \mathfrak{X}_i, G_i]$$

in which $G_{i-1} \neq G_i$ and $\mathfrak{X}_{i-1} \neq \mathfrak{X}_i$.

In a Benz plane the group Π can be generated by proper and affine projectivities, but there are projectivities in classes of Benz planes (e.g. in free Benz planes, see [6]) in whose *irreducible* representation Funk perspectivities are needed. Therefore if we consider irreducible representations of projectivities in Benz planes we have to work with all three types of perspectivities.

Let

$$(*) \qquad \alpha = \prod_{i=1}^{n} \alpha_i$$

(with $\alpha_i = [G_{i-1}, \mathfrak{X}_i, G_i]$) be an irreducible representation of a projectivity of a line (or a circle) onto itself in one of the above geometries.

Let $\{a_i\}$ $(i = 1, \ldots, m)$ be a set of fixed points of α. With this set $\{a_i\}$ and the representation $(*)$ of α we can associate (following Barlotti [2], p. 137) the configuration $\Omega = \Omega(\{a_i\}, \alpha)$. This configuration consists of the centers of α, of the lines (or circles) G_i (which we shall call generators) of the points $a_i^k = \alpha^k(a_i)$ with $\alpha^k = \prod_{j=1}^{k} \alpha_j$, and of the projection lines (or projection circles) which belong to the pencil $\mathfrak{X}_k$ and which are incident with a_i^{k-1} and a_i^k. Obviously Ω defines on a generator G_k a projectivity $\prod_{h=k+1}^{n} \alpha_h \prod_{j=1}^{k} \alpha_j$ which has the points a_i^k as fixed points.

1. The group of projectivities in free affine planes.

LEMMA (1.1). *Let $\mathfrak{A}$ be a free affine plane and G a line in it. Then there is no projectivity α of G onto itself which has four (distinct) fixed points and an irreducible representation:*

$$\bar{\alpha} = \prod_{i=1}^{n} [G_{i-1}, \mathfrak{X}_i, G_i]$$

with $G_0 = G_n = G$ and $n \geqslant 2$.

PROOF. We can assume that among all the projectivities with the above property the line G and the projectivity α of G onto itself are chosen in such a way that α has an irreducible representation $\bar{\alpha}$ with the shortest «length». Let $\mathfrak{J} = \{a_i^0 : i = 1, \dots, 4\}$ be a set of four distinct fixed points of α and consider the configuration $\Omega = \Omega(\mathfrak{J}, \bar{\alpha})$. Since $\mathfrak{A}$ is a free plane, it follows that Ω is an open configuration. Then we can take away the free lines in Ω from Ω (see [17]), and obtain a new configuration Ω^1. Then from Ω^1 we remove all the points which are free in Ω^1 and obtain Ω^2. Proceeding in the same way, step by step, we reach the empty set. We shall prove that if $|\mathfrak{J}| = 4$ this is not possible.

Clearly for any k the four points a_i^k $(i = 1, \dots, 4)$ are all distinct, and the only free elements of Ω are among the projection lines. Also, in order that the «removal» procedure may continue, there should be free projection lines (A_i^k, A_i^{k+1}) which are incident with only one point $(a_i^k = a_i^{k+1})$ of Ω. By removing such lines we get a structure in which only the points a_i^k (intersections of two generators) can be free. We notice also that if the points a_i^k are free and if $a_i^k = a_i^{k+1} = \dots = a_i^{k+t}$, we have $G_{k+t} = G_k$ if t is even, or $G_{k+t} = G_{k+1}$ if t is odd. If the point a_j^k is different from $G_{k-1} \cap G_k$ and $G_k \cap G_{k+1}$, then the lines A_j^k and A_j^{k+1} are themselves distinct and different from G_k; so the points a_j^k different from $G_{k-1} \cap G_k$ and $G_k \cap G_{k+1}$ cannot be free.

Clearly, working on the configuration Ω we can consider the indices k modulo n. There must exist a k with $a_i^{k-1} \neq a_i^k$ since otherwise we should have only two generators and the configuration $\Omega^1 \setminus \{a_i^k\}$ would be closed since no point $a_j \neq a_i^k$ is free. If $a_i^{k-1} \neq a_i^k$ holds then there exists a smallest j (with $0 \leqslant j < n$) such that $a_i^k = a_i^{k+j} \neq a_i^{k+j+1}$ also holds.

Now the points a_i^k can be free only if the equalities expressed here hold:

i) $G_{k+1} = A_i^k$

and

ii) $G_k = A_i^{k+j+1}$ or $G_{k+1} = A_i^{k+j+1}$ (depending on the parity of j).

If the removal procedure continues we must be able to remove a generator in the next step since in Ω^2 no projection line, which is not at the same time also a generator, can be free. Going from Ω^1 to Ω^2 we must obtain a free generator. But a generator G_k can have at most two free points in Ω^1, namely $G_k \cap G_{k-1}$ and $G_k \cap G_{k+1}$. But if this is the case, then G_k must also be a projection line; and since in Ω^1 every parallel class of projection lines contains at least three elements our process stops before reaching Ω^2. Thus the lemma is proved.

COROLLARY (1.2). *If $\mathfrak{A}$ is a free affine plane, then the irreducible representation of the identity is given only by the empty set.*

COROLLARY (1.3). *Every projectivity has exactly one irreducible representation.*

THEOREM (1.4). *Let $\mathfrak{A}$ be a free affine plane, G a line in it and Π the group of projectivities of G onto itself. The stabilizer in Π of any four distinct points of G consists only of the identity. For every two distinct points of G there exists a third point in G such that the stabilizer of these three points has an element different from the identity.*

PROOF. The first part of the theorem is given by lemma (1.1). The second statement of the theorem will be proved by exhibiting a construction. Let us choose on the line G any two points a and b and one point x not on G. We define, in turn, the following elements in $\mathfrak{A}$. The line G_2 is the line joining a and x; the line A_3 is the join of b and x; the line A_2 is the parallel to G through x; the line A_1 is the parallel to G_2 through b; the point y is equal $A_1 \cap A_2$; the line G_1 is the parallel to A_3 through y; the point z is $G_1 \cap G_2$ and the point c is $G \cap G_1$.

The configuration Ω consisting of $G, G_1, G_2, A_1, A_2, A_3, a, b, c, x, y, z$ is free. The projectivity

$$(^0) \qquad \pi = [G, \mathfrak{X}_1, G_1][G_1, \mathfrak{X}_2, G_2][G_2, \mathfrak{X}_3, G]$$

with

$$G_2, A_1 \in \mathfrak{X}_1\,, \qquad G, A_2 \in \mathfrak{X}_2\,, \qquad G_1, A_3, \in \mathfrak{X}_3$$

leaves b fixed and interchanges a and c. The projectivity π^2 has the three fixed points a, b, c and their irreducible representation is non-trivial. This proves the theorem.

THEOREM (1.5). *The group Π and the stabilizers of one and of two points are free groups of rank $\aleph_0$.*

PROOF. The projective completion of the free affine plane $\mathfrak{A}$ is a free projective plane $\hat{\mathfrak{A}}$ and the group Π can be regarded as a subgroup of the group $\hat{\Pi}$ of projectivities of the line $\hat{G} = G \cup \{\infty\}$ of $\hat{\mathfrak{A}}$.

Since $\hat{\Pi}$ is a free group of rank $\aleph_0$ (see [16]), the group Π is also a free group. The generators of Π are the irreducible words (5) $[G, \mathfrak{L}_1][\mathfrak{L}_1, G_1] \dots [\mathfrak{L}_n, G]$ (with $G_{i-1} \neq G_i$ and $\mathfrak{L}_{i-1} \neq \mathfrak{L}_i$). Since $\mathfrak{A}$ has $\aleph_0$ different directions $\mathfrak{L}_i$ and lines G_i the rank of Π is $\aleph_0$. A stabilizer in Π of one or of two points is contained in a stabilizer in $\hat{\Pi}$ of two or of three points and the assertion follows in a similar way.

THEOREM (1.6). *In a free affine plane $\mathfrak{A}$ no affine projectivity of a line H on a line G which has a non trivial irreducible representation can be induced by automorphisms of $\mathfrak{A}$.*

PROOF. Like the preceding, the proof follows from theorem 5 of [16].

2. Groups of projectivities which are ω-transitive.

DEFINITION. A permutation group G is called ω-transitive on an infinite set M if for every pair of ordered n-subsets S_1 and S_2 of M (where $n \geqslant 1$ is any natural number), there is an element in G which maps S_1 onto S_2.

PROPOSITION (2.1). *Let G be a group of permutations on a set S such that G acts transitively on ordered disjoint n-subsets of S. Then G is n-transitive on S if $|S| \geqslant 3n-1$.*

PROOF. Let $\mathfrak{a} = (a_i)_{i=1}^n$ and $\mathfrak{b} = (b_i)_{i=1}^n$ be two ordered n-subsets with $\mathfrak{a} \cap \mathfrak{b} \neq \emptyset$. Since $|S| \geqslant 3n-1$, there is an n-subset $\mathfrak{c} = (c_i)_{i=1}^n$ which is disjoint from both $\mathfrak{a}$ and $\mathfrak{b}$. Then in G there are elements γ and δ such that $\mathfrak{a}^\gamma = \mathfrak{c}$ and $\mathfrak{c}^\delta = \mathfrak{b}$ and we have $\mathfrak{a}^{\gamma\delta} = \mathfrak{b}$.

THEOREM 2.2. *There exist affine [or projective] planes in which the group Π of affine projectivities [or projectivities] of a line G onto itself is at least t-transitive for $t \geqslant 2$ [$t \geqslant 3$]; there are planes in which the group Π is ω-transitive.*

PROOF. Let $\mathfrak{A}_0$ be an affine incidence structure (see [17]) which consists of a line G and of $n_0 \geqslant 3m_0 - 1 \geqslant 5$ different points on G. Starting from $\mathfrak{A}_0$ we define a suitable extension process which consists of the following steps:

(5) We use here the notation given in Scheiermacher und Strambach [16] and notice that a parallel class $\mathfrak{L}_i$ of $\mathfrak{A}$ is a point in $\hat{\mathfrak{A}}$.

We assume that the affine incidence structure $\mathfrak{A}_{7i}$ has already been defined. On $G \subseteq \mathfrak{A}_{7i}$ there are $n_{7i} \geqslant 3m_{7i} - 1$ points (where m_{7i} can be chosen in any way that satisfies the inequality $m_{7i} \geqslant m_{7(i-1)}$ for $i \geqslant 1$).

Starting from $\mathfrak{A}_{7i}$ we define $\mathfrak{A}_{7i+1}$ in the following way. We divide the n_{7i} points on G in all possible (ordered) m_{7i}-tuples $\mathfrak{c}_k = (c_j^k)_{j=1}^{m_{7i}}$. To every pair $\{\mathfrak{c}_{k_1}, \mathfrak{c}_{k_2}\}$ of disjoint ordered m_{7i}-tuples we adjoin a pair $\mathfrak{L}_{k_1k_2} = (\mathfrak{B}_{k_1}, \mathfrak{B}_{k_2})$ of (distinct) pencils of parallel lines, such that $\mathfrak{B}_{k_t} (t \in \{1, 2\})$ consists exactly of the lines $C_j^{k_t}(\mathfrak{L}_{k_1k_2})$ which are chosen in such a way that the line $C_j^{k_t}(\mathfrak{L}_{k_1k_2})$ is incident with exactly the point $c_j^{k_t}$ and is parallel to only the lines of $\mathfrak{B}_{k_t}$. In this way we obtain $\mathfrak{A}_{7i+1}$.

The affine structure $\mathfrak{A}_{7i+2}$ arises from $\mathfrak{A}_{7i+1}$ by simply adjoining the points of intersections $s_j^{k_1k_2}$ of the lines $C_j^{k_1}(\mathfrak{L}_{k_1k_2})$ and $C_j^{k_2}(\mathfrak{L}_{k_1k_2})$.

Now we obtain $\mathfrak{A}_{7i+3}$ by adjoining, for every pair (k_1, k_2) such that the two ordered m_{7i}-tuples are disjoint, a line $S_{k_1k_2}$ which in this stage is incident exactly with the points of the set $\{s_j^{k_1k_2}\}_{j=1}^{m_{7i}}$. In $\mathfrak{A}_{7i+3}$ there exists for every pair (k_1, k_2) of disjoint ordered m_{7i}-tuples $\{\mathfrak{c}_{k_1}, \mathfrak{c}_{k_2}\}$ an affine projectivity which maps $\mathfrak{c}_{k_1}$ onto $\mathfrak{c}_{k_2}$ (namely the product $[G, \mathfrak{B}_{k_1}, S_{k_1k_2}][S_{k_1k_2}, \mathfrak{B}_{k_2}, G]$). The structure $\mathfrak{A}_{7i+3}$ contains lines which are neither parallel nor intersecting. (For instance $C_j^{k_1}(\mathfrak{L}_{k_1k_2})$ and $C_h^{k_2}(\mathfrak{L}_{k_1k_2})$ for $j \neq h$).

For this reason $\mathfrak{A}_{7i+3}$ is a proper substructure of the structure $\mathfrak{A}_{7i+4}$, which arises from $\mathfrak{A}_{7i+3}$ when we adjoin, for every pair of lines of $\mathfrak{A}_{7i+3}$ which are neither parallel nor intersecting, a point of intersection which is incident only with the two lines used to define it.

The structure $\mathfrak{A}_{7i+5}$ is obtained from $\mathfrak{A}_{7i+4}$ by adding new lines in the following way: for any two points which are not joined by a line in $\mathfrak{A}_{7i+4}$ we define a new line incident in $\mathfrak{A}_{7i+5}$ with exactly these two points.

In $\mathfrak{A}_{7i+5}$ there are pairs (p, A), p a point, A a line, with $p \notin A$. We consider the set of all such pairs, and for any element of this set such that there is no line through p and parallel to A we add a new line to the parallel class of A and incident, in this stage, exactly with p. In this way we get the structure $\mathfrak{A}_{7i+6}$.

The structure $\mathfrak{A}_{7(i+1)}$ is obtained from $\mathfrak{A}_{7i+6}$ by adjoining a point of intersection for every two lines which are neither parallel nor intersecting. Each one of these new points is incident with only the two lines used for its definition. It is clear that in $\mathfrak{A}_{7(i+1)}$ for the number of points belonging to the line G we have the strict inequality:

$$n_7(i + 1) > n_{7i} .$$

In the previous steps an extension process is well defined. Let us consider the incidence structure $\mathfrak{A} = \bigcup_{i=0}^{\infty} \mathfrak{A}_i$. Let us denote by t the $\limsup_{i\to\infty} m_{7i}$. It is very easy to prove that $\mathfrak{A}$ is an affine plane in which the group Π of affine projectivities acts in such a way that for any two ordered t-tuples of points, if t is finite, (or for any two finite m-tuples if $t = \aleph_0$) there is in Π an element mapping one of these tuples on the other. The theorem for the affine case follows now from proposition (2.1).

Consider now the projective completion $\hat{\mathfrak{A}}$ of $\mathfrak{A}$ and the group $\hat{\Pi}$ of projectivities of the projective line $\hat{G} = G \cup \{p_\infty\}$. Since p_∞ is not a fixed point of $\hat{\Pi}$, the degree of transitivity of $\hat{\Pi}$ is not less than the degree of transitivity of Π. Thus the theorem holds also in the projective plane.

THEOREM 2.3. *There are Benz planes in which the group Π of projectivities of a circle K onto itself is at least t-transitive, for $t \geqslant 3$. There are Benz planes in which the group Π is ω-transitive.*

PROOF. Let $\mathfrak{A}_0$ be a circle structure which consists of a circle K and $n_0 + 1 \geqslant 3m_0 \geqslant 6$ distinct points on K none of which are parallel. Among these points we select one point, which we denote by ∞. Starting from $\mathfrak{A}_0$ we define a suitable extension process which consists of the following steps:

We assume that the circle structure $\mathfrak{A}_{si}$ has already been defined where $s = 7, 8, 9$ according to whether we construct Möbius, Laguerre or Minkowski planes. On $K \subseteq \mathfrak{A}_{si}$ there are $n_{si} + 1 \geqslant 3m_{si}$ points (where m_{si} can be chosen in any way that satisfies the inequality $m_{si} \geqslant m_{s(i-1)}$ for $i \geqslant 1$).

Starting from $\mathfrak{A}_{si}$ we define A_{si+1} in the following way. We divide the n_{si} points on K different from ∞ into all possible ordered m_{si}-tuples $\mathfrak{c}_k = (c_j^k)_{j=1}^{m_{si}}$. To every pair $\{\mathfrak{c}_{k_1}, \mathfrak{c}_{k_2}\}$ of disjoint ordered m_{si}-tuples we adjoin a pair $\mathfrak{L}_{k_1k_2} = (\mathfrak{B}_{k_1}, \mathfrak{B}_{k_2})$ of distinct pencils of tangent circles: The pencil $\mathfrak{B}_{k_t}$ $(t \in \{1, 2\})$ consists precisely of the circles $C_j^{k_t}(\mathfrak{L}_{k_1k_2})$ such that the circle $C_j^{k_t}(\mathfrak{L}_{k_1k_2})$ is incident with exactly the points $c_j^{k_t}$ and ∞, touches the circles of $\mathfrak{B}_{k_t}$ at ∞, and in this stage is tangent to no other circle. Therefore the circles $C_j^{k_1}$ and $C_j^{k_2}$ in this stage have only the point ∞ in common and are not tangent. In this way we obtain $\mathfrak{A}_{si+1}$.

The circle structure $\mathfrak{A}_{si+2}$ arises from $\mathfrak{A}_{si+1}$ by simply adjoining the points of intersections $s_j^{k_1k_2}$ of the circles $C_j^{k_1}(\mathfrak{L}_{k_1k_2})$ and $C_j^{k_2}(\mathfrak{L}_{k_1k_2})$. Each point $s_j^{k_1k_2}$ in this stage is incident only with the two circles $C_j^{k_1}(\mathfrak{L}_{k_1k_2})$ and $C_j^{k_2}(\mathfrak{L}_{k_1k_2})$ and is parallel to no other point.

Now we obtain $\mathfrak{A}_{si+3}$ by adjoining for every pair (k_1, k_2) such that the two m_{si}-tuples are ordered and disjoint, a circle $S_{k_1k_2}$ which in this stage is incident exactly with the points of the set $\{s_j^{k_1k_2}\}_{j=1}^{m_{si}} \cup \{\infty\}$ and is non-tangent to the other circles. In $\mathfrak{A}_{si+3}$ there exists a projectivity for every pair (k_1, k_2) of disjoint ordered m_{si}-tuples $\{\mathfrak{c}_{k_1}, \mathfrak{c}_{k_2}\}$ which maps $\mathfrak{c}_{k_1}$ onto $\mathfrak{c}_{k_2}$ (namely the product $[K, \mathfrak{B}_{k_1}, S_{k_1k_2}][S_{k_1k_2}, \mathfrak{B}_{k_2}, K]$; each of these two perspectivities is well defined since $K, S_{k_1k_2} \notin \mathfrak{B}_{k_t}$). The structure $\mathfrak{A}_{si+3}$ contains in this stage some circles which have only ∞ in common and are not tangent (for instance $C_j^{k_1}(\mathfrak{L}_{k_1k_2})$ and $C_h^{k_2}(\mathfrak{L}_{k_1k_2})$ for $j \neq h$).

For this reason $\mathfrak{A}_{si+3}$ is a proper substructure of the structure $\mathfrak{A}_{si+4}$ which arises from $\mathfrak{A}_{si+3}$ when we adjoin a new point of intersection for every pair of circles of $\mathfrak{A}_{si+3}$ which have only ∞ in common but are not tangent; the new point is incident only with the two circles used to define it and is not parallel to other points.

The structure $\mathfrak{A}_{si+5}$ is obtained from $\mathfrak{A}_{si+4}$ if we add a circle for every three distinct non-parallel points of $\mathfrak{A}_{si+4}$, which are not incident with a circle; the new circle is incident only with these three points and is not tangent with any other circle in $\mathfrak{A}_{si+5}$.

In $\mathfrak{A}_{si+5}$ there are triples (p, q, A) p, q points, A a circle, with $p \notin A$, $q \in A$ and p not parallel to q. We consider the set of all such triples, and for any element of this set such that there is no circle through p tangent to A at q, we add a new circle belonging to the tangent pencil determined by q and A and which has no other tangency in this stage. In this way we get the structure $\mathfrak{A}_{si+6}$.

The structure $\mathfrak{A}_{si+7}$ arises from $\mathfrak{A}_{si+6}$ in the following way. If L_1, L_2 are distinct circles out of $\mathfrak{A}_{si+6}$ which are not tangent with $|L_1 \cap L_2| = 1$, we add a second point of intersection incident only with L_1 and L_2. To two circles which are not tangent and disjoint we can add, as we like, either zero or two different points of intersection which are only incident with these two circles. In this stage these new points are not parallel to any other point. If the plane we are constructing is a Möbius plane, then in the previous steps an extension process is well defined.

Otherwise from $\mathfrak{A}_{si+7}$ we get the circle structure $\mathfrak{A}_{si+8}$ by considering the equivalence classes of parallel points: If $\mathcal{A}$ is such a parallel class and L is a circle not containing a point of $\mathcal{A}$, then we add a new point which belongs to $\mathcal{A}$ (and to no other parallel class) and which is incident exactly with L. If the geometry we are constructing is a Laguerre plane in the previous steps an extension process is well defined. Otherwise from $\mathfrak{A}_{si+8}$ we get the circle structure $\mathfrak{A}_{si+9}$ in the following way.

If p and q are two distinct non-parallel points of $\mathfrak{A}_{si+8}$ for which one of the two points r, s (or both of them) in $\mathfrak{A}_{si+8}$ for which $p||_+r$ and $r||_-q$ or $p||_-s$ and $s||_+q$ does not exist, we add the missing point (or points) with the prescribed parallelism. These new points in this stage are not parallel with any other point and are not incident with any circle.

Now in the previous steps an extension process is also well defined for Minkowski planes. In $\mathfrak{A}_{s(i+1)}$ we have (by virtue of step $si+6$) the strict inequality for the number of points belonging to the circle K:

$$n_{s(i+1)} > n_{si} .$$

Let us consider the incidence structure $\mathfrak{A} = \bigcup_{i=0}^{n} \mathfrak{A}_i$, and denote by $t-1$ the $\limsup_{i\to\infty} m_{si}$. It is very easy to prove that $\mathfrak{A}$ is a Möbius, Laguerre or Minkowski plane according as $s = 7, 8$ or 9. Now let Π_∞ be the stabilizer of the group of projectivities of K onto itself. Because in every $\mathfrak{A}_{si+3}$ the projectivities have ∞ as a fixed point, Π_∞ acts on $K\setminus\{\infty\} = K'$ and so Π_∞ is $(t-1)$-transitive on K' if t is finite or ω-transitive on K' if $t = \aleph_0$. Since ∞ is no fixed point of Π it follows that Π acts t-transitively on K and the theorem is proved.

Remark (2.4). *The affine perspectivities of a circle K in a Benz plane generate in general a proper subgroup Ψ of the group Π of all projectivities of K onto itself.*

Proof. Funk [6] showed that the stabilizer in Ψ of four points in a free Benz plane consists of the identity only, whereas the stabilizer in the whole group Π of projectivities of every four distinct points is always different from the identity.

Remark (2.5). *There are Benz planes in which the group Ψ is at least k-transitive for $k\geqslant 3$. There are Benz planes in which Ψ is ω-transitive.*

Proof. The property follows from the fact that the projectivities used in the proof of theorem (2.3) are always affine projectivities.

3. Groups of projectivities with the identity as stabilizer of $n+1$ points.

THEOREM 3.1. *For every* $n \geqslant 5$ [*or* $n \geqslant 3$] *there exists a projective* [*affine*] *plane* $\mathfrak{E}$, *such that the group of projectivities* [*affine projectivities*] *of a line* [*affine line*] G *has the following properties*:

The stabilizer in Π *of* $n+1$ *distinct points consists of the identity only, but there are* n *different points* a_i, $i = 1, \dots, n$, *on* G *such that the stabilizer* $\Pi_{a_1 \dots a_n}$ *contains elements different from the identity.*

PROOF. First we shall prove the theorem for the projective case. If $n = 5$, we take a free projective plane for $\mathfrak{E}$, and the theorem holds. (see [16]). Let us assume $n \geqslant 6$. We denote by $\mathfrak{A}_0$ the following configuration. The points of $\mathfrak{A}_0$ are:

$$\{a_i\}_{i=1}^n, \qquad \{a_i^1\}_{i=1}^n, \qquad \{a_i^2\}_{i=1}^n, \qquad s_1, s_2, s_3;$$

the lines are:

$$G, H, K, S, \qquad \{A_i^1\}_{i=3}^n, \qquad \{A_i^2\}_{i=3}^n, \qquad \{A_i^3\}_{i=3}^n;$$

and the incidences are:

$$a_i \in G \text{ for all } i, \quad a_i^1 \in K \text{ for all } i, \quad a_i^2 \in H \text{ for all } i, \quad a_2 = a_2^1 = a_2^2;$$

$$a_1, a_1^1, a_1^2, s_1, s_2, s_3 \in S;$$

$$a_j, a_j^1, s_1 \in A_j^1 \quad \text{for } j = 3, \dots, n; \qquad a_j^1, a_j^2, s_2 \in A_j^2 \quad \text{for } j = 3, \dots, n;$$

$$a_j, a_j^2, s_3 \in A_j^3 \quad \text{for } j = 3, \dots, n.$$

The free extension of $\mathfrak{A}_0$ (see e.g. [8], [13]) is a projective plane $\mathfrak{E}$. Each closed configuration in $\mathfrak{E}$ is contained in the closed subconfiguration $\mathfrak{A}_0'$ of $\mathfrak{A}_0$ which arises from $\mathfrak{A}_0$ by deleting the points a_1, a_1^1, a_1^2. We consider the projectivity $\alpha = [G, s_1, K][K, s_2, H][H, s_3, G]$ and can easily check that α has $a_1, \dots, a_n$ as fixed points. Since every fixed point $f \neq a_i$ of α belongs to a closed subconfiguration of $\mathfrak{E}$, the point f is contained in $\mathfrak{A}_0' \cap G$ and so it is one of the points $a_2, \dots, a_n$. Therefore $\alpha \neq 1$ and it has exactly n fixed points.

Now let τ be a projectivity with $n+1 \geqslant 7$ distinct fixed points $x_1, \ldots, x_{n+1}$ and with an irreducible representation

$$\tau = [G, z_1, G_1][G_1, z_2, G_2] \ldots [G_{m-1}, z_m, G] .$$

If we assume $\tau \neq 1$ it follows that $m \geqslant 3$.

With the set $\mathfrak{S}$ of the fixed points of τ we associate the configuration $\Omega = \Omega(\mathfrak{S}, \tau)$.

Let Ω_0 be the largest closed subconfiguration which is contained in Ω. Ω_0 contains all centers z_i and all lines G_i for otherwise Ω would be open and the number of fixed points would be at most five (see [2]). Every line G_i contains at least $|\mathfrak{S}| - 3 \geqslant n + 1 - 3 \geqslant 4$ different points of Ω_0; in fact, going from Ω to Ω_0 we can drop at most 3 different points onto G_i (that is, at most the points $G_i \cap G_{i-1}$, $G_i \cap G_{i+1}$ and the intersection of G_i with the line joining z_i and z_{i+1}). Therefore every center z_i is incident with at least $|\mathfrak{S}| - 3 \geqslant 4$ different lines of Ω. The configuration Ω_0 is a closed subconfiguration of $\mathfrak{A}'_0$. Since in $\mathfrak{A}'_0$ only the points s_1, s_2, s_3 are incident with more than three different lines, the centers of τ must be chosen from the set of these three points. Since in $\mathfrak{A}'_0$ only the lines G, H and K are incident with more than 3 different points, the lines G_i of τ must be chosen in the set of these three lines. Since in $\mathfrak{A}'_0$ there is no line joining s_2 with a_j ($j \geqslant 2$) the center z_1 must be s_1 or s_3. But in $\mathfrak{A}'_0$ the lines A_j^1 (or A_j^3 have no points of intersection with H or K respectively and this implies that the line G_i of τ must be K or H respectively. Therefore $\tau = \alpha^{\pm 1}$, and we have a contradiction since $\alpha^{\pm 1}$ has exactly n fixed points.

If we delete from $\mathfrak{E}$ the line S and all its points we get an affine plane. Let Γ be the group of affine projectivities of the affine line $G' = G \setminus \{a_1\}$. The stabilizer in G of n different points is always the identity since $\Gamma \subseteq \Pi_{a_1}$. Otherwise α induces an affine projectivity on G' with exactly $n-1$ fixed points and the theorem holds also for affine planes if $n \geqslant 4$. But if $n = 3$ we take a free affine plane, and the theorem holds (see (1.4)).

THEOREM 3.2. *For every $n \geqslant 5$ there are Möbius, Laguerre and Minkowski planes* K, *such that the group of projectivities Π of a circle K onto itself has the following properties: the stabilizer of Π on $n+1$ different points consists of the identity only but there are n distinct points a_i, $i = 1, \ldots, n$, such that the stabilizer $\Pi_{a_1 \ldots a_n}$ contains elements different from the identity.*

PROOF. In case $n = 5$ we take for $\mathbf{K}$ a free Benz plane, and then the theorem holds (see [6]).

Let us assume $n \geqslant 6$. We denote by $\mathfrak{A}_0$ the following configuration:

the points of $\mathfrak{A}_0$ are $\{a_i\}_{i=1}^n$, $\{a_i^1\}_{i=3}^n$, $\{a_i^2\}_{i=3}^n$;

the circles are G, H, K, $\{A_i^1\}_{i=3}^n$, $\{A_i^2\}_{i=3}^n$, $\{A_i^3\}_{i=3}^n$;

and the incidences are:

$$a_i \in G \text{ for all } i,\ a_1 \in H, K, A_i^k \text{ for all possible } i, k;$$

$$a_2 \in H, K; \quad a_i^1 \in K \text{ for all } i \geqslant 3; \quad a_i^2 \in H \text{ for all } i \geqslant 3;$$

$$a_j, a_j^1 \in A_j^1 \text{ for } j \geqslant 3; \quad a_j^1, a_j^2 \in A_j^2 \text{ for } j \geqslant 3; \quad a_j, a_j^2 \in A_j^3 \text{ for } j \geqslant 3\ .$$

Moreover, no two different points of $\mathfrak{A}_0$ are parallel, and the circles A_j^k and A_l^r are tangent in $\mathfrak{A}_0$ at a_1 exactly when $k = r$; in $\mathfrak{A}_0$ no other tangency among circles exists.

The free extension of $\mathfrak{A}_0$ is a Benz plane (see [17], [9], [6]). Every closed configuration of $\mathbf{K}$ in which no hyperfree element exists is contained in $\mathfrak{A}_0$ (see [16], [6]). Note that $\mathfrak{A}_0$ is the maximal closed configuration of $\mathbf{K}$ because the circles G, K, H contain more than 3 points; each of the circles A_j^k has 3 distinct points but each is tangent with some circle because every point is incident with at least 3 distinct circles. We consider the projectivity $\gamma = [G, \mathfrak{X}_1, K][K, \mathfrak{X}_2, H][H, \mathfrak{X}_3, G]$ where $\mathfrak{X}_i$ denotes the tangent pencil determined by a_1 and the circles $\{A_j^i\}_{j=3}^n$. We can easily check that γ has $a_1, a_2, \ldots, a_n$ as fixed points. Since every fixed point f of γ belongs to a closed subconfiguration of $\mathbf{K}$, the point f is contained in $\mathfrak{A} \cap G$ and so it is one of the points $a_1, \ldots a_n$. Therefore $\gamma \neq 1$ and has exactly n fixed points.

Now let τ be a projectivity with $n + 1 \geqslant 7$ distinct fixed points $a_1, \ldots., a_{n+1}$ and with an irreducible representation

$$\tau = [G, \mathfrak{X}_1, G_1][G_1, \mathfrak{X}_2, G_2] \ldots [G_{m-1}, \mathfrak{X}_m, G]$$

where $\mathfrak{X}_i$ denotes a pencil of circles belonging to a proper, affine of Funk perspectivity from G_{i-1} to G_i. The irreducibility of the representation of τ means that $G_{i-1} \neq G_i$ and $\mathfrak{X}_{i-1} \neq \mathfrak{X}_i$ for every i (see [6]).

If we assume $\tau \neq 1$, it follows that $m \geqslant 2$. With the set $\mathfrak{S}$ of fixed points of τ we associate the configuration $\Omega = \Omega(\mathfrak{S}, \tau)$.

Let Ω' be the largest closed subconfiguration which is contained in Ω.

We describe now Funk's method which we follow to go from Ω to Ω'. The procedure is completed in 3 steps. First we note that the following number of circles in Ω is incident with a center of the perspectivity $\alpha_i = [G_{i-1}, \mathfrak{X}_i, G_i]$:

if α_i is a proper perspectivity, then a center p_j^i is incident with at least 6 different circles, and 5 of them are projection circles (6);

if α_i is an affine perspectivity, then the center p_i is incident with at least 8 different circles and at least 6 of them are projection circles;

if α_i is a Funk perspectivity then the free center p_f^i is incident with at least 5 different circles which are projection circles; the intersection center p_s^i is incident with at least 7 different circles, of which at least 5 are projection circles.

In the first step of the construction we remove all projection circles which are free; they are exactly those which are incident with the fixed points of α_i. In such way we obtain Ω_1.

If α_1 is a proper perspectivity, then a center p_j^i is incident with at least 4 circles in Ω_1 of which at least 3 are projection circles. In the case that α_1 is affine, the center p_i is incident with at least 7 different circles, and at least 5 of these are projection circles. If α_i is a Funk perspectivity, the center p_f^i is incident with at least 4 different circles which are projection circles, and p_s^i is incident with at least 6 different circles, where at least 4 of them are projection circles.

The structure Ω_2 arises from Ω_1 by removing the free points; such points can only be images a_j^i of the fixed points a_j of τ under $\prod_{k=1}^{i} \alpha_k$. On G_i there are at most two free fixed points: either Ω has only two generator circles, and the two points are fixed points under all α_k $(k = 1, \ldots, n)$, or one of the two points is a fixed point under α_i and the other is a fixed point under α_{i+1}; Funk proves ([6] lemmas 3.1 to 3.3) that beside these points at most two other points on G_i can be free in Ω_1.

One point h on G_i other than the fixed points may be free since a projection circle of α_i can be also a projection circle of α_{i-1} (or α_{i+1}). Another kind of free points (of which there turns out to be at most two) can arise if the generator G_i is the only circle through them, and they are parallel to only one center. The two circumstances above cannot both be present at the same time.

Funk proves (main theorem) that we now get the closed structure Ω' from Ω_2 if we remove free circles in Ω_2. In Ω_2 every circle G_i contains

(6) In this counting of the number of circles (and the same for those countings which follows) we must take care of the fact that through a center and a point parallel to the center there is no projection circle. Also if a center of an affine projectivity is a point $\alpha^k(a_i)$ then there is no projection circle through it.

at least 3 distinct points a_j^i; the circle G_i contains also a center p_i of α_i. If p_i is one of the points a_j^i, then we have tangency at p_i between G_i and the projection circles A_j^i. Therefore no generator in Ω_2 can be free. The only free circles in Ω_2 are projection circles.

Since Ω_1 contains no free projection circles, every free projection circle of Ω_2 carries a free point of Ω_1, and this free point h is not a fixed point of some α_i, and the point h is then (by lemmas 3.1 and 3.3 of [6]) the only free point on G_i beside the two free fixed points mentioned above. If there are points in Ω_1 on G_i which are parallel to one of the centers of α_i, then there exists no free point h, through which a projection line passes ([6], lemma (3.3)). Then no circle through the centers of α_i can become free in Ω_2. Therefore every center of α_i is incident in Ω_3 with at least 4 different circles.

The same result holds if there does not exist a free projection circle in Ω_2 through the centers.

If there do not exist points in Ω_1 on G_i parallel to one of the centers of α_i, but there is the free point h, then there pass at least 5 circles through the centers of α_i.

In every case any center of each perspectivity α_i is incident with at least 4 different circles of Ω_3.

Funk has proved in his main theorem that the structure Ω_3 is the closed subconfiguration Ω'. Therefore Ω_3 is a subconfiguration of $\mathfrak{A}_0$ and the parallel classes of every point of Ω' are always trivial.

Since a_1 is the only point of $\mathfrak{A}_0$ which is incident with 4 different circles, the point a_1 must be the center of every perspectivity α_1. Therefore τ is a product of affine perspectivities, each of them having the center a_1. Then every point of Ω, different from a_1 is not parallel to a_1, therefore $|\Omega' \cap G_i| \geqslant 4$ and only G, H and K can appear as generators G_i: Since in Ω' there are no points of intersection between G and A_j^2 or H and A_j^1, or K and A_j^3, τ must equal $\gamma^{\pm 1}$.

Remark (3.3). *For every $n \geqslant 5$ there are Möbius, Laguerre and Minkowski planes $\mathbb{K}$ such that the group Ψ of projectivities generated by affine perspectivities of a circle K onto itself (compare remark (2.4)) has the following properties: the stabilizer in Π of $n+1$ distinct points consists of the identity only, but there are n different points such that the stabilizer in Ψ of these points is different from the identity.*

Proof. The property follows from the fact that the projectivities used in the proof of theorem (3.2) are always affine projectivities.

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