RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

LUIGI AMERIO

Continuous solutions of the problem of a string vibrating against an obstacle

Rendiconti del Seminario Matematico della Università di Padova, tome 59 (1978), p. 67-96

<http://www.numdam.org/item?id=RSMUP_1978__59__67_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 59 (1978)

Continuous Solutions of the Problem of a String Vibrating against an Obstacle.

LUIGI AMERIO (*)

RIASSUNTO - Vengono generalizzati i risultati di un precedente lavoro, di Amerio e Prouse, sullo stesso argomento. La soluzione del problema si ottiene ora con sole ipotesi di continuità sui dati, sulla base di un'analisi del supporto della reazione vincolare (distribuzione) e di un'ampliata formulazione della legge di urto elastico.

1. Introduction.

a) The present paper is a generalisation of a preceding one, by Amerio and Prouse [1], which concerns the motion of a string, not subject to any external force and vibrating against a rigid wall, parallel to the position of the same string, at rest.

The *lines of influence* of the wall play, as in [1], an essential role in the study of the motion. However, they are now defined under much wider hypotheses on the *data* than at [1], on the basis, essentially, of the properties of *the set of impact points*; this set coincides with the *support of a distribution* J: the *reaction of the obstacle.*

Moreover, the *elastic impact law* is now formulated in such a way (a weak form of that of *Mechanics*) to allow a wider mathematical utilisation (see (4.11)).

The solution of the problem of the wall *exists and is unique* (in cor-

(*) Indirizzo dell'A.: Istituto Matematico del Politecnico di Milano.

respondance to given boundary und initial values): if, moreover, the initial energy is *finite*, then, $\forall$ time t, the *energy equality* holds.

b) Assume that the string, at rest, is placed on the x axis. The *displacement* $y(x, t) = y(P)$, in the (x, y) plane, satisfies, in the *free* motion, the *homogeneous vibrating string equation* (in the sense of *distributions*):

$$y_{tt} - y_{xx} = 0 \tag{1.1}$$

or, in the *characteristic form*,

$$y_{\xi\eta} = 0 , \tag{1.2}$$

where $\xi \doteq (x + t)2^{-\frac{1}{2}}$, $\eta = (-x + t)2^{-\frac{1}{2}}$ are the characteristic co-ordinates.

Assume now that the free motion of the string is impeded by a rigid wall $y = 0$, that obliges the string to move in the half plane $y \geqslant 0$. By introducing the (*unknown*) *reaction J of the obstacle*, the displacement $y(P)$ satisfies therefore the *non homogeneous vibrating string equation*:

$$y_{\xi\eta} = J . \tag{1.3}$$

We set now the *initial* and *boundary conditions* (in a slightly different form from the usual). Consider, in the (x, t) plane, a domain Z, which is bounded (inferiorly, to the left and to the right) by three lines $(\sigma_0, \sigma', \sigma'')$ of equations:

$$\begin{aligned} &\sigma_0) \quad t = \tau(x) \qquad (a' \leqslant x \leqslant a'') , \\ &\sigma') \quad x = p(t) \qquad (t \geqslant t') , \\ &\sigma'') \quad x = q(t) \qquad (t \geqslant t'') , \end{aligned} \tag{1.4}$$

where $p(t') = a' < q(t'') = a''$.

Assume, moreover, that σ_0 is constituted by a *finite number of characteristic segments*, and that $|p'(t)| \leqslant 1$, $|q'(t)| \leqslant 1$, never being $p'(t) = \pm 1$, or $q'(t) = \pm 1$, on an interval: therefore we exclude that σ' and σ'' contain any characteristic interval. It is $\sigma' \cap \sigma'' = \emptyset$, and we assume, lastly, that the intersection of Z with an arbitrary characteristic straight line r, if $\neq \emptyset$, consists in a segment of *finite* length.

Let now $A(P)$, $P \in \partial Z$, be an arbitrary *continuous* function.

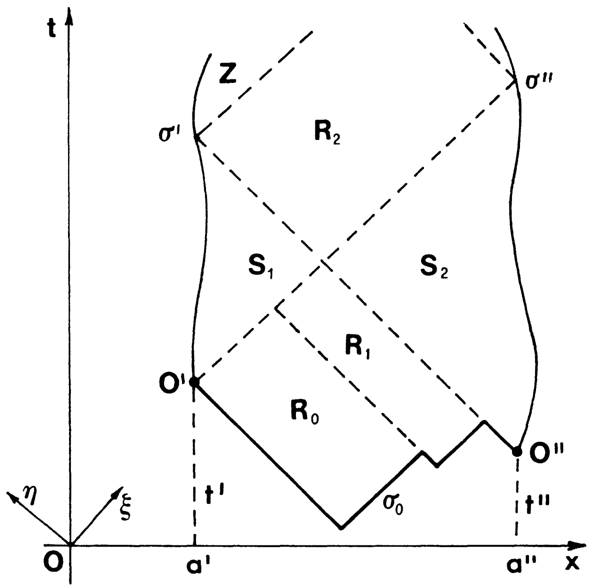

Figure 1

Consider, firstly, the following *problem*: *Find a solution of* (1.2) *such that*

(1.5) $$y(P) \in C^0(Z)\,, \qquad y|_{\partial Z} = A(P)\,.$$

The *restrictions*

$$A|_{\sigma_0}\,, \qquad A|_{\sigma' \cup \sigma''}$$

define, respectively, the *initial* and the *boundary conditions* for the integral $u(P)$.

It is obvious that the given problem has *one and only one solution* ($\in \mathfrak{D}'(\mathring{Z})$), $y(P)$; we may calculate it, by a classical scheme, solving successively Darboux and Goursat problems, in the domains R_0, R_1, S_1, S_2, R_2, ...

Let us consider now the analogous problem, when the motion of the string is *constrained* by the *wall*; we have now to integrate equation (1.3), adding to the boundary condition (1.5) the *unilateral* condition:

$$y(P) \geqslant 0 , \quad \forall P \in Z \quad (\Rightarrow A(P) \geqslant 0) , \tag{1.6}$$

and indicating the *nature of the impact* (elastic, partially elastic, anelastic).

As we shall prove, also *this problem admits one and only one solution, if we assume that*

$$A(P) > 0 \quad \forall P \in \partial Z . \tag{1.7}$$

2. Admissible functions and impact points.

Let us define the functional class Y of the *admissible* functions, to which all solutions $y(P)$ of the problem of the wall must belong, *whatever* is the nature of the impact.

We shall say that $y(P)$ is an *admissible function*, if the following conditions are fulfilled:

I) $y(P) \in C^0(Z)$,

II) $y(P) \geqslant 0 \;\; \forall P \in Z$,

III) $\Gamma = \operatorname{supp} y_{\xi\eta} \subseteq \{P \in Z \colon y(P) = 0\}$

IV) $y_{\xi\eta} \geqslant 0$,

V) (*extension law*, with reference to the « elementary problems » of Darboux and Goursat):

1) *let* $R = ABCD$ *be a characterist rectangle*, $\subseteq Z$ *and of minimum vertex* A. Let moreover $z(P)$ be the solution, in R, of (1.2) satisfying the *Darboux condition*

$$z|_\sigma = y|_\sigma .$$

Then, *if on the whole of* R

$$z(P) \geqslant 0 ,$$

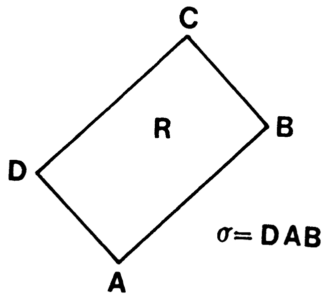

Figure 2

it is also, on the whole of R,

$$y(P) = z(P);$$

2) *let S be a domain* $\subset Z$, bounded to the left by an *arc* $\widehat{AC} \subset \sigma'$ and to the right by two *characteristic segments* issuing from a point B

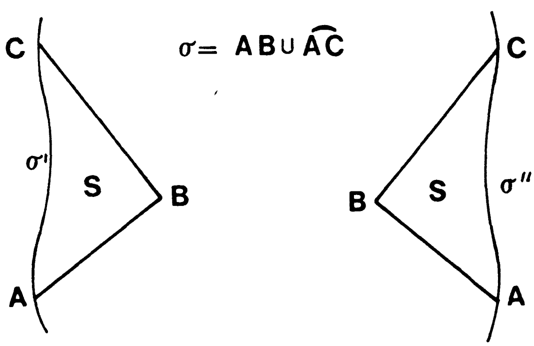

Figure 3

(and analogously for σ''). Let moreover $z(P)$ be the solution, in S, of (1.2) satisfying the *Goursat condition*

$$z|_\sigma = y|_\sigma .$$

Then, *if on the whole of S*

$$z(P) \geqslant 0 ,$$

it is also, on the whole of S,

$$y(P) = z(P) .$$

Observe that condition V) imposes, essentially, that *an admissible function* $y(P)$ *must satisfy the homogeneous vibrating string equation « wherever possible »*: in particular, this occurs on the open set where the string does not touch the wall.

We remark, in comparison with [1], that no reference has been made now to the Cauchy problem.

We shall call *impact points* the points of the set Γ.

Let us indicate some *properties of the admissible functions.*

a) We have $\Gamma \cap \partial Z = \emptyset$; moreover, *on* $\mathring{Z} - \Gamma$, $y(P)$ *satisfies the homogeneous equation* $y_{\xi\eta} = 0$.

b) Let $R = ABCD$ be a characteristic rectangle $\subset Z$ (fig. 2). *Setting*

$$\Delta_R y = y(C) - y(B) - y(D) + y(A) ,$$

it follows from I) and IV)

$$\Delta_R y > 0 \quad \textit{if} \quad \Gamma \cap \mathring{R} \neq \emptyset , \qquad \Delta_R y = 0 \quad \textit{if} \quad \Gamma \cap \mathring{R} = \emptyset . \tag{2.1}$$

c) Γ *is a perfect set.* Assume infact that N is an isolated point of Γ. There exists then, by (2.1), a characteristic square $R_\delta \subset Z$, with center N and edge δ $(0 < \delta \leqslant \delta_0)$, such that it is

$$\Delta_{R\delta} y = \varrho > 0 , \quad \text{independent on } \delta .$$

This is absurd, since, by I),

$$\lim_{\delta \to 0} \Delta_{R\delta} y = 0 .$$

d) Taken $N \in \mathring{Z}$, let $A_1 A_2$ and $B_1 B_2$ be the maximum characteristic segments issuing from N and $\subset Z$.

They determine the sets R_N^+, R_N^-, S_N^+, S_N^- (see fig. 4). In particular, R_N^+ is the maximum characteristic rectangle $\subset Z$ and with N as minimum vertex.

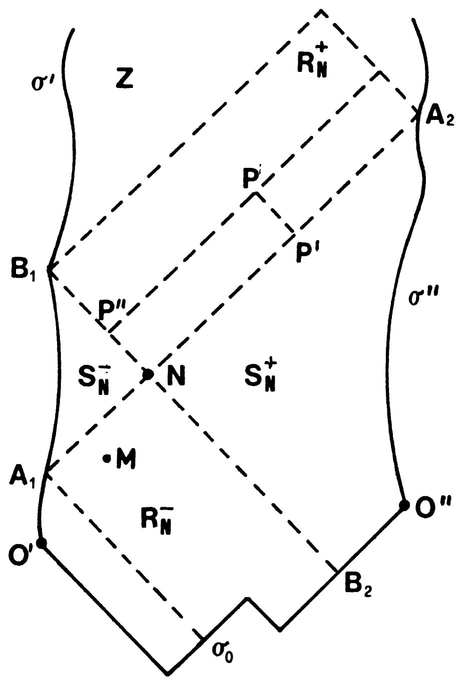

Figure 4

Then:

$$N \in \Gamma \Rightarrow \Gamma \cap (\mathring{R}_N^+ \cup \mathring{R}_N^-) = \emptyset .$$

This property follows from the extension law. Let in fact $z(P)$ be the solution, on R_N^+, of equation (1.2), with the Darboux condition $z|_\sigma = y|_\sigma$, $\sigma = B_1 N A_2$. Since $y(N) = 0$, we have, $\forall P \in R_N^+$, $z(P) = y(P') + y(P'') \geqslant 0$, $\Rightarrow y(P) = z(P)$ on the whole of R, $\Rightarrow y_{\xi\eta} = 0$ on the open set $\mathring{R}_N^+$. It follows $\Gamma \cap \mathring{R}_N^+ = \emptyset$.

Assume now that there exists an impact point $M \in \mathring{R}_N^-$. Then (as $N \in \mathring{R}_M^+$) $N \notin \Gamma$, which is absurd.

e) Let r be an arbitrary ***characteristic straight line. There exist then, on r, two impact points, at most.*** Assume r to be a ξ characteristic; let morevor A and B, $\xi_A < \xi_B$, be two impact points, $\in r$. No impact point $\in \breve{R}_A^+ \cup \mathring{R}_B^-$: therefore the only impact points $\in R_\delta = UVCD$, are placed on the segment AB. If $\Gamma \cap (AB)^0 \neq \emptyset$, we have then

$$\Delta_{R\delta} = \varrho > 0 \,, \quad \text{independent on } \delta \,,$$

which is absurd, by I).

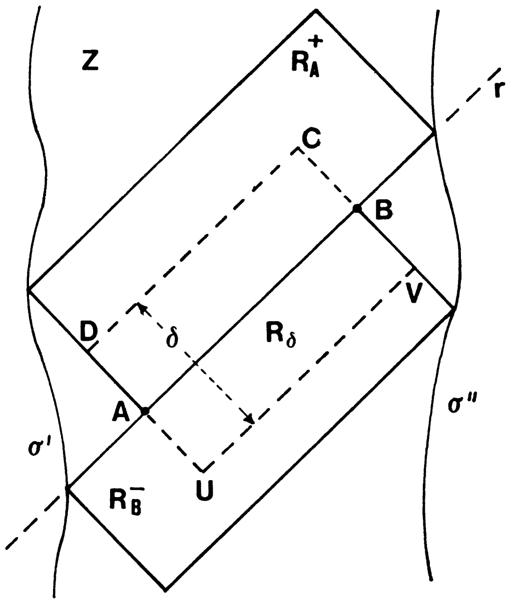

Figure 5

The thesis follows. Hence, *$N \in \Gamma$ implies that N is a limit point for the set* $(\mathring{S}_N^+ \cup \mathring{S}_N^-) \cap \Gamma$.

f) Let $R = ABCD$ be a characteristic rectangle $\subset Z$ (see fig. 5). Then, *if B and $D \in \Gamma$, neither A nor $C \in \Gamma$.* Assume infact, for instance, that $C \in \Gamma$. We have

$$\Gamma \cap \mathring{R}_B^+ = \Gamma \cap \mathring{R}_D^+ = \Gamma \cap \mathring{R}_C^+ = \Gamma \cap \mathring{R}_C^- = \emptyset \,,$$

and the characteristics through C contain the only impact points D and B. The point C would therefore be an isolated point of Γ, contrary to *c*).

g) Let $A \in \Gamma$ be a limit point *both* for $\Gamma \cap S_A^+$ and $\Gamma \cap S_A^-$. Then *no other impact point is placed on the characteristics through* A. Let infact B be a second impact point (see fig. 5). Since $\Gamma \cap \mathring{R}_B^- = \emptyset$, A cannot then be a limit point of $\Gamma \cap S_A^+$.

h) *Setting*

$$\mu_0 = \inf_\Gamma \{\xi\} , \tag{2.2}$$

there exists one and only one point $M_0(\xi_0, \eta_0) \in \Gamma$ *such that*

$$\xi_0 = \mu_0 . \tag{2.3}$$

The existence is obvious, since Γ is closed and inferiorly bounded. In order to prove the uniqueness, assume that there are $M_0(\mu_0, \eta_0)$

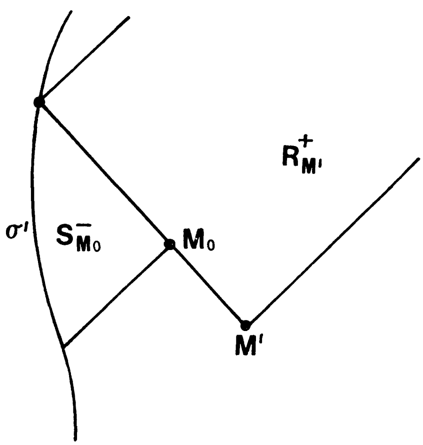

Figure 6

and $M'(\mu_0, \eta')$, $\in \Gamma$ and with $\eta' < \eta_0$. Since $\Gamma \cap \mathring{R}_{M'}^+ = \emptyset$ and, by construction, $\Gamma \cap \mathring{S}_{M_0}^- = \emptyset$, M_0 would be an isolated point of Γ, contrary to *c*).

In the same way, *setting*

$$\nu_1 = \inf_\Gamma \{\eta\} , \tag{2.4}$$

we prove the existence and the uniqueness of a point $M_1(\xi_1, \eta_1) \in \Gamma$, *such that*

$$\eta_1 = \nu_1 \,. \tag{2.5}$$

Therefore:

$$\xi_0 < \xi_1 \,, \quad \eta_0 > \eta_1 \,, \quad \Rightarrow x_0 < x_1 \,. \tag{2.6}$$

It follows that $M_0(x_0, t_0)$ *and* $M_1(x_1, t_1)$ *are opposite vertices of a characteristic rectangle* $R_1^* = M_0 Q_0 M_1 Q_1$, *where the boundary does not contain any impact point, besides* M_0 *and* M_1 (if $M \in \Gamma$, since $\Gamma \cap \breve{R}_M^- = \emptyset$, M_0 would be an isolated impact point). It may occur that $Q_0 \notin Z$,

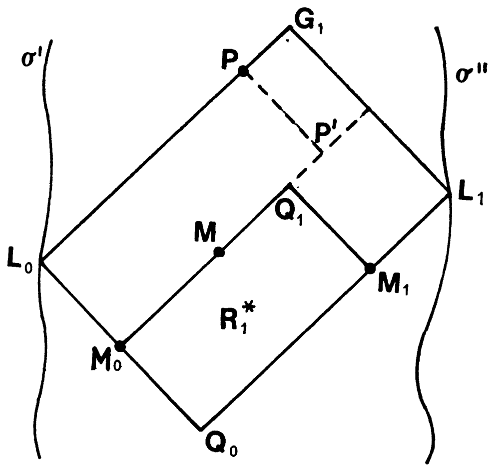

Figure 7

but $\Gamma \cap \sigma_0 = \emptyset$. Let us now extend the edges $Q_0 M_0$ and $Q_0 M_1$ as far as we meet the lines σ' and σ'', at the points L_0 and L_1 respectively.

Drawing moreover through L_0 and L_1 the forward half characteristics ξ and η respectively, we obtain a (ξ, η) *pair* with *vertex* G_1. Set $\sigma_1 = L_0 G_1 L_1$ and let W_1 be the domain with boundary $L_0 M_0 Q_1 M_1 L_1 G_1$. Observe that

$$\mathring{W}_1 \subset \mathring{R}^+_{M_0} \cup \mathring{R}^+_{M_1} \Rightarrow \Gamma \cap \mathring{W}_1 = \emptyset \,.$$

We prove, more precisely, that M_0 *and* M_1 *are the only impact points* $\in W_1$. We have, firstly (by construction and by the property of R_1^* mentioned before):

$$\Gamma \cap \{L_0 M_0 Q_1 M_1 L_1\} = \{M_0\} \cup \{M_1\} .$$

We have afterwards $\big($as $y(M_0) = 0\big)$, $\forall P \in L_0 G_1$,

$$y(P) = y(P') + y(L_0) - y(M_0) \geqslant y(L_0) = A(L_0) > 0 ,$$

from what it follows

$$(2.7) \quad y(P) \geqslant \min\{A(L_0), A(L_1)\} > 0 \qquad \forall P \in \sigma_1, \quad \Rightarrow \Gamma \cap \sigma_1 = \emptyset .$$

We conclude that

$$(2.8) \qquad \Gamma_1 = \Gamma \cap R_1^*$$

is a perfect set. $\Gamma - \Gamma_1$ is a *perfect* set too, since Γ_1 and $\Gamma - \Gamma_1$ are *separated* by $W_1 - \big(\{M_0\} \cup \{M_1\}\big)$.

Replace now σ_0 by σ_1 and repeat, with respect to $\Gamma - \Gamma_1$, the same construction before made for Γ. We obtain a second rectangle, R_2^*, which contains a *perfect set* $\Gamma_2 = (\Gamma - \Gamma_1) \cap R_2^*$, and so on, obtaing a sequence of *perfect and separated sets* $\Gamma_1, \Gamma_2, \ldots, \Gamma_n, \ldots$, *all* $\subseteq \Gamma$. Let us now *prove that*

$$(2.9) \qquad \Gamma = \bigcup_n \Gamma_n$$

(as obviously occurs if the Γ_n's are in a finite number).

Consider, on σ', the points $L_0(l_0, t_0)$, $L_2(l_2, t_2), \ldots, L_{2n}(l_{2n}, t_{2n}), \ldots$, that is the left end points of the (ξ, η) pairs

$$\sigma_1 = L_0 G_1 L_1 , \quad \sigma_2 = L_2 G_2 L_3, \ldots, \sigma_{n+1} = L_{2n} G_{n+1} L_{2n+1}, \ldots;$$

consider, analogously, the right end points $L_{2n+1}(l_{2n+1}, t_{2n+1})$, on σ''. Therefore:

$$l_{2n} = p(t_{2n}),\ l_{2n+1} = q(t_{2n+1}) .$$

Observe that $t_{2n} < t_{2n+2}$, $t_{2n+1} < t_{2n+3}$. For obtaining (2.9), it is sufficient to prove that

$$\lim_{t\to\infty} t_{2n} = +\infty\,, \qquad \lim_{t\to\infty} t_{2n+1} = +\infty\,. \tag{2.10}$$

Assume the contrary, supposing, for istance, that

$$t_{2n} \uparrow \bar{t} < +\infty \; (\Rightarrow L_{2n} \uparrow \bar{L}(p(\bar{t}), \bar{t}))\,.$$

This implies also that $t_{2n+1} \uparrow t^* < +\infty$; all points L_{2n+1} are in fact beneath the point $\bar{M}$, intersection of σ with the ξ foreward half characteristic issuing from $\bar{L}$. Hence $L_{2n+1} \uparrow \tilde{L}(q(\tilde{t}), \tilde{t}) \in \sigma''$, $G_{n+1} \uparrow G^*$, where G^* is the intersection of the ξ and η foreward half characteristics issuing from $\bar{L}$ and $\tilde{L}$ respectively (it may occur that $G^* = \bar{L}$ or $G^* = \tilde{L}$). It follows that $\sigma_n \uparrow \sigma^*$, where $\sigma^* = \bar{L}G^*\tilde{L}$ is a (ξ, η) pair (like every σ_n), or a characteristic segment.

Observe now that M_{2n} belongs, $\forall n$, to that part of Z which is inferiorly bounded by σ_n, and superiorly by σ^*. Therefore every limit point M^*, of the sequence $\{M_{2n}\}$, $\in \sigma^*$. As $y(M_{2n}) = 0$, it follows $y(M^*) = 0$, which is absurd. Denote infact by $\bar{\sigma}'$ and by $\tilde{\sigma}''$ the parts of σ' and of σ'' with end points $0', \bar{L}$ and $0'', \tilde{L}$ respectively. We have

$$\min_{\bar{\sigma}' \cup \tilde{\sigma}''} A(P) = a > 0$$

and we deduce from (2.7):

$$y(P) \geqslant a > 0 \qquad \forall P \in \sigma^*\,.$$

Hence $y(M^*) \geqslant a$, and (2.10) is proved.

h) Let K_1 be the t-projection, on the x axis, of the set Γ_1; we have $K_1 \subseteq m_0 \mapsto\!\!\dashv m_1$, where m_0 and m_1 are the abscissae of M_0 and M_1 respectively.

Assuming $x_1 \in K_1$, let us prove that *the straight line* $x = x_1$ *contains only one point of* Γ_1.

Suppose infact that there exist two such points, $H_1(x_1, t_1)$ and $H_2(x_1, t_2)$, with $t_1 < t_2$. Then $H_2 \in \overset{\circ}{R}{}^{+}_{H_1}$, which is absurd.

We conclude that Γ_1 *can be represented by an equation*

$$t = t_1(x) \qquad (x \in K_1) \tag{2.11}$$

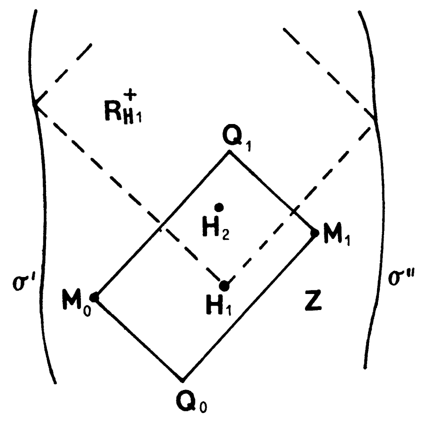

Figure 8

($\Rightarrow K_1$ *perfect set*). Moreover, the *Lipschitz condition*

$$(2.12) \qquad |t_1(x_2) - t_1(x_1)| \leqslant |x_2 - x_1| \qquad (\forall x_1, x_2 \in K_1)$$

holds. Assume infact, for instance, $t_1(x_2) - t_1(x_1) > x_2 - x_1 > 0$ (for a pair x_1, x_2) and consider the points $H_1(x_1, t(x_1))$, $H_2(x_2, t_1(x_2))$. Then $H_2 \in \mathring{R}^+_{H_1}$, which is absurd.

Observe, lastly, that, by *e*) and *f*), Γ_1 *does not contain any triplet of points* $H_1(x_1, t_1)$, $H_2(x_2, t_2)$, $H_3(x_3, t_3)$, where $x_1 < x_2 < x_3$, *such that*

$$|t_2 - t_1| = x_2 - x_1, \qquad |t_3 - t_2| = x_3 - x_2 .$$

The same properties hold for the sets $\Gamma_2, \ldots, \Gamma_n, \ldots$.

3. The lines of influence of the wall.

a) Let us *extend*, in the following way, *the definition of the function* $t_1(x)$, *on the whole of the interval* $l_0 \mapsto l_1$.

First of all, we define $t_1(x)$ on the intervals $l_0 \mapsto m_0$ and $m_1 \mapsto l_1$, by making the graph coincide with the characteristic segments $L_0 M_0$ and $M_1 L_1$ (η and ξ respectively).

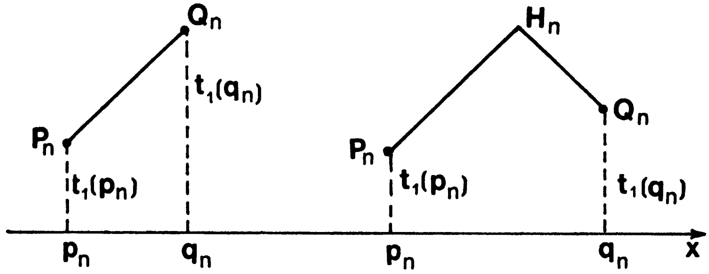

Figure 9

Setting then $\Delta = m_0 \mapsto m_1 - K_1$, we have $\Delta = \bigcup_n \delta_n$, where $\{\delta_n\}$ si a sequence of open and disjoint intervals: *no pair of such intervals can have an end point in common*, since K_1 has no isolated points. Setting then $\delta_n = p_n \rightarrow q_n$, there occur only two cases:

$$|t_1(q_n) - t_1(p_n)| = q_n - p_n\,, \qquad or \quad |t_1(q_n) - t_1(p_n)| < q_n - p_n\,.$$

In the first case we make the graph of $t_1(x)$ coincide, on $p_n - q_n$, with the characteristic segment P_nQ_n; in the second case the graph coincides with the (ξ, η) pair $P_nH_nQ_n$.

The function $t_1(x)$ *is defined*, in such a way, $\forall x \in l_0 \mapsto l_1$ and *it satisfies*, by construction and by (2.12), *the Lipschitz condition*

$$(3.1) \qquad |t_1(x_2) - t_1(x_1)| \leqslant |x_2 - x_1| \qquad (\forall x_1, x_2 \in l_0 \mapsto l_1)\,.$$

We shall call the *line* λ_1, *of equation*

$$(3.2) \qquad t = t_1(x) \qquad (x \in l_0 \mapsto l_1)\,,$$

the (first) line of influence of the wall. Observe moreover that the sign $=$ does *not* occur, at (3.1), for any $x_2 \neq x_1$, *if* x_1 *is a limit point of* K_1, *both from the left and from the right.*

The *physical meaning* of the line λ_1 is clear (see [1]). Observe infact that when a point $\bar{x}$ of the string, at the time $\bar{t}$, hits the wall, an *impulse* is created, which influences the motion of the string in the forward characteristic semicone issuing from the point $\bar{P}(\bar{x}, \bar{t})$.

Therefore, by the construction made, *the line* λ_1 *delimitates, from* *above, the largest part of Z, in which the free motion of the string is not influenced by the wall.*

In the same way, considering the set Γ_2, we obtain the *second line of influence*, λ_2 (and, successively, $\lambda_3, \ldots, \lambda_n, \ldots$). The line λ_{n+2}, $n \geqslant 0$, is *above* the (ξ, η) pair $\sigma_{n+1} = L_{2n} G_{n+1} L_{2n+1}$.

b) We can *characterize* the lines of influence on the basis of the following considerations.

Let us denote, in correspondance to an arbitrary point $P_0 \in Z$, by Z_{P_0} the intersection of Z with the *backward characteristic semicone* with vertex in P_0: Z_{P_0} will be still called the *backward characteristic semicone* relative to the point P_0 (see fig. 10).

Let Λ_1 be the following domain:

$$\Lambda_1 = \{P \in Z : \Gamma \cap \mathring{Z}_P = \emptyset\} . \tag{3.3}$$

One proves that Λ_1 *coincides with that part Λ of Z which is superiorly bounded by λ_1.*

Assume infact $P_0 \in \Lambda$: we have then (by (3.1), see also fig. 9) $\Gamma \cap \mathring{Z}_{P_0} = \emptyset$, $\Rightarrow P_0 \in \Lambda_1$, $\Rightarrow \Lambda \subseteq \Lambda_1$. Assume now $P_0 \in \Lambda_1$; no impact point can then belong to $\mathring{Z}_{P_0}$: hence P_0 cannot be *above* $\lambda_1 \Rightarrow \Lambda_1 \subseteq \Lambda$.

We can also obtain, as in [1], the line λ_1 by the following method. This method has practical importance too, as it refers to the *solution*,

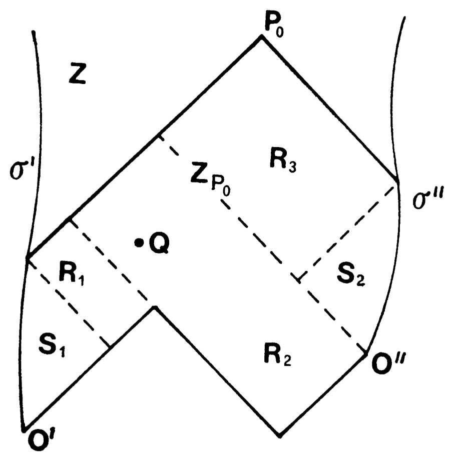

Figure 10

$w(P)$, *of the free problem*, with the *same* boundary condition as $y(P)$:

(3.4) $$w(P)|_{\partial Z} = y(P)|_{\partial Z} = A(P)\,.$$

Let Λ' be the following domain:

(5.3) $$\Lambda' = \{P \in Z\colon w(Q) \geqslant 0 \ \ \forall Q \in Z_P\}\,.$$

We prove then that $\Lambda_1 = \Lambda'$.

It is obvious that $\sigma_0 \subset \Lambda_1 \cap \Lambda'$. Assume now $P_0 \in \Lambda_1 - \sigma_0$. We have $y_{\xi\eta} = 0$ on the whole of $\mathring{Z}_{P_0}$: therefore $y(P)$ is determined, on the whole of Z_{P_0}, by the boundary values $A(P)$ (fig. 10) and necessarily coincides with the solution of the free problem: $y(Q) = w(Q)$, $\forall Q \in Z_{P_0}$. As $y(Q) \geqslant 0$ $\forall Q \in Z_{P_0}$, it follows $w(Q) \geqslant 0$ $\forall Q \in Z_{P_0}$, $\Rightarrow P_0 \in \Lambda'$, $\Rightarrow \Lambda_1 \subseteq \Lambda'$.

Assume now $P_0 \in \Lambda' - \sigma_0$ $\big(\Rightarrow w(Q) \geqslant 0,\ \forall Q \in Z_{P_0}\big)$. Let us observe that the value $w(P_0)$ is obtained from the data, by solving, successively, a finite number of Darboux and Goursat problems, for the equation $w_{\xi\eta} = 0$ (in the domains S_1, R_1, R_2, S_2, R_3, with reference to fig. 10). By (3.4) and since $w(Q) \geqslant 0$ on S_1, it follows from the extension law $y(Q) = w(Q)$ on the whole of S_1; the same equality then holds on R_1 and, successively, on R_2, S_2, R_3. We have therefore $y_{\xi\eta} = w_{\xi\eta} = 0$ on the whole of $\mathring{Z}_{P_0}$, $\Rightarrow \Gamma \cap \mathring{Z}_{P_0} = \emptyset$, $\Rightarrow P_0 \in \Lambda_1$. Hence $\Lambda' \subseteq \Lambda_1$ and the thesis follows.

Quite analogous considerations can be made for the line λ_2, assuming as initial values the values of $y(P)$ on σ_1 (which is possible as $y(P)|_{\sigma_1} > 0$).

4. Elastic impact law.

a) In order *to extend* the solution $y(P)$ *above* λ_1 we have to fix an *impact law* (in the present paper, we shall consider the *elastic impact*).

Let Z_1 be that part of Z which is superiorly bounded by σ_1, and inferiorly by σ_0.

Assume $C \in Z_1 - \Lambda_1$. Let us draw from C the ξ and η backward half characteristics and let D and B be their intersections with λ_1.

Denote, moreover, by A the intersection of the η and ξ backward half characteristics issuing from D and B respectively. Since DAB is a (η, ξ) pair, DAB cannot be a part of λ_1: the point $A \notin \lambda_1$ and is placed, necessarily, *beneath* λ_1 (it may occur that two segments DD_1 and BB_1, or one segment, $\subset \lambda_1$).

Assume now that the characteristic rectangle $R = ABCD$, *is* $\subset Z_1$, and observe that *the free solution*, $w(P)$, *has*, on the whole of R, *the expression*:

$$w(P) = \alpha(\xi) + \beta(\eta)\,, \tag{4.1}$$

where $\alpha(\xi) \in C^0(\xi_A \vdash\!\dashv \xi_B)$, $\beta(\eta) \in C^0(\eta_A \vdash\!\dashv \eta_D)$, *We have* moreover

$$y(P) = w(P) \qquad \forall P \in R \cap \Lambda_1\,. \tag{4.2}$$

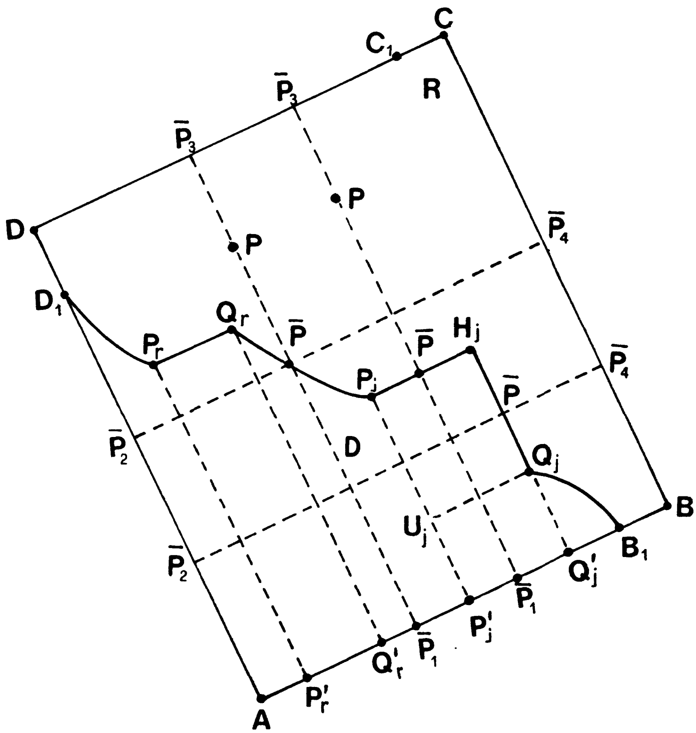

Figure 11

b) Assume, firstly, that $\alpha(\xi)$ and $\beta(\eta)$ are *absolutely continuous* functions. Therefore:

$$(4.3)\qquad w_\xi(P) = \alpha'(\xi) \in L^1(\xi_A \mapsto \xi_B)\,, \qquad w_\eta(P) = \beta'(\eta) \in L^1(\eta_A \mapsto \eta_D)\,.$$

We shall utilize now the *elastic impact law.* To this purpose, we shall *formulate* it, at every impact point, with respect to the *characteristic directions*, rather than with respect to the *time t* and adding the condition $y(P)|_{\Gamma_1} = 0$ (as it has been done in [1]). Consider now $\bar{P} \in \Gamma_1$, assuming that $\bar{P}$ is a limit point of Γ_1, both from the left and from the right (see fig. 11). *We shall then define the derivatives* $y_\xi(P)$ *and* $y_\eta(P)$, *on the segments* $\bar{P}\bar{P}_3$ *and* $\bar{P}\bar{P}_4$, *setting respectively*:

$$(4.4)\qquad \begin{aligned} y_\xi(P) &= -w_\xi(\bar{P}) = -w_\xi(\bar{P}_1) = -\alpha'(\bar{\xi}) \qquad \forall P \in \bar{P} \dashv \bar{P}_3\,,\\ y_\eta(P) &= -w_\eta(\bar{P}) = -w_\eta(\bar{P}_2) = -\beta'(\bar{\eta}) \qquad \forall P \in \bar{P} \dashv \bar{P}_4 \end{aligned}$$

(in the «regular case», this follows from (3.2), [1]).

Assume now $\bar{P}$ to be *inside* a characteristic segment ξ (or a characteristic segment η). *Since, in this case,* $\bar{P} \notin \Gamma_1$, *we have to set*:

$$(4.5)\qquad \begin{aligned} y_\xi(P) &= w_\xi(\bar{P}) = w_\xi(\bar{P}_1) = \alpha'(\bar{\xi}) \qquad \forall P \in \bar{P} \dashv \bar{P}_3\,,\\ y_\eta(P) &= w_\eta(\bar{P}) = w_\eta(\bar{P}_2) = \beta'(\bar{\eta}) \qquad \forall P \in \bar{P} \dashv \bar{P}_4\,. \end{aligned}$$

The derivatives y_ξ and y_η are therefore a.e. defined on R; by (4.3) they are defined a.e. on the edges DC and BC too.

Let us calculate now the value $y(C)$, *assuming* $y(\xi, \eta_D)$ and $y(\xi_B, \eta)$ to be *a.c.* functions.

Setting $\lambda_1 \cap R = \mu$, let k' be the η-projection of $\Gamma_1 \cap \mu$, on the segment AB_1. The complementary set, $h' = AB_1 - k'$, consists of a sequence $\{\delta'_m\}$ of open intervals. We prove that, for every such interval, it is (fig. 11):

$$(4.6)\qquad w(P'_r) - w(Q'_r) = 0\,, \qquad w(P'_j) - w(Q'_j) = 0\,.$$

The first of (4.6) is obvious. Since P_r and $Q_r \in \Gamma_1$, we have infact $w(P_r) = w(Q_r) = 0$, $\Rightarrow w(P'_r) - w(Q'_r) = 0$.

In the second of (4.6), H_j is the vertex of a (ξ, η) pair $P_jH_jQ_j$. Since P_j and $Q_j \in \Gamma_1$, we obtain

$$(4.7)\qquad 0 = w(H_j) - w(P_j) - w(Q_j) + w(U_j) = w(H_j) + w(U_j)\,.$$

Moreover, as H_j and $U_j \in R \subset Z$, it is necessarily $w(H_j) \geqslant 0$, $w(U_j) \geqslant 0$, which implies, by (4.7), $w(H_j) = w(U_j) = 0$.

The second of (4.6) is therefore proved. It follows

$$\int_{h'} w_\xi \, d\xi = \sum_m \int_{\delta'_m} w_\xi \, d\xi = 0 ,$$

and we have

$$w(B) - w(A) = \int_{\xi_A}^{\xi_B} w_\xi \, d\xi = w(B) - w(B_1) + \int_{k'} w_\xi \, d\xi + \int_{h'} w_\xi \, d\xi =$$
$$= w(B) + \int_{k'} w_\xi \, d\xi ,$$

as $w(B_1) = 0$, since $B_1 \in \Gamma_1$. Hence

$$\int_{k'} w_\xi \, d\xi = \int_{k'} \alpha'(\xi) \, d\xi = - w(A) . \tag{4.8}$$

We obtain moreover, by (4.4) (4.5) and (4.8),

$$y(C) - y(D) = \int_{\xi_A}^{\xi_B} y_\xi \, d\xi = y(C) - y(C_1) + \int_{\xi_A}^{\xi_{B_1}} y_\xi \, d\xi = \tag{4.9}$$
$$= w(B) - w(B_1) - \int_{k'} w_\xi \, d\xi + \int_{h'} w_\xi \, d\xi = w(B) - \int_{k'} w_\xi \, d\xi .$$

We deduce from (4.8) and (4.9) the *value*:

$$y(C) = w(B) + w(D) + w(A) . \tag{4.10}$$

c) *Suppose* now $w(P) \in C^0(R)$, $R \subset Z_1$. In this hypothesis, *we assume the formula* (4.10) *to express the definition of the displacement for the elastic impact case.*

On this basis, the displacement $y(P)$ is *defined*, on the whole of R, by the equalities:

$$\begin{aligned} y(P) &= w(P) & \forall P \in R \cap \Lambda_1 , \\ y(P) &= w(P_1) + w(P_2) + w(P_0) & \forall P \in R - \Lambda_1 , \end{aligned} \tag{4.11}$$

where the caracteristic rectangle $R_P = P_0P_1PP_2$ is obtained by utilising, for the point P, the same procedure as for C.

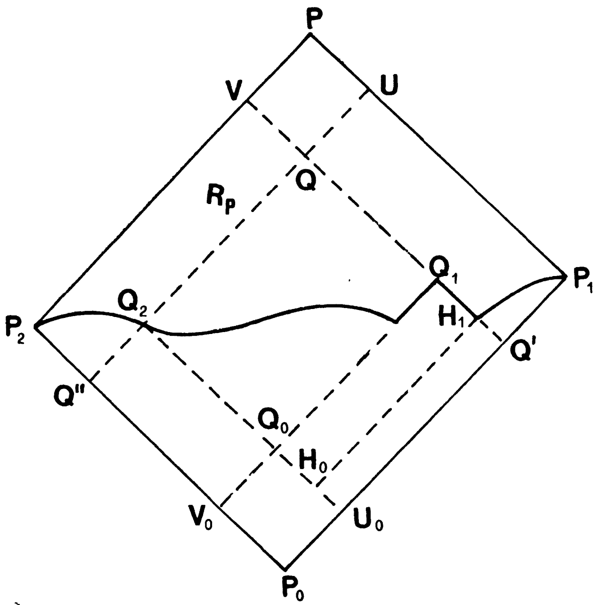

Figure 12

Observe, lastly, that (4.11) is *equivalent* to the equality:

$$\Delta_{R_P} y = 2w(P_0)\,. \tag{4.12}$$

d) Let us prove that *the function* $y(P)$, defined by (4.11), *is* actually *endowed with the following properties* (all *necessary*, with respect to the definition of *admissible* functions).

1) *We have*

$$w(P_0) > 0 \qquad \forall P \in R - \Lambda_1\,. \tag{4.13}$$

Therefore, by (4.11) and (4.12),

$$\begin{aligned} &y(P) \geqslant w(P_0) > 0 \\ &\Delta_{R_P} y = 2w(P_0) > 0 \qquad \forall P \in R - \Lambda_1\,. \end{aligned} \tag{4.14}$$

It is infact $w(P_0) \geqslant 0$ and it cannot be $w(P_0) = 0$. It would be, infact, in such case, $\forall Q \in R_P$,

$$w(Q) = w(Q') + w(Q'') - w(P_0) = w(Q') + w(Q'') \geqslant 0 .$$

Hence: $w(Q) \geqslant 0$ on the whole of Z_P, $\Rightarrow P \in \Lambda_1$, contrary to $P \in R - \Lambda_1$.

2) $y(P) \in C^0(R)$.

The continuity is obvious at the points beneath λ_1. Assume now P and $Q \in R - \Lambda_1$: we have

$$(4.15) \quad y(P) - y(Q) = w(P_1) + w(P_2) + w(P_0) - w(Q_1) - w(Q_2) - w(Q_0) .$$

Supposing then Q_1 and Q_2 not to be vertices of (ξ, η) pairs, the right hand side of (4.15) $\to 0$ as $P \to Q$. Let now, for example, Q_2 be vertex of a (ξ, η) pair. We have (as in *b*)) $w(H_1) = w(Q_1) = 0$, $w(H_0) - w(Q_0) = w(H_1) - w(Q_1) = 0$; it follows then from (4.15):

$$\lim_{P \to Q} \{y(P) - y(Q)\} = \lim_{P \to Q} \{w(P_1) - w(H_1) + w(P_2) - w(Q_2) + w(P_0) - w(H_0)\} = 0 .$$

The same equality holds if $Q \in \lambda_1$.

3) *It is*

$$(4.16) \qquad \operatorname{supp} y_{\xi\eta} = \Gamma_1 \cap R .$$

Therefore $y(P)$ *satisfies the homogeneous equation* $y_{\xi\eta} = 0$, *on the whole of* $\mathring{R} - \Gamma_1$.

Since (fig. 12)

$$y(U) = w(P_1) + w(Q_2) + w(U_0) ,$$

$$y(V) = w(Q_1) + w(P_2) + w(V_0) ,$$

we have infact

$$(4.17) \qquad y(P) - y(U) - y(V) + y(Q) = w(P_0) - w(U_0) - w(V_0) + w(Q_0) = 0 .$$

Observe now that (4.17) holds even if the edge QU, or QV, of the rectangle $QUPV$ is placed on a characteristic segment $\subset \lambda_1$. It holds, more generally, $\forall$ rectangle $R^* \subset R$, with $\Gamma_1 \cap \mathring{R}^* = \emptyset$ (we prove this, by decomposing R^* into two, or four, characteristic rectangles, each of them being contained in Λ_1, or in $R - \mathring{\Lambda}_1$).

We have therefore $y_{\xi\eta} = 0$, on the whole of $\mathring{R} - \Gamma_1$. If, lastly, $\bar{P} \in \Gamma_1 \cap \mathring{R}$, it follows from (4.14) $\Delta_{R_P} y > 0$, $\forall$ rectangle $R_P \ni \bar{P}$, with $P \in R - \Lambda_1$: therefore $\bar{P} \in \operatorname{supp} y_{\xi\eta}$.

OBSERVATION. Assume that *the trace of $w(P)$ on DAB satisfies the conditions set at b)*: assume therefore *$\alpha(\xi)$ and $\beta(\eta)$ to be a.c. functions on the intervals $\xi_A \vdash\!\dashv \xi_B$, $\eta_A \vdash\!\dashv \eta_D$ respectively.* From these *hypotheses* we can deduce that, *if the function $y(P)$ is defined by* (4.11), *then* (4.4) *and* (4.5) *hold.* In other words: *$y(P)$ satisfies the mechanical law of elastic impact* (in the form set at *a*)).

Consider, for instance, the derivative y_ξ. Observe that, on R, the derivative $w_\xi(\xi, \eta)$ is independent on η.

Setting then

$$(4.18) \qquad \varphi(\xi) = \begin{cases} w_\xi(\xi, \eta) = \alpha'(\xi) & \forall \xi \in B_1 \dashv B \cup h' \\ -w_\xi(\xi, \eta) = -\alpha'(\xi) & \forall \xi \in k' \end{cases}$$

(and bearing in mind that $w(B_1) = 0$ and that, $\forall \eta$, $\int_{h'} w_\xi(\tau, \eta)\, d\tau = 0$) we obtain (fig. 13):

$$(4.19) \qquad \int_{\xi_A}^{\xi_B} \varphi(\tau)\, d\tau = w(B) - \int_{k'} w_\xi(\tau, \eta_A)\, d\tau =$$

$$= w(B) - \left\{ \int_{k'} w_\xi(\tau, \eta_A)\, d\tau + \int_{h'} w_\xi(\tau, \eta_A)\, d\tau \right\} =$$

$$= w(B) - \int_{\xi_A}^{\xi_{B_1}} w_\xi(\tau, \eta_A)\, d\tau = w(B) + w(A) = y(C) - w(D),$$

by (4.10).

We deduce afterwards from (4.18) and (4.19), $\forall P(\xi, \eta_D) \in DC$:

$$\int_{\xi_A}^{\xi} \varphi(\tau)\, d\tau = \int_{\xi_{P_0}}^{\xi_{P_1}} \varphi(\tau)\, d\tau = w(P_1) + w(P_0) = y(P) - w(D),$$

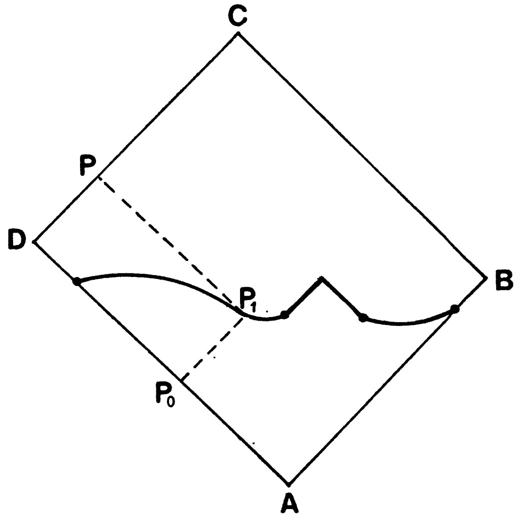

Figure 13

by (4.11). The function $y(\xi, \eta_D)$ is therefore a.c. on $\xi_D \mapsto \xi_C$, and it is, a.e.:

$$y_\xi(\xi, \eta_D) = \varphi(\xi) .$$

It follows then (4.4), by (4.18) (and analogously for (4.5)).

5. Solution of the problem.

The *elastic impact law* and the *extension law* allow *to extend* the solution $y(P)$ from Λ_1 to that part Z_1 of Z, which is superiorly bounded by the line $\sigma_1 = L_0 G_1 L_1$. Let $m_0 \mapsto m_1$ be the t-projection on the x axis of that part, λ, of λ_1, with end points M_0 and M_1. Let us call, moreover, (m) *pair* every (ξ, η) pair, $PUQ \subset \lambda$ $(P, Q \in \Gamma)$, such that *it may be* $w(U) > 0$ at its vertex U. This case can occur only when the vertex V, opposite to U in the characteristic rectangle $PUQV$, is placed beneath the polygonal line σ_0. Let π_0 be the number of the (ξ, η)-vertices of σ_0: it is then obvious that *the number of* (m) *pairs is finite* $(\leqslant \pi_0)$. Let $d > 0$ be the *distance* of λ from ∂Z (fig. 14).

Let us divide now λ in a finite number of parts, of lengths $\leqslant d$, by means of the points (with increasing abscissae) $N_0(x_0, t(z_0)), \ldots, \ldots, N_r(x_r, t(x_r))$, $x_0 = m_0$, $x_r = m_1$ (fig. 15).

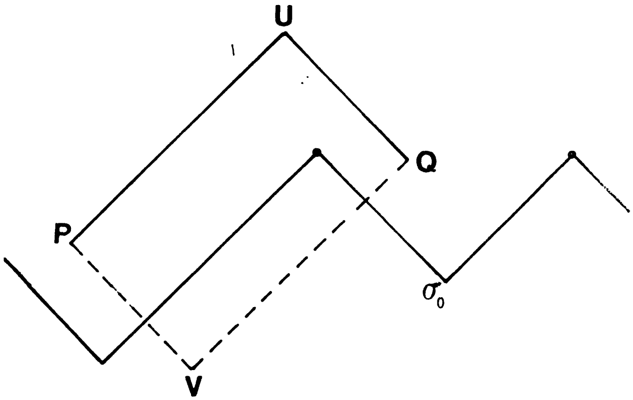

Figure 14

All characteristic rectangles (or segments) with left and right vertices N_0 and $N_1, ..., N_{r-1}$ and N_r, are contained in Z.

It may occur that some group of points $(N_h, ..., N_{h+i}; i \geqslant 0)$ belong to the same characteristic segment, or (ξ, η) pair: in such case we shall substitute the group $(N_h, ..., N_{h+i})$ by the end points of that segment, or (ξ, η) pair. Calling again N_k the points of subdivision, we have now subdivided λ in a finite number of partial arcs, γ_i, the end points of which are, $\forall j$, *impact points*. For every arc γ_j there occurs then *one of the following circumstances*:

1) γ_j is a characteristic segment,

2) γ_j is a (ξ, η) pair,

3) $\gamma_j = \widehat{D_j B_j}$ (where $D_j = N_{k_j}$, $B_j = N_{k_j+1}$) is an arc of length $\leqslant d$.

The γ_j's of type 1) or 2) are necessarily *separated*, one from the other, by arcs of type 3); all (m) pairs are among those pairs considered at 2). In the case 3), the characteristic rectangle $R_j = A_j B_j C_j D_j$ is $\subset Z$; moreover, on the edge $D_j C_j$, the only point $D_j \in \gamma_j$ (on the contrary, by the construction made, the segment $D_j U_j$, or the pair $D_j U_j Q_j$, would have been included in the types 1) or 2)). There cannot therefore occur the cases at the fig. 16_1, 16_2. The only possible case corresponds to the fig. 16_3. The same occurs for the edge $B_j C_j$.

Let us assume now that the problem of the wall has a solution, and let us calculate it.

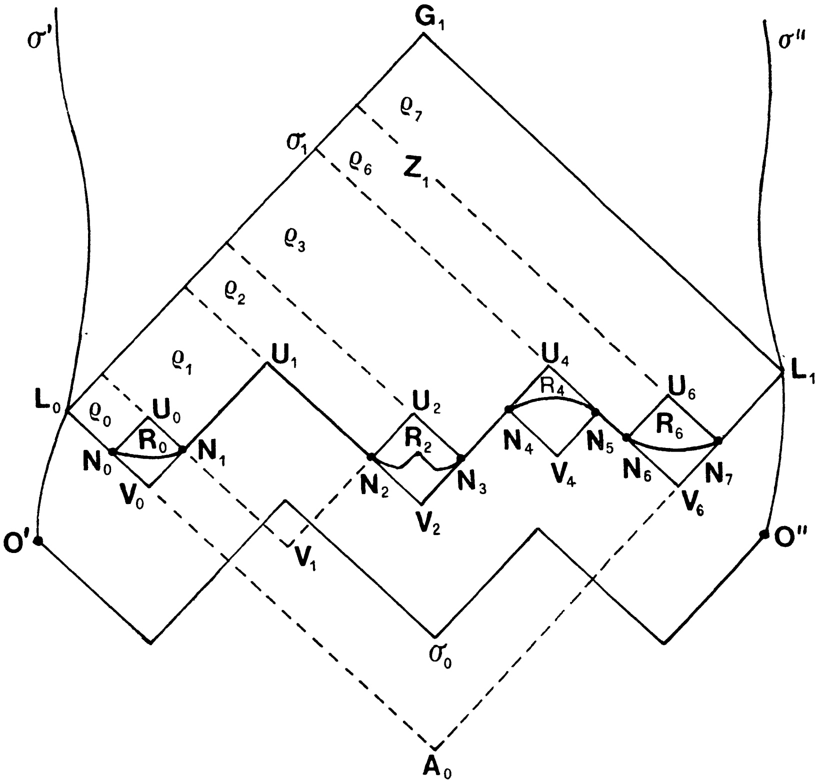

Figure 15

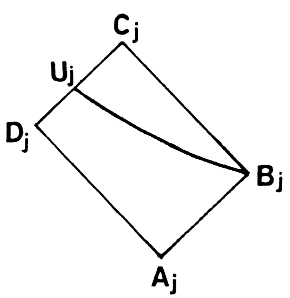

Figure 16_1

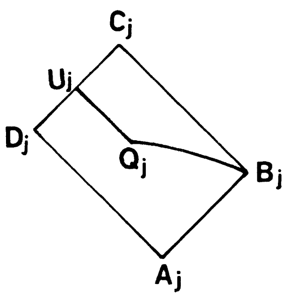

Figure 16_2

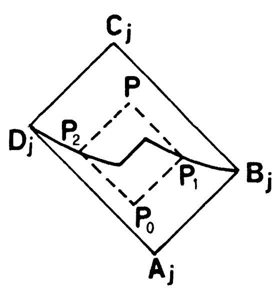

Figure 16_3

We have $y = w(P)$, $\forall P \in \Lambda_1$. We have moreover (cfr. (4.11) and fig. 16_3)

$$y(P) \geqslant w(P_0) > 0 \qquad \forall P \in R_j - \Lambda_1 . \tag{5.1}$$

The displacement $y(P)$ is now determined at all points of a poligonal characteristic line ϑ (cfr. fig. 15, where $\vartheta = L_0 N_0 U_0 N_1 U_1 N_2 U_2 \cdot$ $\cdot N_3 N_4 U_4 N_5 N_6 U_6 N_7 L_1$). Moreover $\vartheta \subset Z_1 - \mathring{\Lambda}_1$ and all minimum points of ϑ are impact points, $N_k (\Rightarrow y(N_k) = 0)$. By the *elastic impact law* (and with reference to the rectangles R_0, R_2, R_4, R_6), we have:

$$y(P) > 0 \qquad \forall P \in (R_0 \cup R_2 \cup R_4 \cup R_6) - \Lambda_1 . \tag{5.2}$$

The same inequality holds in the remaing part of Z_1 (cf. [1]). Consider infact, on the rectangle ϱ_0, the Darboux problem for the equation $z_{\xi\eta} = 0$, with the boundary condition $z|_{L_0 N_0 U_0} = y|_{L_0 N_0 U_0}$. Since $y(N_0) = 0$, we have $z(P) > 0$ on the whole of $\varrho_0 - L_0 \dashv N_0$: this implies (by the extension law) $y(P) = z(P) > 0$ on the same set. We prove, in the same way, that $y(P) > 0$ on the rectangles $\varrho_1, \varrho_2, \varrho_3, \varrho_6, \varrho_7 -$ $- L_1 \dashv N_7$, with the only exception of points $\in \vartheta$ (where it can be $y(P) = 0$). We obtain, in particular:

$$y(P) > 0 \qquad \forall P \in \sigma_1 . \tag{5.2}$$

The uniqueness and the existence of the solution $y(P)$ of our problem, on the whole of Z_1, is therefore proved.

Assuming now as initial value of $y(P)$, $P \in Z - \mathring{Z}_1$, the *restriction* $y(P)|_{\sigma_1}$, we can proceed in calculating the solution, which *exists and is unique*, on the whole of Z.

OBSERVATION I. The expression of $y(P)$, $P \in Z_1 - \Lambda_1$, is very simple. Let firstly be $P = G_1$. We have (with reference to fig. 15, in which $N_1 U_1 N_2$ is an (m) pair):

$$\begin{aligned} y(G_1) - y(L_0) &= \\ &= y(U_0) - y(N_0) + y(U_1) - y(N_1) + y(U_2) - y(N_2) + \\ &+ y(U_4) - y(N_3) + y(U_6) - y(N_6) + y(L_1) - y(N_7) = \\ &= y(U_0) + y(U_1) + y(U_2) + y(U_4) + y(U_6) + y(L_1) , \end{aligned} \tag{5.4}$$

that is:

$$(5.5)\qquad y(G_1) = w(L_0) + w(L_1) + \\ + w(V_0) + w(V_2) + w(V_4) + w(V_6) + w(U_1) .$$

We have moreover (by extending the free solution $w(P)$ beneath σ_0)

$$w(V_6) - w(A_0) = w(N_5) - w(V_4) + w(N_4) - w(V_2) + w(U_1) - w(V_0) ,$$

that is:

$$(5.6)\qquad w(A_0) = w(V_6) + w(V_4) + w(V_2) + w(V_0) - w(U_1) .$$

It follows then from (5.5) and (5.6):

$$y(G_1) = w(L_0) + w(L_1) + w(A_0) + 2w(U_1) .$$

More generally, if $P \in Z_1 - \Lambda_1$, the following formula holds (with obvious notations):

$$(5.7)\qquad y(P) = w(P_1) + w(P_2) + w(P_0) + 2\sum_j w(U_j)$$

where the summation must be extended to all vertices of the (m) pairs which are *inside* the characteristic rectangle $P_0P_1PP_2$.

OBSERVATION II. Let us consider, as in [1], the problem of the wall with the classical *initial* and *boundary conditions*:

$$y(p(t), t) = A_1(t) > 0 , \quad y(q(t), t)) = A_2(t) > 0 \qquad (t \geqslant 0)$$

$$y(x, 0) = A_0(x) > 0 , \quad y_t(x, 0) = \psi(x) \qquad (p(0) \leqslant x \leqslant q(0)) ,$$

assuming

$$A_1(t) , \quad A_2(t) \in C^0(0 \vdash + \infty); \qquad A_0'(x) , \quad \psi(x) \in L^1(p(0) \mapsto q(0)) .$$

In such hypotheses the solution $w(P)$ of the *free* problem $(P \in Z^* = = \{t \geqslant 0, p(t) \leqslant x \leqslant q(t)\})$ is > 0 in a part of Z^*, $Z^*_\delta = \{0 \leqslant t \leqslant \delta, p(t) \leqslant$

$\leqslant x \leqslant q(t);\ \delta > 0\}$. If the polygonal line $\sigma_0 \subset Z^*_\delta$, and setting

$$y(P)|_{\sigma_0} = w(P)|_{\sigma_0},$$

this case can therefore be reduced to that before treated.

OBSERVATION III. Assume, in observation II, $p(t) \equiv 0$, $q(t) \equiv \equiv l > 0$, $A_1(t) \equiv A_2(t) \equiv K > 0$: therefore Z^* reduces to a half strip in the (x, t) plane.

Assume now the *initial energy* of the string to be *finite*, that is:

$$A_0'(x) \in L^2(0 \vdash\!\dashv l), \qquad \psi(x) \in L^2(0 \vdash\!\dashv l).$$

We recognize then easily that *energy equality holds*:

$$\int_0^l \{y_x^2(x, t) + y_t^2(x, t)\}\, dx = \int_0^l \{A_0'^2(x) + \psi^2(x)\}\, dx \qquad (\forall t \geqslant 0). \tag{5.8}$$

Observe, first of all, that $y_x^2 + y_t^2 = y_\xi^2 + y_\eta^2$. Observe, moreover, that, in crossing the impact points, the ξ and η derivatives change their sign, but not their absolute value. We have lastly (since $y_t(0, t) \equiv y_t(l, t) \equiv 0$): $y_\xi(0, t) = -y_\eta(0, t)$, $\Rightarrow y_\xi^2(0, t) = y_\eta^2(0, t)$; analo-

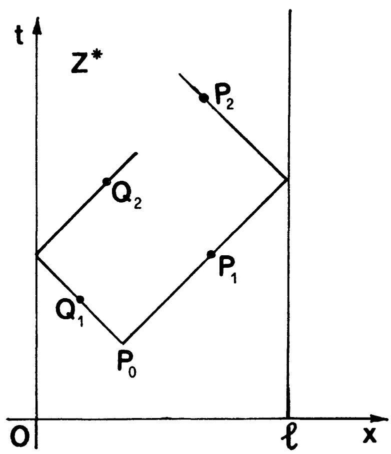

Figure 17

gously, $y_\xi^2(l, t) = y_\eta^2(l, t)$. Taken an arbitrary $P_0 \in Z^*$, its follows:

(5.9) $\quad y_\xi^2(P_0) = y_\xi^2(Q_1) = y_\eta^2(Q_2) = \dots \qquad y_\eta^2(P_0) = y_\eta^2(P_1) = y_\xi^2(P_2) \dots .$

Assume now $0 < \bar{t} \leqslant l$. We have:

$$\int_0^l y_\eta^2(x, 0)\, dx = \int_0^{l-\bar{t}} y_\eta^2(x, 0)\, dx + \int_{l-\bar{t}}^l y_\eta^2(x, 0)\, dx = \int_{\bar{t}}^l y_\eta^2(x, \bar{t})\, dx + \int_{l-\bar{t}}^l y_\xi^2(x, \bar{t})\, dx\,,$$

$$\int_0^l y_\xi^2(x, 0)\, dx = \int_{\bar{t}}^l y_\xi^2(x, 0)\, dx + \int_0^{\bar{t}} y_\xi^2(x, 0)\, dx = \int_0^{l-\bar{t}} y_\xi^2(x, \bar{t})\, dx + \int_0^{\bar{t}} y_\eta^2(x, \bar{t})\, dx\,.$$

It follows:

$$\int_0^l \{y_\eta^2(x, 0) + y_\xi^2(x, 0)\, dx = \int_0^l \{y_\eta^2(x, \bar{t}) + y_\xi^2(x, \bar{t})\}\, dx\,, \qquad \forall \bar{t} \in 0 \vdash\!\dashv l\,.$$

Therefore the same equality holds in the interval $l \vdash\!\dashv 2l$ and, successively, $\forall t \geqslant 0$.

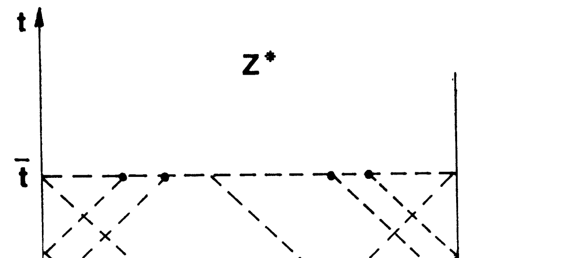

Figure 18

More generally, assuming $A_0'(x)$ and $\psi(x) \in L^m(0 \vdash\!\dashv l)$, $1 \leqslant m \leqslant +\infty$, we obtain the *equality*:

$$\int_0^l \{|y_\xi(x, 0)|^m + |y_\eta(x, 0)|^m\}\, dx = \int_0^l \{|y_\xi(x, t)|^m + |y_\eta(x, t)|^m\}\, dx\,.$$

REFERENCES

[1] L. Amerio - G. Prouse, *Study of the motion of a string vibrating against an obstacle*, Rend. di Mat., (2), vol. **8**, serie VI, 1975.

Manoscritto pervenuto in redazione il 18 maggio 1978.