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Rings Radical Over P.I. Subrings (*).

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A ring R is said to be *radical* over a subring A if a power of every element in R falls in A, that is, if given $r \in R$ there is an integer $n(r) \geqslant 1$ such that $r^{n(r)} \in A$. One of the first results in the direction of studying the nature of rings radical over certain subrings is a result due to Kaplansky [6]. He showed that a semisimple ring radical over its center must be commutative. Herstein [3] extended this by showing that if R is any ring radical over its center, then the commutator ideal is nil; hence, if R should be without nil ideals it must be commutative. Lihtman [7] substantially generalized this last result by showing its conclusion remains valid if we merely assume R to be radical over a commutative subring.

One can look at Lihtman's theorem from the following point of view: if R is a ring without nil ideals and is radical over a subring A which satisfies the polynomial identity $p(x_1, x_2) = x_1x_2 - x_2x_1$ then R itself must satisfy the same identity (that is, R is commutative).

One can naturally ask if there is anything particular about the identity $p(x_1, x_2) = x_1x_2 - x_2x_1$. What if A satisfies any polynomial identity; does it then follow that if R is without nil ideals it must satisfy this same identity?

This is the question to which we address ourselves here. It would be reasonable to expect the answer to the above question to be yes.

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We don't quite prove that conjecture here—certain difficulties tied in with the Köthe conjecture prevent us from reaching this ultimate result—but we do prove the result in case R has no nil right ideals. Our techniques don't seem to provide a means of skirting the assumption « no nil right ideals » to « no nil ideals ».

In what follows R will be a ring, A a subring of R, such that R is radical over A and A satisfies a polynomial identity $p(x_1, ..., x_n)$. We assume, as we may without loss of generality, that $p(x_1, ..., x_n)$ is homogeneous and multilinear. Thus the form of $p(x_1, ..., x_n)$ is $p(x_1, ..., x_n) = q(x_1, ..., x_{n-1})x_n + h(x_1, ..., x_n)$ where $q(x_1, ..., x_{n-1})$ is multilinear and homogeneous of degree $n-1$, and where x_n is never the last variable in any nonomial of $h(x_1, ..., x_n)$. We shall always use the symbol $Z(M)$ to denote the center of M.

We begin with

LEMMA 1. *Suppose that R is a prime ring with non non-zero nil ideals. If A is semi-prime then R satisfies the polynomial identity of A.*

PROOF. - Since A is a semi-prime P.I. rings, by [5, Th. 1.4.2] $F = Z(A) \neq 0$. Since R is radical over A, from the very definition of the hypercenter of R, (see [4]) $Z(A)$ is contained in the hypercenter of R. However, since R has no nil ideals, by the main theorem of [4] the hypercenter of R coincides with the center of R, and so $F \subset Z(R)$. Since R is prime, the elements of F are not zero divisors in R. This immediately implies that A is also prime. Localize A and R at F to get rings A_1, R_1 respectively. Then R_1 is prime with no nil ideals, is radical over A_1 and A_1 satisfies the polynomial identity $p(x_1, ..., x_n)$ satisfied by A. Moreover, since A is a prime P.I. rings, by Posner's theorem [5, Th. 1.4.3] A_1 is a simple algebra finite-dimensional over its center.

We claim that R_1 is simple; for, if $U \neq 0$ is an ideal of R_1 then $U \cap A_1 = 0$ or $U \supset A_1$ since A_1 is simple. The second possibility forces $1 \in U$, hence $U = R_1$. On the other hand, if $U \cap A_1 = 0$ U must be nil since it is radical over A_1, which is not possible. Thus R_1 is simple.

Thus, without loss of generality, R is simple and A is a simple algebra finite-dimensional over its center F, where F is a field and $F \subset Z(R)$.

If A is a division ring, since R is radical over A we have that every element in R is either invertible or nilpotent. But in that case the nilpotent elements of R form an ideal of R. Since R has no nil ideals we have that R has no nilpotent elements; thus R is a division ring. By a result of Faith [1] we have that R is commutative or $R = A$. In either case R satisfies the P.I. of A.

So suppose that A is not a division ring; then A has an idempotent e such that eAe is a division ring. But then eRe is simple and is radical over eAe; by the above eRe is a division ring and satisfies the polynomial identity of A, hence that of A. Therefore R is a minimal right ideal of R. Since R is simple, with 1, and has a minimal right ideal, R must be a simple artinian ring. Thus $R = D_n$, the ring of all $n \times n$ matrices over the division ring D. Since R is radical over A clearly all the idempotents of R are in A. If e_{ij} denotes the usual matrix units of D_n we have that e_{ii} and all $e_{ii} + \delta e_{ij}$, for $i \neq j$ and $\delta \in D$, are in A, since they are idempotents. From this we get $\delta e_{ij} \in A$ for all $\delta \in D$, all $i \neq j$. From this we get that $D \subset A$ and all $e_{ij} \in A$. Hence $R = A$ and R then certainly satisfies the identity of A.

We pass to

LEMMA 2. *If R has no nil ideals and A has no nilpotent elements then R satisfies the polynomial identity of A.*

PROOF. By theorem 6 of [2] R has no nilpotent elements. Therefore, by a theorem of Andrunakievitch and Rjahubin [5, Th. 1.1.1], R is a subdirect product of rings R_α which are without zero divisors. Each R_α is radical over A_α, the image of A in R_α, so by Lemma 1 —since A_α satisfies the identity of A—each R_α satisfies the polynomial identity of A. Hence R does.

Prior to passing to the proof of our main theorem we need a simple remark about prime rings. Let R be a prime ring and $\varrho \neq 0$ a right ideal of R. Suppose that ϱ satisfies a polynomial identity; then R has no nil right ideals.

To see this, let $J \neq 0$ be a nil right ideal and $s \neq 0 \in J$. Then $\varrho s \neq 0$; if $t \neq 0$ is in ϱs then tR is a nil right ideal of R and, being in ϱ, satisfies a polynomial identity. By a result due to Kaplansky and Levitzki (see Lemma 2.1.1 in [5]) R would have a non-zero nilpotent ideal; this is not possible in a prime ring.

We are now able to prove our principal result.

THEOREM. *Let R be a ring having no non-zero nil right ideals, and suppose that R is radical over A. If A satisfies a polynomial identity then R satisfies the same identity.*

PROOF. We proceed by induction on the degree of the homogeneous, multilinear polynomial identity $p(x_1, \dots, x_n)$ satisfied by A.

By a theorem of Felzenswalb [2, Th. 2] A is semi-prime, hence $Z(A) \neq 0$. Since $Z(A)$ is in the hypercenter of R, $Z(A) \subset Z(R)$.

Let $a \neq 0$ be in A. Let $M = \{x \in R | ax \text{ is not nilpotent.}\}$ Because R has no nil right ideals, M is not empty. If $x \in M$ let $P_{a,x}$ be an ideal of R maximal with respect to exclusion of $\{(ax)^n\}$; $P_{a,x}$ is a prime ideal of R. Let $P_a = \bigcap_{x \in M} P_{a,x}$. We claim that $P_a \cap aR = 0$. For, suppose suppose that $0 \neq ay \in P_a$; then $(ay)z$ is not nilpotent for some $z \in R$. Thus $ayz \notin P_{a,yz}$ where $yz \in M$; this contradicts $ayz \in P_a \subset P_{a,yz}$. Since $P_a \cap aR = 0$ and R is semi-prime we have that $P_a RaR = 0$ and so $P_a \cap RaR = 0$.

By Lemma 2 we may assume that A has nilpotent elements, otherwise we are done. Let $a \neq 0$ be in A such that $a^2 = 0$. Hence aR is not nil and is radical over $A_1 = aR \cap A$. If $b_1, \ldots, b_{n-1} \in A_1$ then $ab_i = 0$; consequently $0 = p(b_1, \ldots, b_{n-1}, a) = q(b_1, \ldots, b_{n-1})a + h(b_1, \ldots, b_{n-1}, a) = q(b_1, \ldots, b_{n-1})a$ (since $ab_i = 0$ and a is never the last term of a monomial of $h(x_1, \ldots, x_{n-1})$, $h(b_1, \ldots, b_{n-1}, a) = 0$).

Consider $T = \{x \in aR | xa = 0\}$; T is an ideal of aR and $B = aR/T$ is without nil right ideals, is radical over the image of A_1, and moreover, by the above, satisfies $q(x_1, \ldots, x_{n-1})$. By our induction hypothesis, B satisfies $q(x_1, \ldots, x_{n-1})$, hence for all $r_1, \ldots, r_{n-1} \in R$ $q(ar_1, \ldots, ar_{n-1})a = 0$. Therefore aR satisfies the P.I. $q(x_1, \ldots, x_{n-1})x_n$.

Let $\bar{R} = R/P_{a,x}$ for $x \in M$; the non-zero right ideal aR of $\bar{R}$ satisfies a P.I. Since $\bar{R}$ is prime, by our remark preceding this theorem, $\bar{R}$ has no nil right ideals. By Theorem 2 of [2] $\bar{A}$, the image of A in $\bar{R}$, is semi-prime. Since $\bar{R}$ is radical over $\bar{A}$, by Lemma 1 $\bar{R}$ must satisfy $p(x_1, \ldots, x_n)$. Thus $p(r_1, \ldots, r_n) \in P_{a,x}$ for every $x \in M$, and so

$$p(r_1, \ldots, r_n) \in P_a = \bigcap_{x \in M} P_{a,x}$$

for every $r_1, \ldots, r_n$ in R.

We know that $P_a \cap RaR = 0$; so, if $r_1, \ldots, r_n \in RaR$ then $p(r_1, \ldots, r_n) \in P_a \cap RaR = 0$. In other words, RaR satisfies $p(x_1, \ldots, x_n)$ for every $a \in A$ such that $a^2 = 0$.

Let W be an ideal of R maximal with respect to the property of satisfying $p(x_1, \ldots, x_n)$. Since $W \supset RaR$ for $0 \neq a \in A$ with $a^2 = 0$, $W \neq 0$. Since W is semi-prime and P.I., $0 \neq Z(W) \subset Z(R)$. If $r_1, \ldots, r_n \in R$ and $\lambda \neq 0 \in Z(W)$ then, since $\lambda r_i \in W$, $0 = p(\lambda r_1, \ldots, r\lambda_n) = \lambda^n p(r_1, \ldots, r_n)$. Because R has no nil ideals and $\lambda \in Z(R)$ we have that $\lambda p(r_1, \ldots, r_n) = 0$ for all $\lambda \in Z(W)$ and all $r_1, \ldots, r_n \in R$.

Let $T = \{x \in R | Z(W)x = 0\}$. If $T = 0$ then, by the above, R

satisfies $p(x_1, \ldots, x_n)$. On the other hand, if $T \neq 0$ then $T \cap Z(W) = 0$ since R is semi-prime. We claim that $T \cap W = 0$; otherwise, since $T \cap W \subset W$ is a non-zero ideal and satisfies a polynomial identity, then $Z(T \cap W) \neq 0$. But $Z(T \cap W) \subset Z(R)$, and since it lies in T and W, it must lie in $T \cap Z(W) = 0$. Hence $T \cap W = 0$. T is an ideal of R, hence has no nil right ideals, and since T is radical over $A \cap T$ and is not nil, $T \cap A \neq 0$. If $0 \neq b \in A \cap T$ is such that $b^2 = 0$ we saw that RbR must satisy $p(x_1, \ldots, x_n)$. Because $WRbR \subset \subset WT = 0$, $W + RbR$ satisfies $p(x_1, \ldots, x_n)$ and is properly larger than W. This contradicts the choice of W. Hence $A \cap T$ has no nilpotent elements. By Lemma 2, T satisfies $p(x_1, \ldots, x_n)$ leading to the contradiction that $T + W \neq W$ satisfies $p(x_1, \ldots, x_n)$. Hence $T = 0$ and so R satisfies $p(x_1, \ldots, x_n)$. The theorem is now proved.

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