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Structure Theorems for Modifications of Complex Spaces.

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In this paper we are concerned with the modification of complex spaces. Given such a modification $f: (Y', X') \to (Y, X)$, $Y' \subset X'$, $Y \subset X$, we consider the problem of a « description » of f. In this direction the main problem is the following: under what hypothesis is the given modification isomorphic to the monoidal transformation of X along Y? The main results of the paper are that this is the case when:

a) X' is normal, Y' is an irreducible projective bundle $\mathbb{P}(\mathcal{L})$ on Y and the ideal $I_{Y'}$ of Y' is invertible (Theorem 3.2), or

b) Y' is irreducible, $I_{Y'}$ is invertible, Y and X are smooth (Theorem 3.7).

When X' is smooth Theorem 3.7 was proved by Moĭšezon ([6]). An algebraic analogue of the theorem was proved by A. Lascu ([5]).

In § 1, 2 we prove some results on meromorphic maps between complex spaces and on the dimension of the exceptional set Y' of a modification.

1. Preliminaries.

1) Let $(X, \mathcal{O}_X)$ be a (reduced and connected) complex space. Let $\mathcal{M}_X$ be the sheaf of the germs of meromorphic functions on X.

We say that a morphism $f: X \to Y$ of complex spaces is *bimero-*

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morphic if the homomorphism $M_Y \to f_* \mathcal{M}_X$ is an isomorphism. It can be proved that if Y is normal and $f^{-1}(y)$ is finite for every $y \in Y$, then f is an open embedding. Moreover the fibres of a bimeromorphic morphism $f\colon X \to Y$ (where Y is normal) are connected.

Let X, Y be irreducible. A *meromorphic* map $F\colon X \to Y$ is an irreducible analytic subset F of $X \times Y$ such that: there are an analytic subset $A \subsetneq X$ and an analytic subset $F_1 \subset F$ such that $F \setminus F_1$ is the graph of a morphism $X \setminus A \to Y$.

In particular one has $F_1 = \mathrm{pr}_X^{-1}(A) \cap F$ (F being irreducible). For every subset $Z \subset X$ we put $F(Z) = \mathrm{pr}_Y\big(\mathrm{pr}_X^{-1}(Z) \cap F\big)$ and we call $F(Z)$ the *image* of Z by F. A point $x \in A$ is said to be *regular* for F if there is a neighborhood U of x and a morphism $f\colon U \to Y$ such that $f|_{U \setminus A} = F_{U \setminus A}$.

Let $\Omega = \Omega(F)$ be the subset of regular points of F: Ω is open and $\mathrm{Sing}(F) = X \setminus \Omega$ is called the *singular locus* of F. Let X be normal. Then it can be proved ([11]) that:

(i) if $F(x)$ is compact and $\neq \emptyset$ for every x, then $\mathrm{Sing}(F)$ is an analytic subset of codimension $\geqslant 2$;

(ii) a point x is regular for F iff $F(x)$ has a connected component of dimension 0.

2) Let X be a complex space. We shall say that X is *meromorphically separated* if for $x, y \in X$, $x \neq y$, there is a meromorphic function f on X, regular at x, y, such that $f(x) \neq f(y)$.

Let $\mathfrak{L}$ be an invertible sheaf on X and denote by $\mathbb{A}(\mathfrak{L})$ the graded algebra $\bigoplus_{n=0}^{+\infty} \Gamma(X, \mathfrak{L}^{\otimes n})$ and by $Q(\mathfrak{L})$ the quotient field of $\mathbb{A}(\mathfrak{L})$. $Q(\mathfrak{L})$ is a field of meromorphic functions.

PROPOSITION 1.1. *Let X be compact and normal and $Q(\mathfrak{L})$ separates the points of X. Then X is projective.*

PROOF. Let $s_0, \ldots, s_k \in \Gamma(X, \mathfrak{L}^{\otimes r})$ be such that:

$$\bigcup_{i=0}^{k} \{x \in X\colon s_i(x) = 0\} = \emptyset \qquad \text{and} \qquad f_{ij} = s_i/s_j\,,\ i, j = 0, \ldots, k$$

separate points of X. Let $\mathbb{P}^k = \mathbb{P}^k(\mathbf{C})$ and f be the morphism $X \to \mathbb{P}^k$ defined by $x \mapsto \big(s_0(x), \ldots, s_k(x)\big)$. f is a one-to-one, proper map and f^{-1} is continuous from $f(X)$ to X. Let $N = N(X)$ be the open subset of the normal points of $f(X)$; $g = f^{-1}$ is holomorphic on N. Let $\nu\colon f(X)^* \to f(X)$ be the normalization of $f(X)$; $f(X)^*$ is a projective variety and $\varphi = \nu^{-1} \circ f$ is a memomorphic map $X \to f(X)^*$ which is a morphism on $X \setminus f^{-1}(f(X) \setminus N)$. We have $\mu(x) \subset \nu^{-1}(f(x))$ for every $x \in X$ and furthermore $\mathrm{Sing}\,(\varphi)$ is an analytic subset of codimension $\geqslant 2$. Let $x \in \mathrm{Sing}\,(\varphi)$ and $y_1, \ldots, y_l \in \nu^{-1}(f(x))$. Let H be a hyperplane section of $f(X)^*$ such that $y_i \notin H$, $i = 1, \ldots, l$. Then $V = f(X)^* \setminus H$ is an affine variety, $x \in f^{-1}(\nu(H))$ and $\varphi\big(X \setminus f^{-1})\nu(H))\big) \setminus \mathrm{Sing}\,(\varphi) \subset f(X)^* \setminus H$ for every $x \in X$. It follows that p extends to a morphism $\tilde{\varphi}\colon X \setminus f^{-1} \cdot \cdot (\nu(H)) \to V$. This proves that φ extends on X and $\varphi(x) \in \nu^{-1}(f(x))$. Hence φ is one-to-one and so is an isomorphism between X and $f(X)^*$.

2. Modifications.

1) Let X be a (connected) complex space, Y a complex subspace, I_Y the ideal of Y and $\pi\colon \tilde{X} \to X$ the monoidal trasformation of X with center Y ([6]). The universal property of $\pi\colon \tilde{X} \to X$ is the following: for every complex space Z and for every morphism $f\colon Z \to X$ such that $f^* I_Y$ is an invertible ideal there is a morphism $g\colon Z \to \tilde{X}$ (unique up to isomorphisms) such that $\pi \circ g = f$. In particular if $\tilde{X} = \pi^{-1}(Y)$ one has $I_{\tilde{Y}} = \pi^* I_Y$.

Remark. If $f^* I_Y$ is invertible on the complement of a proper analytic subset A of Z, then g is a meromorphic map $Z \to \tilde{X}$.

We denote by $f\colon (Y', X') \to (Y, X)$ a modification of irreducible complex spaces and we will refer to Y' as to the *exceptional subset* of the given modification ([6], [9]).

We say that the modification is

a) *regular* if Y and X are both smooth,

b) a *point-modification* if Y is zero-dimensional.

In the sequel we shall be concerned with the following problem: under what hypothesis is the modification $f\colon (Y', X') \to (Y, X)$ isomorphic to the monoidal transformation of X with center Y? As we shall see later, conditions may be placed on properties of the embedding $Y' \hookrightarrow X'$ or on properties of the embedding $Y \hookrightarrow X$.

2) Now let us establish some geometrical properties of regular modifications.

THEOREM 2.1. *Let* $f\colon (Y', X') \to (Y, X)$ *be a regular modification of n-dimensional complex spaces. Then*

(i) *if* $\dim_{\mathbf{C}} Y = 0$, Y' *is of pure dimension* $n-1$,

(ii) Y' *is of dimension* $n-1$ *and it is of dimension* $\geqslant n-2$ *at every point* $x \in Y'$,

(iii) *the connected components of* Y' *of dimension* $n-2$ *are fibres.*

In particular if $\dim_{\mathbf{C}} \operatorname{Sing}(X') \leqslant n-3$ *then* Y' *is of pure dimension* $n-1$.

PROOF. We first remark that for algebraic varieties (or for algebraic spaces as well) it can be proved that Y' is actually of pure dimension $n-1$ ([5]). From this remark the affirmation (i) follows immediately.

We shall prove (ii) by induction on n. Let $d = \dim_{\mathbf{C}} Y$, $a \in Y$ and $p = \dim_{\mathbf{C}} f^{-1}(a)$. Let U be a neighborhood of a in X such that: $\dim_{\mathbf{C}} f^{-1}(y) \leqslant p$ for every $y \in U$, U is a fibration $\varphi\colon U \to \gamma$, where γ is an analytic curve, and $Y_\lambda = U_\lambda \cap Y$, $U_\lambda = \varphi^{-1}(\lambda)$, is a submanifold of dimension $d-1$. Let us assume U_λ is defined by $h_\lambda = 0$, h_λ holomorphic, and let $V_\lambda = \overline{f^{-1}(U_\lambda \setminus Y)}$. V_λ is an irreducible analytic subset of $f^{-1}(U)$ and $f_\lambda = f|_{V_\lambda}$ gives a modification $V_\lambda \to U$ with exceptional subset $E_\lambda = V_\lambda \cap f^{-1}(Y)$.

Let $\lambda_0 \in \gamma$; by the induction hypothesis one has two possibilities: *a*) E_{λ_0} is of pure dimension $n-2$; *b*) E_{λ_0} is reduced to a point and f_{λ_0} is an isomorphism.

In the case *b*), for every point 0 of Y_{k_0} the corresponding fibre of f is either of dimension 0 or it has an irreducible component of dimension 1 (actually $\dim_{\mathbf{C}} V_\lambda \cap f^{-1}(y) = 0$). In the first case we have that $\dim_{\mathbf{C}} f^{-1}(y_0) = 0$ for an $y_0 \in Y_{\lambda_0}$ and therefore for all y in a neighborhood. It follows that f is a local isomorphism. In the second one $f^{-1}(Y_{\lambda_0})$ has an irreducible component of dimension $d \leqslant n-2$. This is impossible because then the analytic subset defined by $h_{\lambda_0} \circ f = 0$ would have an irreducible component of codimension >1.

Let us suppose that case *a*) holds so that E_{λ_0} is of pure dimension $n-2$. From the above discussion it follows that E_λ is of pure dimension $n-2$ for every $\lambda \in \gamma$, thus $\dim_{\mathbf{C}} Y' = n-1$. Now assume

$$Y' = Y'_1 \cup \ldots \cup Y'_l \cup Z_1 \cup \ldots \cup Z_k$$

where Y'_j is irreducible and $(n-1)$-dimensional for $j = 1, \ldots, l$ and

Z_i is irreducible of dimension $\leqslant n-2$ for $i=1,\ldots,k$. We have $f(Y'_i)=Y$ for at least one i (and suppose $i=1$) and $Y'_i \cap Y'_l \neq \emptyset$, $Z_j \cap Y'_l \neq \emptyset$ for every i, j (the fibres being connected). Let $y_1=f(x_1)$ where $x_1 \in Z_1 \setminus Y'_1$ and let V_0 be a submanifold of U through y_0 defined by $h=0$. The analytic subset Y, defined by $h \circ f=0$, is of pure dimension $n-1$ and $\overline{f^{-1}(V_0 \setminus Y)}$ is an irreducible component of W. Let W_0 be an irreducible component of W containing x_0; then: $W_0 \subset Z_1$ and $f|_{W_0}$ gives a modification $W_0 \to V_0$. It follows that $W_0 \cap Z_1$ is of pure dimension $n-2$ or that $f|_{W_0}$ is an isomorphism. In view of the fact that $W_0 \cap Z_1$ is the zero-set of $h \circ f|_{Z_1}$ and that Z_1 is irreducible, we have $Z_1 \subset W_0$ and $\dim_{\mathbf{C}} Z_1 = n-2$. This proves part (ii) of the statement.

If $x'_1 \in Z_1 \setminus Y'_1$ is another point such that $f(x'_1)=y'_1 \neq y_1$ then, by repeating the above argument with respect to a variety V_1 through x'_1 parallel to V_0, we get a contradiction. Therefore we have $f(Z_j)=y_j$ for $j=1,\ldots,k$. In particular every Z_j is compact and the connected components of $\bigcup_{j=1}^{k} Z_j$ are fibres. This proves part (iii) of the statement. Finally, if $\dim_{\mathbf{C}} \operatorname{Sing}(X') \leqslant n-3$, then $Z_j \not\subset \operatorname{Sing}(X')$, $j=1,\ldots,k$; in view of the jacobian criterium f is an isomorphism at every point of $Z_j \setminus \operatorname{Sing}(X')$, $j=1,\ldots,k$, therefore $Z_1=\ldots=Z_k=\emptyset$ and Y' is of pure dimension $n-1$.

REMARK. It was proved in [10] that if X' is meromorphically separated and X is *locally factorial* (i.e., the local rings $\mathcal{O}_{X,x}$ are U.F.D.) then Y' is of pure codimension 1.

COROLLARY 2.2. *Let $f\colon X' \to X$ be a proper morphism of irreducible complex spaces and let $Y' \subset X'$, $Y \subset X$ be irreducible complex subspaces of codimension 1 such that $f(Y')=Y$. Assume X smooth and that $f|_{X' \setminus Y'}$ is an isomorphism onto $X \setminus Y$. Then f is an isomorphism.*

3. Structure theorems.

1) Let us go back to the initial problem, i.e., the description of modification of complex spaces.

If X is a complex space and Y is a complex subspace we shall denote by $\pi\colon (\tilde{Y}, \tilde{X}) \to (Y, X)$ the monoidal transformation of X with center Y.

PROPOSITION 3.1. *Let $f\colon (Y', X') \to (Y, X)$ be a modification where Y' is irreducible, $I_{\tilde{Y}}$ and f^*I_Y are invertible. Assume $\tilde{X}$ is locally factorial and that $\tilde{Y}$ is irreducible. Then the modifications $f\colon (Y', X') \to$ $\to (Y, X)$ and $\pi\colon (\tilde{Y}, \tilde{X}) \to (Y, X)$ are isomorphic.*

PROOF. Assume that X' is normal and consider the meromorphic map $g\colon X' \to \tilde{X}$ determined by f^*I_Y. For a generic $x \in \tilde{Y}$, the fibre $g^{-1}(x)$ is discrete and therefore reduced to a single point x'. Thus g is an isomorphism at x'. The subset A of the points where g is not a local ismorphism is of codimension $\geqslant 1$ in Y' (Y' being irreducible) and of codimension $\geqslant 2$ in X'. We have $A = \{a \in \tilde{Y}\colon \dim_{\mathbf{C}} g^{-1}(a) \geqslant 1\}$.

Let $b \in B = g^{-1}(A)$ and $a = g(b)$ and let ξ be a generator of $I_{\tilde{Y},a}$. Let η be $\xi \circ g$ and let h be a generator of $I_{Y',b}$; $h/\eta = \lambda$ is a holomorphic function on $U \cap (X' \setminus B)$ (U being a neighborhood of b in X') therefore λ is holomorphic on U. It follows that the pull-back $g_a^*\colon \mathcal{O}_{\tilde{X},g(a)} \to \mathcal{O}_{X',a}$ induces an isomorphism $I_{\tilde{Y},g(a)} \approx I_{Y',a}$. This implies that g_a^* is an isomorphism ($I_{\tilde{Y},g(a)}$ and $I_{Y',a}$ are invertible!). Thus $A = \emptyset$ and g is an isomorphism.

In the general case let $\nu\colon X'^* \to X'$ be the normalization of X', $W = \nu^{-1}(Y)$ and $z \in W \cap \operatorname{Sing}(X'^*)$. Let $h \in I_{W,z}$ be holomorphic on U, $x = \nu(z)$ and $'h$ be a generator of $I_{Y',x}$. The function $\mu = h/{}'h\circ\nu$ is holomorphic on $U \setminus \operatorname{Sing}(X'^*)$ and, therefore, on U. This proves that I_W is an invertible ideal.

From the first part of the proof it follows that there is an isomorphism $\theta\colon X'^* \to \tilde{X}$ such that $\nu\circ\theta^{-1}\circ g = \mathrm{id}_{X'}$, $\theta^{-1}\circ g\circ\nu = \mathrm{id}_{X}$. Thus ν and g are isomorphisms and this concludes the proof.

Now let X' be normal, Y' be an irreducible complex projective bundle $\mathbf{P}(\mathcal{L})$ on Y where $\mathcal{L}$ is a locally free sheaf on Y of rank $r+1$ and $r + \dim_{\mathbf{C}} Y = n - 1$ ($n = \dim_{\mathbf{C}} X$). Let $f\colon (Y', X') \to (Y, X)$ be a modification such that $f|_{Y'}$ is the natural projection $\mathbf{P}(\mathcal{L}) \to Y$. Let $\mathcal{O}_{\mathbf{P}(\mathcal{J})}(1)$ be the fundamental sheaf on $\mathbf{P}(\mathcal{L})$.

THEOREM 3.2. *Let $I_{Y'}$ be invertible. Then*

(i) *$I_{Y'}/I_{Y'}^2$ is locally isomorphic to $\mathcal{O}_{\mathbf{P}(\mathcal{J})}(m)$ where $m > 0$.*

(ii) *$I_{Y'}$ is an ample sheaf with respect to f and the modification is isomorphic to the monoidal transformation $\pi\colon (\tilde{Y}, \tilde{X}) \to (Y, X)$.*

PROOF. (i) Since the problem is local with respect to Y we can assume that $Y' = Y \times \mathbf{P}^r$. Let $y \in Y$. Then there are two invertible sheaves $\mathfrak{L}_1$ on Y and $\mathfrak{L}_2$ on $\mathbf{P}^r$ such that

$$I_{Y'}/I_{Y'}^2 \approx p_1^*\mathfrak{L}_1 \otimes_{\mathcal{O}_Y} p_2^*\mathfrak{L}_2$$

(p_1, p_2 natural projections) ([7]) so that we can assume $\mathfrak{L}_2 \approx \mathcal{O}_{\mathbf{P}^r}(m)$ and $\mathfrak{L}_1 \approx \mathcal{O}_Y$. It follows that $I_{Y'}/I_{Y'}^2 \approx \mathcal{O}_{\mathbf{P}(\mathfrak{J})}(m)$. One has $m \geqslant 0$. If not, as $\Gamma(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(m)) = 0$ for $m < 0$, we have $\Gamma(Y', I_{Y'}/I_{Y'}^{2\otimes k}) = 0$ for every $k \geqslant 1$. Then, from the exact sequence

$$0 \to I_{Y'}^k/I_{Y'}^{k+1} \to I_{Y'}/I_{Y'}^{k+1} \to I_{Y'}/I_{Y'}^k \to 0$$

it follows that $\Gamma(Y', I_{Y'}/I_{Y'}^k) = 0$ for every $k=1$.

Let $u \not\equiv 0$ be an element of $\Gamma(X, I_{Y'})$ and $y' \in Y'$: there is $k \geqslant 2$ such that $v = u \circ f \notin I_{Y',y'}^k$. Thus v gives a non zero element in $\Gamma(X'$, $I_{Y'}/I_{Y'}^k)$: contradiction.

Now assume $m = 0$. Then $Y_{Y'}/I_{Y'}^2$ is isomorphic to $\mathcal{O}_{Y'}$. Let $Y = \bigcup_{i \in I} U_i$, where U_i is open in X' and such that $I_{Y'|U_i}$ is generated by h_i.

We can assume that $h_i/h_{j U_i | \cap U_j} = 1$. Let h be a holomorphic function on a neighborhood of Y' vanishing on Y' and let $\beta_i = h/h_i$. We have $\lambda_i \in \mathcal{O}(U_i)$ and $\lambda_i = \lambda_j$ on $U_i \cap U_j \cap Y'$. Thus h determines a holomorphic function λ on Y' (which is constant on each fibre). The zero-set Z of h has Y' as an irreducible component; let Z be $Y' \cup Z'$: Z' is of pure codimension 1 and $\dim_{\mathbf{C}} Z' \cap Y' = n-2$. Take $h = g \circ f$ where g is a holomorphic function on X vanishing on Y. Then $Z' \cap Y'$ intersects each fibre of f but it does not contain all fibres. This is a contradiction because then λ would have different values on a fibre. Thus $I_{Y'}/I_{Y'}^2$ is locally isomorphic to $\mathcal{O}_{\mathbf{P}(\mathfrak{J})}(m)$ where $m > 0$.

(ii) Let us denote by $I_{(y)}$ the algebraic restriction of $I_{Y'}$ to $f^{-1}(y)_0$. Part (i) implies that the reduced sheaf $I_{(y)}^{\text{red}}$ is isomorphic to $\mathcal{O}_{\mathbf{P}^r}(m)$. Therefore $I_{(y)}$ is ample on $f^{-1}(y)_0$.

In view of a result of Schneider ([9]) $I_{Y'}$ is ample with respect to f, hence we can assume that there exists a closed embedding $\varphi\colon X' \hookrightarrow X \times \mathbf{P}^N$ (for a suitable N) such that $\varphi^* \mathcal{O}_{\mathbf{P}^N}(1) \approx I_{Y'}^l$.

In view of the theorem of Grauert and Remmert on projective morphisms (cf. [4]), for every coherent sheaf $\mathcal{F}$ on X' and for every compact $K \subset X$ there is an integer n_0 such that $R^1 f_*(\mathcal{F} \otimes I_{Y'}^{nl})|_K = 0$

for every $n = n_0$. From the exact sequence

$$0 \to I_{Y'}^{k+1} \to I_{Y'}^{k} \to \mathcal{O}_{\mathbf{P}(\mathfrak{J})}(km) \to 0$$

decreasing induction on k implies that $R^1 f_*(I_{Y'}^l)|_X = 0$ for every $l = 0$. Arguing as in [10] (Théorème 2.2.3) we get part (ii) of the statement.

REMARK. The above theorem tells us that a modification which « blows-down a projective bundle » Y' is always isomorphic to a monoidal transformation (provided $I_{Y'}$ is invertible).

2) In this final part we shall prove that, under natural hypothesis, every regular modification $f\colon (Y', X') \to (Y, X)$ is isomorphic to the monoidal transformation $\pi\colon (\tilde{Y}, \tilde{X}) \to (Y, X)$.

This was proved in [5] for algebraic normal varietis and that proof extends to normal algebraic spaces as well, by passing to an « étale » covering and applying the « descent property » ([3]).

For complex manifolds the theorem was proved in [6].

We proceed in several steps.

LEMMA 3.3. *A regular point-modification $f\colon (Y', X') \to (y_0, X)$ of irreducible complex spaces such that $I_{Y'}$ is invertible, is isomorphic to the monoidal trasformation $\pi\colon (\tilde{Y}, \tilde{X}) \to (y_0, X)$.*

PROOF. We can assume X is $\mathbf{P}^n$ and that X' is a compact Moĭšezon space therefore a complete $\mathbb{C}$-algebraic space ([2]). We have $\dim_{\mathbb{C}} \operatorname{Sing}(X') \leqslant n-2$ because $I_{Y'}$ is invertible. Let $\nu\colon X'^* \to X$ be the normalizati on of X' and put $W = \nu^{-1}(Y')$: W is irreducible. Let $z \in W \cap \operatorname{Sing}(X'^*)$, $x = \nu(z)$ and let $h \in I_{W,z}$ be holomorphic on a neighborhood U of z and $'g$ a generator of $I_{Y',x}$. The function $h/'g\circ\nu$ is holomorphic on $U \setminus \operatorname{Sing}(X'^*)$ and therefore on U. It follows that $'g\circ\nu$ generates locally I_W. Then, by the previous remark, the modification $g\colon (W, X'^*) \to (y_0, X)$ is isomorphic to the monoidal transformation. Let $x \in W \approx \mathbf{P}^{n-1}$ and let $z_1, \dots, z_n$ be local coordinates at y_0 such that $z_1(y_0) = \dots = z_n(y_0) = 0$. Let $x_\alpha = z_\alpha \circ g$, $\alpha = 1, \dots, n$, and let us assume that x_1 generates $I_{W,x}$. Let $y = \nu(x)$ and let ξ be a generator of $I_{Y',y}$. On a neighborhood of y the zero-sets of ξ, $z_1\circ f$ coincide, so that $\xi^s = (\lambda(t_1\circ f)$, where λ is invertible and $s \in \mathbf{N}$, and therefore $(\xi\circ\nu)^s = (\lambda\circ\nu)x_1$. On the other hand, as $\xi\circ\nu$ generates $I_{W,x}$, we have also $x_1 = \mu(\xi\circ\nu)$ where μ is invertible. Thus $s = 1$ and $z_1\circ f$ generates $I_{Y',y}$. In particular if I_0 denotes the ideal sheaf of $\{y_0\}$, $f^* I_0 = I_{Y'}$ is invertible and, in view of the Proposition 2.1, $(Y', X') \to (y_0, X)$ is isomorphic to the monoidal transformation.

REMARKS. In the previous statement, the hypothesis that $I_{Y'}$ is invertible can be replaced by the following ones: Y' is *geometrical principal* (i.e., Y' is locally a zero-set of a holomorphic function) and X' is a regular in codimension 1. Namely we have the

LEMMA 3.4. *Let $(Y', X') \to (y_0, X)$ be a regular point-modification of irreducible algebraic varieties. Assume that Y' is geometrically principal and that X' is regular in codimension* 1. *Then the sheaf $f^* I_0$ is invertible.*

PROOF. We can assume that X and X' are complete. Let $y \in Y'$ and h be a local equation for Y' on a neighborhood U of y. Let u be a rational function on X such that $h = u \circ f$ and put $u = q/r$ where q, r are rational functions on X without common factors in $\mathcal{O}_{X,y_0}$. We observe that $q(y_0) = 0$. Let (h) denote the divisor of h. On U we have $(h) = lY'$, $l > 0$, and therefore $(h) = (f \circ q) - (f \circ r) > 0$. As q and r have no common factor in $\mathcal{O}_{X,y_0}$, $f \circ q$ is a positive divisor on a neighborhood V of y and on V one has: $(f \circ q) = mY'$, $m > 0$. Let ψ be in $f^* I_0^m$ (or in $I^m_{Y',y}$): we have $(f \circ \psi) - (f \circ q) \geqslant 0$ on V, so that $f \circ \psi = \beta f \circ q$, $\beta \in \mathcal{O}_{X',y}$. This proves that $f^* I_0^m$ (and $I^m_{Y'}$) are invertible and therefore that $f^* I_0^m$ (and $I_{Y'}$) are invertible ($\mathcal{O}_{X',y}$ being local).

LEMMA 3.5. *Let $(Y', X') \xrightarrow{f} (y_0, X)$ be a regular point-modification of complex compact surfaces. Assume that X' is normal. Then the modification is isomorphic to a product of monoidal transformations.*

PROOF. Let $Y' = C_1 \cup \ldots \cup C_k$ be the irreducible decomposition of Y' and let $\hat{X}' \xrightarrow{\pi} X'$ be a desingularization of X': in view of the fundamental theorem of surface theory ([8]), $F = f \circ \pi \colon \hat{X}' \to X$ is a product of monoidal trasformations. Furthermore the exceptionl set E of F is

$$C_1^* \cup \ldots \cup C_k^* \cup D_1 \cup \ldots \cup D_l$$

where C_j^*, D_i^* are projective lines and

$$(C_j^{*2}) = -1\,, \qquad (D_i^2) = -1\,, \qquad 1 \leqslant j \leqslant k\,,\ 1 \leqslant i \leqslant l\,.$$

We may blow-down the curves $D_1, \ldots, D_l$ in such a way as to get a regular surface X_0' with a morphism $\pi_0 \colon X_0' \to X'$ which is actually an isomorphism (X' being normal).

LEMMA 3.6. *Let $f: (Y', X') \to (y_0, X)$ be a regular point-modification of complex compact surfaces. Assume Y' is geometrically principal. Then the modification is a product of monoidal transformations.*

PROOF. Let us assume for simplicity that Y' is irreducible. We may restrict ourselves to the following case: X is $\mathbf{P}^2$ and X' is algebraic. By passing to a non-singular model of X' and arguing as in the previous lemma we find a modification $\pi: (\tilde{Y}, \tilde{\mathbf{P}}^2) \to (Y', X')$ (where $g: (\tilde{Y}, \tilde{\mathbf{P}}^2) \to (y_0, X)$ is a product of monoidal transformations and $f \circ \pi = g$). Let I_0 be the ideal sheaf of $\{y_0\}$ and let z_1, z_2 be rational functions on X giving local coordinates at y_0 (and $z_1(y_0) = z_2(y_0) = 0$). Let $y_1 = z_1 \circ f$, $y_2 = z_2 \circ f$, $x_1 = z_1 \circ g$ and $x_2 = z_2 \circ g$. The invertible ideal $I_{\tilde{Y}}$ is generated by x_1 or x_2 and there are two points $b_1, b_2 \in \tilde{Y}$, such that $I_{\tilde{Y},x} = x_1 \mathcal{O}_{\tilde{X},x} = x_2 \mathcal{O}_{\tilde{X},x}$ for $x \neq b_1, b_2$. Let $c = \pi(x) \neq \pi(b_1), \pi(b_2)$: $y_2 = 0$ is a local equation for Y' at c. We have $y_2 \circ \pi = u x_1$ where $u = (p/q) \circ g$ is invertible in $\mathcal{O}_{\tilde{X},y}$ and p, q are polynomials in z_1, z_2 without common factors. Further

$$p/q = \frac{\alpha_0 z_1 + \beta_0 z_2 + p_1}{\alpha_1 z_1 + \beta_1 z_2 + q_1}$$

where p_1, q_1 are polynomials of degree $\geqslant 2$ and $\alpha_0, \beta_0, \alpha_1, \beta_1 \in \mathbf{C}$, $\alpha_0 \neq 0$, $\beta_0 \neq 0$. It follows that

$$(p/q) \circ f = \frac{\alpha_0 y_1 + \beta_0 y_2 + p_1 \circ f}{\alpha_1 y_1 + \beta_2 y_2 + p_2 \circ f};$$

p and q are coprime therefore $p \circ f$, $q \circ f$ can vanish only on Y' (locally at x). It follows that either $p \circ g$ and $q \circ g$ vanish on Y' or are invertible at x (because $(p/q) \circ g$ is invertible). In the first case $p = z_1 P_1$, $q = z_1 Q_1$ which implies $\beta_0 = \beta_1 = 0$ and $p_1 = z_1 P_2$, $q_1 = z_1 Q_2$ where $P_2(0) \neq 0$, $Q_2(0) \neq 0$. Thus $y_2 = v y_1$ where v is a unit of $\mathcal{O}_{X',c}$. It follows that the ideal $f^* I_0$ is invertible on $X'_0 = X' \setminus \{\pi(b_1)\} \cup \{\pi(b_2)\}$. The morphism $X'_0 \to \tilde{\mathbf{P}}^2$ determinated by $f^* I_0$ is an inverse of $\pi|_{X'_0}$ and this proves that X' is non singular in codimension 1. Now the result follows from Lemma 3.4.

REMARKS. 1) Let A_j be the analytic set defined by $z_j = 0$ and let $W_j = \overline{f^{-1}(A_j) \setminus Y'}$, $j = 1, 2$. As a consequence of the above lemma we have $W_1 \cap W_2 = \emptyset$.

2) The assumption that Y' is geometrically principal cannot be dropped.

Now we are in position to prove the

THEOREM 3.7. *Let $f: (Y', X') \to (Y, X)$ be a regular modification of irreducible complex spaces. Assume that Y' is irreducible and that $I_{Y'}$ is invertible. Then the modification is isomorphic to the monoidal transformation of X wich center Y.*

PROOF. From the hypothesis it follows that X' is nonsingular in codimension 1. The problem is local with respect to X along Y so we may assume X is a ball in $\mathbf{C}^n$ centered at 0 and Y is defined by $z_{d+1} = \dots = z_n = 0$. Let ζ_j be the function $z_j \circ f$, $j = d+1, \dots, n$ and let W_j be the analytic set $\overline{f^{-1}(V_j \setminus Y)}$ where $V_j = \{z \in X : z_j = 0\}$, $j = d+1, \dots, n$. In view of Remark 1 it is easy to prove that $W_{d+1} \cap \cap \dots \cap W_n = \emptyset$. Let $y \in Y'$ and let U be a neighborhood of y and ζ_j such that $\zeta_{j|U \setminus Y} \neq 0$. Let h be a generator of $I_{Y,y}$. Then we have $\zeta_j = \lambda h^m$ where λ is a unit of $\mathcal{O}_{X',y}$. Let $y' \in Y' \cap U$ be a regular point of X' and Δ a one dimensional analytic disk such that $\Delta \cap Y' = \{y'\}$. On Δ we have $h^m = \zeta_j/\lambda$ and $\zeta_j/\lambda(y') = 0$ i.e., $\zeta_j/\lambda|\Delta$ is a holomorphic function vanishing at y' and admitting a holomorphic root. This implies that $m = 1$ and therefore that ζ_j is a generator of $I_{Y',y}$. In particular $f^* I_Y$ is invertible. The statement is now a consequence of the Proposition 3.1.

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