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Some Remarks on an Operational Time Dependent Equation.

G. DA PRATO (*)

Introduction.

Let E be a complex Hilbert space and $\{A(t)\}_{t\in[0,T]}$; $\{B(t)\}_{t\in[0,T]}$ two families of linear operators (generally not bounded) in E.

Consider the Cauchy problem:

$$\begin{cases} U'(t) = A(t)\,U(t) + U(t)B^*(t) + f(t, U(t))\,, \\ U(0) = U_0\,, \end{cases}$$

where f is a mapping $[0, T]\times Q \to \mathcal{L}(E)$ and $Q\subset \mathcal{L}(E)$.

Problems of this kind arise in several fields as Optimal Control theory ([2], [3], [7], [8], [9]) and the Hartree-Fock time dependent problem in the case of finite Fermi system ([1]).

In this paper we generalize the results contained in [3] and we give some new regularity result for the case where $A(t)$ and $B(t)$ generates « hyperbolic » semi-groups.

1. The semi-group $T \to e^{tA}\,T\;e^{tB}$.

Let E be a complex Hilbert space (norm $|\;|$, inner product $(\;,\;)$). We note by $\mathcal{L}(E)$ (resp. $H(E)$) the complex (resp. real) Banach space of linear bounded (resp. hermitian) operators $E\to E$ and by $H_+(E)$ the cone of positive operators.

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Let A and B be the infinitesimal generators of two semi-groups e^{tA} and e^{tB}; we assume that:

$$(1.1) \qquad |e^{tA}| \leqslant M_A \exp(w_A t) , \qquad |e^{tB}| \leqslant M_B \exp(w_B t) .$$

We note finally by $\mathcal{L}_s(E)$ (resp. $H_s(E)$) the set $\mathcal{L}(E)$ (resp. $H(E)$) endowed by the strong topology; $\mathcal{L}_s(E)$ is a locally convex space.

Consider the following semi-group in $\mathcal{L}_s(E)$:

$$(1.2) \qquad G_t(T) = e^{tA} T e^{tB} , \qquad \forall T \in \mathcal{L}(E) , \ t \geqslant 0 ,$$

G_t is not strongly continuous in $\mathcal{L}(E)$, but it is sequentially strongly continuous in $\mathcal{L}_s(E)$, that is:

$$T_n \to T \text{ in } \mathcal{L}_s(E) \Rightarrow G_t(T_n) \to G_t(T) \text{ in } \mathcal{L}_s(E)$$

and the mapping:

$$\overline{\mathbf{R}}_+ \to \mathcal{L}_s(E) , \qquad t \to G_t(T)$$

is continuous $\forall T \in \mathcal{L}_s(E)$.

If $B = A^*$ (1) it is:

$$(1.3) \qquad G_t(T) \in H(E) , \qquad \forall T \in H(E) .$$

Put:

$$(1.4) \qquad D(L) = \left\{ T \in \mathcal{L}(E) ; \ \exists \lim_{h \to 0} \frac{1}{h} (G_h(T)x - Tx), \ \forall x \in E \right\} ,$$

$$(1.5) \qquad L(T)x = \lim_{h \to 0^+} \frac{1}{h} (G_h(T)x - Tx) , \qquad \forall T \in D(L) , \ \forall x \in E .$$

LEMMA 1.1. *If $T \in D(L)$ and $x \in D(B)$ then $Tx \in D(A)$ and it is:*

$$(1.6) \qquad L(T)x = ATx + TBx .$$

PROOF. Let $T \in D(L)$, $x \in D(B)$, $y \in D(A^*)$; it is:

$$(L(T)x, y) = \frac{d}{dh} (Te^{hB}x, e^{hA^*}y)|_{h=0} = (TBx, y) + (Tx, A^*y) .$$

(1) A^* is the adjoint of A.

It follows that the mapping:

$$D(A^*) \to \mathbf{C}\,, \qquad y \to (Tx,\, A^*y) = \big(L(T)x,\, y\big) - (TBx,\, y)$$

is continuous, $Tx \in D(A)$ and:

$$(ATx,\, y) = \big(L(T)x,\, y\big) - (TBx,\, y) \qquad \neq$$

The following proposition is clear:

PROPOSITION 1.2. *If $T \in D(L)$ then $G_t(T) \in D(L)$ and it is*:

$$L\big(G_t(T)\big) = e^{tA}L(T)\,e^{tB}\,, \tag{1.7}$$

$$\frac{d}{dt}\big(G_t(T)x\big) = e^{tA}L(T)\,e^{tB}x\,. \tag{1.8}$$

PROPOSITION 1.3. *L is closed in $\mathfrak{L}_s(E)$ and in $\mathfrak{L}(E)$.*

PROOF. Let $T_n \in D(L)$, $T_n \to T$, $S_n = L(T_n) \to S$ in $\mathfrak{L}_s(E)$; due to (1.8) it is:

$$e^{tA}\,T_n\,e^{tB}x - T_n x = \int_0^t e^{sA}\,S_n\,e^{sB}x\,ds$$

recalling the dominate convergence theorem we obtain:

$$\frac{1}{t}\,\big(G_t(T)x - Tx\big) = \frac{1}{t}\int_0^t G_s(S)x\,ds$$

it follows $T \in D(L)$ and $L(T) = S$. Therefore L is closed in $\mathfrak{L}_s(E)$ and consequently in $\mathfrak{L}(E)$. $\neq$

PROPOSITION 1.4. *$D(L)$ is dense in $\mathfrak{L}_s(E)$.*

PROOF. Put:

$$Q_t x = \frac{1}{t}\int_0^t G_s(T)x\,ds\,, \qquad \forall T \in \mathfrak{L}(E)\,,\ \forall x \in E\,,$$

it is:

$$\lim_{t\to 0^+} Q_t = I \qquad \text{in } \mathcal{L}_s(E)\,,$$

moreover

$$\frac{1}{h}\left(G_h(Q_t) - Q_t\right)x = \frac{1}{th}\left[\int_t^{t+h} - \int_0^h G_s(T)x\,ds\right]$$

it follows $D_t \in D(L)$ and therefore $D(L)$ is dense in $\mathcal{L}_s(E)$. $\neq$

PROPOSITION 1.5. *$\varrho(L) \supset\,]w_A + w_B, \infty[$ and it is* (2):

$$(1.9) \qquad R(\lambda, L)(T)x = \int_0^\infty e^{-\lambda t} e^{tA} T e^{tB} x\,dt\,, \qquad \forall x \in E\,,\ \forall \lambda > w_A + w_B\,,$$

$$(1.10) \qquad \|R(\lambda, L)\|_{\mathcal{L}(\mathcal{L}(E))} \leqslant M_A M_B (\lambda - w_A - w_B)^{-1}\,, \qquad w_A + w_B < \lambda \ (3)\,.$$

PROOF. Put

$$F(T)x = \int_0^\infty e^{-\lambda t} e^{tA} T e^{tB} x\,dt\,, \qquad \forall T \in \mathcal{L}(E)\,.$$

For every $T \in D(L)$ it is:

$$F(L(T))x = \int_0^\infty e^{-\lambda t} G_t'(T)x\,dt = (\lambda F(T) - T)x$$

moreover if $T \in \mathcal{L}(E)$ it is:

$$\frac{1}{h}\{G_h(F(T)) - F(T)\}x = \frac{e^{\lambda h} - 1}{h}\int_h^\infty e^{-\lambda t} G_t(T)x\,dt - \frac{1}{h}\int_0^h e^{-\lambda t} G_t(T)x\,dt$$

it follows

$$L(F(T))x = (\lambda F(T) - T)x\,. \qquad \neq$$

(2) If L is a linear operator, $\varrho(L)$ is the resolvent set and $R(\lambda, L)$ the resolvent of L.

(3) $\mathcal{L}(\mathcal{L}(E))$ is the Banach space of the linear bounded operators $\mathcal{L}(E) \to \mathcal{L}(E)$. We note $\|\ \|$ the norm in $\mathcal{L}(\mathcal{L}(E))$.

PROPOSITION 1.6. *If* $T_n \to T$ *in* $C([0, T]; \mathfrak{L}_s(E))$ (4) *then*

$$G_t(T_n(t)) \to G_t(T(t)) \qquad in\ C([0, T]; \mathfrak{L}_s(E)) .$$

PROOF. Let $x \in E$; for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbf{N}$, $f_1, \ldots, f_{n_\varepsilon}$ in $C([0, T])$ and $x_1, \ldots, x_{n_\varepsilon} \in E$ such that:

$$\left|e^{tB}x - \sum_{i=1}^{n_\varepsilon} f_i(t)x_i\right| < \varepsilon , \qquad \forall t \in [0, T] ,$$

it follows:

$$|G_t(T(t) - T_n(t))x| \leqslant M_B \exp(|w_B|T)|(T(t) - T_n(t))e^{tB}x| \leqslant$$
$$\leqslant M_B \exp(|w_B|T)\varepsilon(|T(t)| + |T_n(t)|) +$$
$$+ M_B \exp(|w_B|T)\sum_{i=1}^{n_\varepsilon}|f_i(t)|\,|T(x)x_i - T_n(t)x_i| .$$

Choose N such that $|T_n(t) \leqslant N|$, then:

$$|G_t(T(t) - T_n(t))x| \leqslant 2NM_B \exp(|w_B|T)\varepsilon +$$
$$+ M_B \exp(|w_B|T)\sum_{i=1}^{n_\varepsilon}|\varphi_i(t)|\,|T(x)x_i - T_n(x)x_i| .$$

Choose n'_ε such that:

$$|T(t)x_i - T_n(t)x_i| \leqslant \varepsilon/(n_\varepsilon \operatorname{Max}\{|\varphi_i|;\ i = 1, 2, \ldots, n_\varepsilon\}) , \qquad \forall n > n_\varepsilon ,$$

then

$$n > n'_\varepsilon \Rightarrow |G_t(T(t)\, H\, T_n(t))x| \leqslant (2N + 1)M_B \exp(|w_B|T)\varepsilon . \qquad \neq$$

2. The linear problem.

Let $\mathcal{A} = \{A(t)\}_{t\in[0,T]}$, $\mathcal{B} = \{B(t)\}_{t\in[0,T]}$ be two families of linear operators in E.

Let F be a Hilbert space (norm $\|\ \|$, inner product $((\ ,\ ))$) continuously and densely embedded in E.

(4) $C([0, T]; \mathfrak{L}_s(E))$ is the set of the mappings $[0, T] \to \mathfrak{L}_s(E)$ continuous; due to the Banach-Steinhaus theorem every $u \in C([0, T]; \mathfrak{L}_s(E))$ is bounded.

Let finally Z be an isometric isomorphism in $\mathcal{L}(F, E)$.
We assume:

(2.1)

a) $\mathcal{A}$ *(resp.* $\mathcal{B}$*) is* (M_A, w_A)*-stable and* w_A*-measurable (resp.* (M_B, w_B)*-stable and* w_B*-measurable) in* E [5].

b) It is $F \subset D(A(t))$ *(resp.* $D(B(t))$*),* $A(t)$ *(resp.* $B(t)$*)* $\in \mathcal{L}(F, E)$ *and* $|A(t)|$ *(resp.* $|B(t)|$*) is bounded in* $[0, T]$ [6].

c) The mapping $A(\cdot)x$ *(resp.* $B(\cdot)x$*) is continuous* $\forall x \in F$.

d) There exists a mapping H *(resp.* K*):* $[0, T] \to \mathcal{L}(E)$ *such that:*

d₁) H *(resp.* K*) is bounded in* $[0, T]$ *and strongly measurable in* E.

d₂) It is:

$$ZA(t)Z^{-1}x = A(t)x + H(t)x\,, \qquad \forall x \in D(A(t))\,,$$
$$ZB(t)Z^{-1}x = B(t)x + K(t)x\,, \qquad \forall x \in D(B(t))\,.$$

If [2.1) is fulfilled it is known ([4], [6]) that there exists an evolution operator $G_A(t, s)$ (resp. $G_B(t, s)$) for the problem:

$$u' = A(t)u\,, \quad u(0) = x \qquad (\text{resp. } u' = B(t)u\,, \quad u(0) = x)\,.$$

Moreover G_A (resp. G_B): $\Delta = \{(t, s) \in [0, T]^2;\ t \geq s\} \to \mathcal{L}(E)$ is strongly continuous and $G(r, s) \in \mathcal{L}(F)$.

Finally it is:

$$(2.2) \qquad \begin{cases} \lim\limits_{n\to\infty} G_{A,n} = G_A\,, \\ \lim\limits_{n\to\infty} G_{B,n} = B_B\,, \end{cases} \qquad \text{in } C(\Delta; \mathcal{L}_s(E))\,,$$

(5) $\mathcal{A}$ is w_A-measurable in E if $\varrho(A(t)) \supset\,]w_A, +\infty[$ and $R(\lambda, A(\cdot))$ is strongly measurable $\forall \lambda > w_A$.

$\mathcal{A}$ is (M_A, w_A)-stable in E if $\varrho(A(t)) \supset\,]w_A, +\infty[$ and it is:

$$\left|\prod_{i=1}^{k} R(\lambda, A(t_i))\right| \leq M_A/(\lambda - w_A)^k$$

$\forall k \in \mathbf{N}$, $t_1 \geq t_2 \geq \ldots \geq t_k$, $t_i \in [0, T]$, $i = 1, \ldots, n$.

(6) With the topology of $\mathcal{L}(F, E)$.

where $G_{A,n}$ (resp. $G_{B,n}$) is the evolution operator associated to the problem:

$$u_n' = A_n(t)u_n\,, \quad u_n(0) = x \qquad (\text{resp. } u_n' = B_n(t)u_n\,, \quad u_v(0) = x)$$

where $A_n(t) = n^2R(n, A(t)) - n$ and $B_n(t) = n^2R(n, B(t)) - n$.

Consider now the problem:

$$(2.3)\quad \begin{cases} T'(t) = A(t)T(t) + T(t)B^*(t) + F(t)\,, & F \in C([0, T]; \mathcal{L}_s(E))\,, \\ T(0) = T_0 \in \mathcal{L}(E)\,. \end{cases}$$

We define $L(t)$ as in (1.4), (1.5) and write (2.3) in the following form:

$$(2.4)\quad \begin{cases} T'(t) = L(t)(T(t)) + F(t)\,, \\ T(0) = T_0\,. \end{cases}$$

We consider also the approximate problem:

$$(2.5)\quad \begin{cases} T_n'(t) = L_n(t)(T(t)) + F(t)\,, \\ T_n(0) = T_0\,, \end{cases}$$

where $L_n(t)(T) = A_n(t)T + TB_n^*(t)$.

We say that T is a *strong solution* of (2.4) if there exists:

$$(2.6)\quad \{T_k\} \subset D(L(t)) \cap C^1([0, T]; \mathcal{L}_s(E))\ (^7)$$

such that:

$$\begin{cases} T_k' - L(T_k) \to F & \text{in } C([0, T]; \mathcal{L}_s(E))\,, \\ T_k(0) \to T_0 & \text{in } \mathcal{L}(E)\,. \end{cases}$$

If $T \in D(L(t)) \cap C^1([0, T]; \mathcal{L}_s(E))$ and (2.4) is fulfilled we say that T is a *classical solution* of (2.4).

THEOREM 2.1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two family of linear operators in* E *verifying* (2.1). *Then for every* $T_0 \in \mathcal{L}(E)$ *and* $F \in C([0, T]; \mathcal{L}_s(E))$ *the*

(7) $C^1([0, T]; \mathcal{L}_s(E))$ is the set of the mappings $[0, T] \to \mathcal{L}_s(E)$ strongly continuously differentiable.

problem (2.4) *has a unique strong solution given by*:

$$T(t)x = G_A(t,0)T_0G_B^*(t,0)x + \int_0^t G_A(t,s)F(s)G_B^*(t,s)x\,ds\,. \tag{2.7}$$

If $T_0 \in \mathcal{L}(F)$ *and* $F \in C([0,Y];\mathcal{L}_s(F))$ *then the solution* T *is classical.*

PROOF. Let first $T_0 \in \mathcal{L}(F)$ and $F \in C([0,T];\mathcal{L}_s(F))$; in this case we can easily verify that T is a classical solution.

In the general case by approximating T_0 and F we can show that $T(t)$, given by (2.7) is a strong solution.

Assume finally that T is a strong solution of (2.4) and take $\{T_k\}$ as in (2.6). Put $F_k = T_k' - L(T_k)$; it is:

$$\frac{d}{ds}\big(G_A(t,s)T_k(s)G_B(t,s)x\big) = G_A(t,s)F_k(s)G_B(t,s)x\,, \qquad \forall x \in E\,,$$

by integration in $[0,t]$ it follows:

$$T_k(t)x = G_A(t,0)T_k(0)G_B(t,0)x + \int_0^t G_A(t,s)F_k(s)G_B(t,s)x\,ds$$

and, taking the limit for $k \to \infty$, the conclusion follows.

3. The quasi-linear problem.

Let Q a closed convex set in $\mathcal{L}(E)$ and f a strongly continuous mapping

$$f\colon [0,T]\times Q \to \mathcal{L}(E)\,, \qquad (t,S) \to f(t,S)\,.$$

Consider the problems:

$$\begin{cases} U'(t) - L(t)\big(U(t)\big) + f\big(t, U(t)\big) = 0\,, \\ U(0) = U_0\,, \end{cases} \tag{3.1}$$

$$\begin{cases} U_n'(t) - L_n(t)\big(U(t)\big) + f\big(t, U_n(t)\big) = 0\,, \\ U_n(0) = U_0\,. \end{cases} \tag{3.2}$$

We say that U is a *strong solution* of (3.1) if there exists $\{U_k\} \subset$ $\subset D(L(t)) \cap C^1([0, T]; \mathcal{L}_s(E))$ such that:

$$\begin{cases} U'_k - L(U_k) + f(t, U_k) \to 0 & \text{in } C([0, T]; \mathcal{L}_s(E)) , \\ U_k(0) \to U_0 & \mathcal{L}_s(E) . \end{cases}$$

If U belongs to $D(L(t)) \cap C^1([0, T]; \mathcal{L}_s(E))$ and fulfils (3.1) we say that U is a *classical solution* of (3.1).

The following proposition is an immediate consequence of the Theorem 2.1.

PROPOSITION 3.1. *U is a strong solution of* (3.1) *if and only if it is*:

$$(3.3) \qquad U(t)x = G_A(t, 0)\, U_0 B^*_B(t, 0)x - \int_0^t G_A(t, s) f(s, U(s)) G^*_B(t, s) x\, ds .$$

We remark now that $C([0, T]; \mathcal{L}_s(E))$ is not a metric space, but we can define in it the following norm:

$$(3.4) \qquad \|U\| = \operatorname{Sup}\{|U(t)|, t \in [0, T]\} , \qquad \forall U \in C([0, T]; \mathcal{L}_s(E)) ,$$

by virtue of the Banach-Steinhaus theorem.

$C([0, T]; \mathcal{L}_s(E))$ endowed by the norm (3.4) is a Banach space which we note by $B([0, T]; \mathcal{L}_s(E))$.

LEMMA 3.2. *Let K be a closed subset of $B([0, T]; \mathcal{L}_s(E))$ and γ_n, γ mappings $K \to K$. Assume that*:

$$(3.5) \qquad \|\gamma_n(U) - \gamma_n(V)\| \leqslant \alpha \|U - V\| , \qquad \alpha \in]0, 1[, \ U, V \in K ,$$

$$(3.6) \qquad \gamma_n(U) \to \gamma(U) \qquad \text{in } C([0, T]; \mathcal{L}_s(E)) , \ \forall U \in K .$$

Then there exists $\{U_n\}$ and U unic in K such that:

$$(3.7) \qquad \gamma_n(U_n) = U_n , \qquad \gamma(U) = U ,$$

$$(3.8) \qquad U_n \to U \text{ in } C([0, T]; \mathcal{L}_s(E)) .$$

PROOF. By virtue of the contractions principle there exists U_n and U such that (3.7) is fulfilled.

To prove (3.8) fix Z in K; it is:

$$U_n = \lim_{m\to\infty} \gamma_n^m(Z)\,, \qquad U = \lim_{m\to\infty} \gamma^m(Z) \qquad \text{in } B([0,T];\mathfrak{L}_s(E))$$

and

$$\|U_n - \gamma_n^m(Z)\| \leqslant \frac{\alpha^m}{1-\alpha}\left(\|\gamma_n(Z)\| + \|Z\|\right)$$

therefore there exists $M>0$ such that:

$$\|U_n - \gamma_n^m(Z)\| \leqslant M\alpha^m\,. \tag{3.9}$$

It is easy to show that:

$$\lim_{n\to\infty} \gamma_n^m(U) = \gamma^m(U) \qquad \text{in } C([0,T];\mathfrak{L}_s(E))\,,\ \forall U\in K\,,\ m\in\mathbf{N} \tag{3.10}$$

if $x\in E$ and $t\in[0,T]$ it follows:

$$|U(t)x - U_n(t)x| \leqslant |U(t)x - \gamma^m(Z)(t)x| + \\ + |\gamma^m(Z)(t) - \gamma_n^m(Z)(t)x| + |U_n(t)x - \gamma_n^m(Z)(t)x|$$

due to (3.9) it follows:

$$|U(t)x - U_n(t)x| \leqslant 2M\alpha^m|x| + |\gamma^m(Z)(t)x - \gamma_n^m(Z)(t)x|$$

and the conclusion follows from (3.10). $\neq$

We prove now the existence of the maximal solution for the problem (3.1).

We assume:

$$\left\{\begin{array}{l} a)\ f\in C([0,T]\times Q_s\,(^8);\ \mathfrak{L}_s(E))\cap C([0,T]\times Q;\ \mathfrak{L}(E)),\\ b)\ \exists\mu\colon \mathbf{R}_+\to\mathbf{R}_+ \textit{ such that:}\\ \quad |f(t,T)-f(t,S)\mathbf{Z}\leqslant\mu(r)|T-S| \textit{ if } |T|\leqslant r,\ |S|\leqslant r\,,\\ c)\ \exists\alpha\colon \mathbf{R}_+\to\mathbf{R}_+ \textit{ such that:}\\ \quad r>0\,,\ |T|\leqslant r\,,\ T\in Q\,,\ \beta\in]0,\alpha(r)[\ \Rightarrow T-\beta f(t,T)\in Q\,. \end{array}\right. \tag{3.11}$$

We remark that c) is trivial if $Q=\mathfrak{L}_s(E)$ or $H(E)$.

(8) Q_s is endowed by the topology of $\mathfrak{L}_s(E)$.

LEMMA 3.3. *Assume that:*

i) $\mathcal{A}$ *and* $\mathcal{B}$ *verify* (2.1), $U_0 \in Q$,

ii) $T \in Q \Rightarrow \exp(sA(t))\,T\exp(sB(t)) \in Q$, $\forall t \in [0, T]$,

iii) f *verifies* (3.11).

Take α, β *such that:*

$$\begin{cases} a \geqslant M_A M_B \exp\big((|w_A| + |w_B|)\,T\big)|U_0|\,, \\ \beta \leqslant \alpha(2a)\,. \end{cases}$$

Then there exists $\tau > 0$ *such that the problem* (3.1) *has a unique strong solution in* $[0, \tau]$.

PROOF. Put:

$$\varphi(t, T) = T - \beta f(t, T)$$

then φ maps $[0, T]\times\big(Q \cap P(0, 2a)\big)$ in Q (9) and it is:

$$|\varphi(t, T) - \varphi(t, S)| \leqslant (1 + \beta\mu(2a)(|T - S|\,, \quad \forall T, S \in Q \cap P(0, 2a)\,. \tag{3.13}$$

Problem (3.1) is equivalent to:

$$\begin{cases} U' - L(t)\,U + \dfrac{1}{\beta}\,\varphi(t, U) = 0\,, \\ U(0) = U_0\,, \end{cases} \tag{3.14}$$

put $U = \exp(-t/\beta)\,V$, then it is:

$$V(t)x = G_A(t, 0)\,U_0 G_B^*(t, 0)x + \frac{1}{\beta}\int_0^t e^{s/\beta} G_A(t, s)\varphi\big(s, U(s)\big)\,G_B^*(t, s)x\,ds \tag{3.15}$$

which is equivalent to the equation:

$$U(t) = G_A(t, 0)\,U_0 G_B^*(t, 0)\,e^{-t/\beta} + \frac{1}{\beta}\int_0^t e^{-(t-s)/\beta} G_A(t, s)\varphi\big(s, U(s)\big)\,G_B^*(t, s)\,ds = \gamma(U)(t)\,. \tag{3.16}$$

(9) $P(0, r) = \{T \in \mathfrak{L}(E);\ |T| \leqslant r\}$.

It is:

$$\|\gamma(U)-\gamma(V)\| \leqslant M_A M_B \exp\Big((|w_A|+|w_B|)T\Big)(1-e^{-t/\beta})\|U-V\| ,$$
$$\forall U, V \in C\Big([0,\tau]; (Q\cap P(0,2a))_s\Big)$$

and

$$\|\gamma(U)\| \leqslant a + M_A M_B \exp\Big((|w_A|+|w_B|)T\Big)(1+\mu(2a))2a +$$
$$+ \operatorname{Sup}\{|\varphi(t,0)|,\, t\in[0,T]\}(1-e^{-t/\beta}) .$$

Therefore there exists $\tau>0$ such that γ is a contraction in

$$C\Big([0,\tau]; (Q\cap P(0,2a))_s\Big) . \qquad \neq$$

The following theorem is an immediate consequence of Lemma 3.3, Proposition 1.6, Lemma 3.2 and standard arguments.

THEOREM 3.4. *Assume that $\mathcal{A}$, $\mathcal{B}$, f verify the hypotheses of Lemma* 3.3. *Then there exists the maximal solution U of the problem* (3.1). *If I is the interval where U is defined it is*:

$$U_n \to U \qquad in \qquad C(I, \mathcal{L}_s(E))$$

U_n being the solution of (3.2). *Finally if $\|U\|$ is bounded it is* $I=[0,T]$.

PROPOSITION 3.5. *Assume that the hypotheses of Theorem* 3.4 *are fulfilled. Assume moreover*:

$$(3.18)\qquad \begin{cases} \text{i) } M_A = M_B = 1 , \\ \text{ii) } \exists \omega_1 \in \mathbb{R} \textit{ such that} \\ |T| \leqslant |T+\alpha(f(t,T)-f(t,0)+\omega_1 T)| , \\ \qquad\qquad \forall \alpha \geqslant 0 ,\ t\in[0,T] ,\ T\in Q . \end{cases}$$

Then the maximal solution of (3.1) *verifies the following inequality*:

$$(3.19)\qquad |U(t)| \leqslant \exp\big((w_A+w_B+\omega_1)t\big)|U_0| + \\ + \int_0^t \exp\big((w_A+w_B+\omega_1)(t-s)\big)|f(s,0)|\, ds .$$

PROOF. We remember (Kato [5]) that (3.18)-ii) is equivalent to:

$$\langle f(t, T) - f(t, 0), \Gamma \rangle \geqslant -\omega_1 |T| \,, \qquad \forall \Gamma \in \partial |T| \,, \tag{3.20}$$

$\partial |T|$ being the sub-differential of the norm in $\mathfrak{L}(E)$.
Due to (3.18) for every $T \in D(L(s))$ there exists $\Gamma \in \partial |T|$ such that

$$\langle L(s)(T), \Gamma \rangle \leqslant (w_A + w_B) |T| \,. \tag{3.21}$$

Suppose first that U is a classical solution of (3.1); then

$$\frac{d^-}{dt} |U(t)| = \inf \{\langle U(t), \Gamma \rangle, \Gamma \in \partial |U(t)|\} \leqslant \tag{3.22}$$

$$\leqslant \langle L(t)(U(t)), \Gamma \rangle - \langle f(t, U(t)) - f(t, 0), \Gamma \rangle + \langle f(t, 0), \Gamma \rangle$$

if we take Γ such that

$$\langle L(t)(U(t)), \Gamma \rangle \leqslant (w_A + w_B) |U(t)|$$

it is

$$\frac{d^-}{dt} |U(t)| \leqslant (w_A + w_B + \omega_1) |U(t)| + |f(t, 0)| \tag{3.13}$$

which implies (3.19). If U is a strong solution the conclusion follows by approximation. $\neq$

4. Regularity.

If for every $V \in \mathfrak{L}(F)$ it is $f(t, V) \in \mathfrak{L}(F)$ we put

$$f_Z(t, V) = Z f(t, Z^{-1} V Z) Z^{-1} \,.$$

THEOREM 4.1. *Assume that the hypotheses of Theorem* 3.4 *are fulfilled. Moreover assume that f maps* $[0, T] \times \mathfrak{L}(F)$ *in* $\mathfrak{L}(F)$ *and that f_Z verifies* (3.11); *then if* $U_0 \in \mathfrak{L}(E) \cap \mathfrak{L}(F)$ *the maximal solution of* (3.1) *is classical and* $U(t) \in \mathfrak{L}(F)$, $\forall t \in [0, T]$.

Proof. Consider the problems:

$$(4.1)\quad \begin{cases} V'(t) = \big(A(t) + H(t)\big)\, V(t) + V(t)\big(B(t) + K(t)\big) + f_z(t, V)\,, \\ B(0) = Z U_0 Z^{-1}\,, \end{cases}$$

$$(4.2)\quad \begin{cases} V_n'(t) = \big(A_n(t) + H_n(t)\big)\, V_n(t) + \\ \qquad\qquad + V_n(t)\big(B_n(t) + K_n(t)\big) + Z f_n(t, U_n) Z^{-1}\,, \\ V_n(0) = Z U_0 Z^{-1}\,, \end{cases}$$

where

$$(4.3)\quad \begin{cases} H_n(t) = n^2 R\big(n, A(t)\big) H(t) R\big(n, A(t)\big) + H(t)\,, \\ K_n(t) = n^2 R\big(n, B(t)\big) K(t) R\big(n, B(t)\big) + K(t)\,. \end{cases}$$

By virtue of Theorem 3.4 the problems (4.1) and (4.2) have maximal solutions in $[0, \tau[$, τ being the maximal time for U; moreover

$$V_n \to V \qquad \text{in } C\big([0, \tau[;\, \mathcal{L}_s(E)\big)\,.$$

It is easy to see that $V_n = Z U_n Z^{-1}$, therefore

$$U_n \to U \quad \text{in } C\big([0, \tau[;\, \mathcal{L}_s(E)\big)\,, \qquad Z U_n Z^{-1} \to V \quad \text{in } \mathcal{L}_s(E)$$

it follows $U \in \mathcal{L}(F)$, $V = Z U Z^{-1}$. $\neq$

Remark. If A and B are independent of t we have the following result (cf. [3]).

Theorem 4.2. *Assume that the hypotheses of Theorem* 3.4 *are fulfilled. Suppose moreover that* $f \in C^1\big([0, T], \mathcal{L}_s(E)\big)$ *and* $U_0 \in D(L)$.
Then the maximal solution of (3.1) *is classical.*

5. Exemples.

1) Let $f \in C^2(\mathbf{R})$, put:

$$(5.1)\qquad f(T) = \int_{-\infty}^{+\infty} f(\lambda)\, dE_\lambda\,, \qquad \forall T \in H(E)\,,$$

E_λ being the spectral projector attached to T.

If we choose $Q = H(E)$, $B = A$ then f fulfils (3.11) (cf. Tartar [8]) and (3.1) has a unique maximal solution.

Assume now

$$Q = \{T \in H(E);\ a \leqslant T \leqslant b\}\,, \qquad a, b \in \mathbf{R}\,. \tag{5.2}$$

LEMMA 5.1. *If* $f(a) \leqslant 0$ *and* $f(b) \geqslant 0$ *then* $\forall r > 0$, $\exists \beta_r > 0$ *such that*:

$$|x| \leqslant r\,, \qquad x \geqslant a\,, \qquad \beta \in]0, \beta_r[\ \Rightarrow x - \beta f(x) \geqslant a\,. \tag{5.3}$$

PROOF. If $f(a) < 0$ the thesis is evident. Assume $f(a) = 0$; then it is $f(x) = (x - a)\psi(x)$ and if $x \geqslant a$ it is

$$x - a - \beta f(x) = (x - a)\big(1 - \beta\psi(x)\big) \geqslant 0$$

for suitable β. $\neq$

The following proposition is now evident

PROPOSITION 5.2. *Assume that* (2.1) *is fulfilled with* $B = A$. *Assume moreover that* $f \in C^2(R)$, $f(a) \leqslant 0$, $f(b) \geqslant 0$. *Then if* $a \leqslant U_0 \leqslant b$ *there exists a unique global solution* U *such that* $a \leqslant U(t) \leqslant b$.

2) *Riccati equation.*

Assume $Q = H_+(E)$, $B = A$, $|e^{tA}| \leqslant 1$ and (2.1) fulfilled; assume $f(T) = TPT - F(t)$ where $P \geqslant 0$, $F(t) \geqslant 0$; then it is easy to see that f verifies (3.11); therefore (3.1) has a maximal solution in Q. Moreover it is

$$|T| \leqslant |T + \alpha TPT|\,, \qquad \forall \alpha > 0\,,\ \forall T > 0\,,$$

because

$$\big((T + \alpha TPT)x, x\big) \geqslant (Tx, x)$$

therefore if $U_0 \geqslant 0$ (3.1) has a global solution.

Finally assume $P \in \mathcal{L}(F)$, put $\overline{P} = ZPZ^{-1}$ then $f_z(V) = V\overline{P}V$ and the hypotheses of the Theorem 4.1 are fulfilled and the solution is classical.

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