Rendiconti del Seminario Matematico della Università di Padova

David P. Ellerman

Gian-Carlo Rota

A measure theoretic approach to logical quantification

Rendiconti del Seminario Matematico della Università di Padova, tome 59 (1978), p. 227-246

<http://www.numdam.org/item?id=RSMUP_1978__59__227_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1978, tous droits réservés.

L'accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

REND. SEM. MAT. UNIV. PADOVA, Vol. 59 (1978)

A Measure Theoretic Approach to Logical Quantification.

DAVID P. ELLERMAN (*) - GIAN-CARLO ROTA (**)

1. Introduction.

It is an old philosophical thesis that logic is, in some sense, a limiting case of probability theory. However, this thesis has not given rise to a mathematical treatment of quantification theory (i.e., first-order logic) by the methods of probability theory because of certain measure-theoretic difficulties. The principal difficulty is that it is impossible to put a *positive* finitely-additive real-valued probability measure on an uncountable set X (i.e., on the power-set Boolean algebra of X). If there was such a measure μ on a set X, then the existential quantification of a unary predicate on X could be obtained as the support of the expectation of the indicator or characteristic function of the predicate and the quantification of n-ary relations could be similarly treated by means of conditional expectation operators. However, if X is uncountable, then consider the sets $X_n = \{x \in X | \mu(\{x\}) > 1/n\}$ where n is a positive integer. Since $X = \bigcup_n X_n$ and since a countable union of countable sets is countable, at least one of the X_n is uncountable. But that contradicts $\mu(X) = 1$, so such a positive measure μ is impossible. A non-negative finitely-additive measure (e.g., an ultrafilter) can always be defined on any set X, but if the measure is not positive, then it ignores certain non-empty subsets

(*) Indirizzo dell'A.: University of Massachusetts, Boston, Mass. U.S.A.
(**) Indirizzo dell'A.: Massachusetts Institute of Technology, Cambridge, Mass. U.S.A. Work sponsored by NSF Grant No. MCS 78-02743.

and thus it does not permit a faithful treatment of logical quantification.

This difficulty can be circumvented by associating with each Boolean algebra B, a ring $G(B)$, called the *generator ring* of B, and a universal measure $|\ |: B \to G(B)$. The measure is universal in the sense that any finitely-additive measure $\mu: B \to R$, with values in any ring R (all rings are taken to be commutative with unity), factors through the universal measure by a unique ring homomorphism $\hat{\mu}: G(B) \to R$. In addition to allowing a measure-theoretic approach to logic, generator rings and the associated universal positive measures should be of use in probability theory itself.

It has long been known (e.g., Wright [12]) that there is an analogy between averaging operators such as the conditional expectation operators of probability theory and the algebraic (existential) quantifiers in Halmos' theory of polyadic algebras (Halmos [5]) or the cylindrifications used by Tarski and his co-workers in the theory of cylindric algebras (Henkin, Monk, and Tarski [7]). All these operators satisfy an averaging condition which has the general form: $A(f)\cdot A(g) = A\cdot(f\cdot A(g))$. We will provide some theoretical underpinning for this « analogy » by constructing the logical quantifiers using an abstract rendition of conditional expectation operators on a ring of simple random variables.

The propositional operations will be treated ring-theoretically using the *valuation rings* defined by Rota ([10] and [11]). The valuation rings of Boolean algebras might be viewed by logicians as a generalization of Boolean rings which occur as a trivial special case and they might be viewed by probability theorists as an intrinsic algebraic construction which generalizes the rings of simple random variables found in probability theory.

2. Valuation rings.

Let L be a distributive lattice with maximal element u and minimal element z, and let A be a ring (always commutative with unity). Let $F(L, A)$ be the free A-module on the elements of L which consists of all the finite formal sums $\sum_i a_i x_i$ for $a_i \in A$ and $x_i \in L$. A ring structure is put upon $F(L, A)$ by defining multiplication as $x\cdot y = x\wedge y$ for lattice elements $x, y \in L$ and then extending by linearity to all the elements of $F(L, A)$. Let J be the submodule generated by all the elements of the form $x\vee y + x\wedge y - x - y$ for $x, y \in L$.

LEMMA. J is an ideal.

PROOF. Since J is a submodule, it suffices to check that J is closed under multiplication by lattice elements w. Now

$$w(x\vee y+x\wedge y-x-y)=w\wedge(x\vee y)+w\wedge(x\wedge y)-w\wedge x-w\wedge y\,.$$

Since L is a distributive lattice, we have

$$w\wedge(x\vee y)=(w\wedge x)\vee(w\wedge y)\quad\text{and}\quad w\wedge(x\wedge y)=(w\wedge x)\wedge(w\wedge y)\,.$$

Hence

$$w(x\vee y+x\wedge y-x-y)=$$
$$=(w\wedge x)\vee(w\wedge y)+(w\wedge x)\wedge(w\wedge y)-w\wedge x-w\wedge y$$

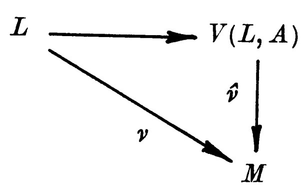
which is a generator of J.

The *valuation ring of L with values in A* is then defined as

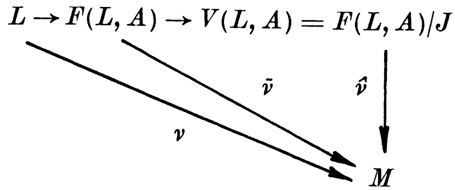
$$V(L,A)=F(L,A)/J\,.$$

A *valuation* on L with values in an abelian group G is a function $\nu\colon L\to G$ such that $\nu(x\vee y)+\nu(x\wedge y)=\nu(x)+\nu(y)$ for all x and y in L. The injection $L\to(L,A)$ is a valuation and it is universal for valuations on L with values in an A-module.

THEOREM 1. Let M be an A-module and let $\nu\colon L\to M$ be a valuation. Then there exists a unique linear transformation (i.e., A-module homomorphism) $\hat{\nu}\colon V(L,A)\to M$ such that the following diagram commutes.



PROOF. By the universality property of the free module $F(L,A)$, there is a unique linear transformation $\bar{\nu}\colon F(L,A)\to M$ such that the left-hand triangle in the following diagram commutes.



Since ν is a valuation, the kernal of $\bar{\nu}$ contains J, so $\bar{\nu}$ extends to the unique linear transformation $\hat{\nu}$: $V(L, A) \to M$ as desired.

Corollary. The valuations on L with values in A are in one-to-one correspondence with the linear functionals on $V(L, A)$. The construction $L \to V(L, A)$ defines a functor from the category of distributive lattices (with u and z, and (u, z)-preserving lattice homomorphisms) to the category of A-algebras.

For any x in L, $u \cdot x = u \wedge x = x$ so the maximal element u of L serves as the unity of the valuation ring $V(L, A)$ (although the minimal element z is not the zero of the ring). In the situation of Theorem 1, if the A-module M is also an A-algebra and if $\nu(x \wedge y) = \nu(x) \cdot \cdot \nu(y)$ for all x and y in L, then the factor map $\hat{\nu}$ is an A-algebra homomorphism. The map $L \to A$ which carries each lattice element to 1 is a valuation satisfying that multiplicative condition, so there is an A-algebra homomorphism ε: $V(L, A) \to A$ which takes $\sum_i a_i x_i$ to $\sum_i a_i$. Hence $V(L, A)$ is an augmented algebra with the augmentation ε. Since $z \cdot \sum_i a_i x_i = \sum_i a_i (z \wedge x_i) = z \sum a_i$, the minimal element z of the lattice functions in the valuation ring as an integral which « computes » the augmentation $\varepsilon(\sum a_i x_i) = \sum a_i$ of any ring element.

As in any augmented algebra, there is another natural multiplication that can be put on the A-module structure of $V(L, A)$ in order to obtain a ring, i.e., $f \vee g = \varepsilon(g) \cdot f + \varepsilon(f) g - f \cdot g$ for all f and g in $V(L, A)$. This join notation for the dual multiplication is appropriate since if f and g are lattice elements, then

$$\varepsilon(g) f + \varepsilon(f) g - fg = f + g - f \wedge g = f \vee g$$

is the join of f and g. In the dual valuation ring, endowed with the $\vee$-multiplication, the roles of u and z are reversed, i.e., z is the unit and u is the integral.

The dual role of the « meet » and « join » multiplications is not the only aspect of Boolean duality which extends to arbitrary valuation rings. Even if the distributive lattice L is not complemented, a *complementation* endomorphism $\tau\colon V(L, A) \to V(L, A)$ can be defined on the valuation ring by $\tau(f) = \varepsilon(f)(u + z) - f$ for any f in $V(L, A)$. If x is a lattice element, then $\tau(x) = u + z - x$ so $\tau(u) = z$ and $\tau(z) = u$. Moreover if x does have a complement x' in L, then $x \vee x' + x \wedge x' - x - x' = u + z - x - x' =$ in $V(L, A)$ so $\tau(x) = x'$. Complementation is also idempotent in the sense that $\tau^2(f) = f$ for any f in $V(L, A)$. Let us denote the valuation ring with the usual « meet » multiplication by $(V(L, A), \wedge)$ and let $(V(L, A), \vee)$ denote the ring with the « join » multiplication. Then complementation

$$\tau\colon \ (V(L, A), \wedge) \to (V(L, A), \vee)$$

is an isomorphism of augmented algebras and, since τ^2 is the identity mapping,

$$\tau\colon \ (V(L, A), \vee) \to (V(L, A), \wedge)$$

is also an isomorphism.

A valuation ν is *normalized* if $\nu(z) = 0$. Given a valuation ring $V(L, A)$, the ring obtained by setting z equal to zero, i.e., $V(L, A)/(z)$, will be called the *normalized valuation ring*. If « normalized valuation » is substituted throughout for « valuation », then Theorem 1 will describe the universality property enjoyed by normalized valuation rings. Boolean algebras, when constructed as Boolean rings, occur as a rather special case of normalized valuation rings. When L is a Boolean algebra B and $A = 2$ $(= Z_2)$, then $(V(B, 2), \wedge)/(z)$ is simply B constructed as a Boolean ring with « meet » multiplication and $(V(B, 2), \vee)/(u)$ is B as a Boolean ring with « join » multiplication. For a history of the two interpretations of a Boolean algebra as a Boolean ring, with either the « meet » or the « join » multiplication, consult Church [11], pp. 103-104. In this special case of $L = B$ and $A = 2$, the complementation isomorphisms given above reduce to the familiar De Morgan's laws, i.e., $(x \wedge y)' = x' \vee y'$ and $(x \vee y)' = x' \wedge y'$. The generalization of Boolean duality to arbitrary valuation rings is due to Ladnor Geissinger, whose papers ([2], [3] and [4]) should be consulted for further analysis and applications of valuation rings.

In view of the above isomorphism, we can ignore the « join » multiplication, and always consider $V(L, A)$ as being endowed with the usual « meet » multiplication. We have heretofore refrained from normalizing the valuation ring because it is only the unnormalized valuation ring which enjoys the extensive duality theory given above. However, for a variety of reasons we will henceforth consider all valuation rings as being normalized unless otherwise specified. Hence $V(L, A)$ will now denote the *normalized* valuation ring on L with values in A. The normalization eliminates the augmentation, the « join » multiplication and the idempotent endofunction of complementation. However when a lattice element x has a complement x' in L, then $u - x = x'$.

3. Propositional calculus.

The conventional algebraic treatment of the propositional calculus utilizes the free Boolean algebra $B = B(P)$ which is free on the set P of propositional variables. We will formulate a treatment in the general setting of the (normalized) valuation ring $V(B, A)$, whose A is any commutative ring—instead of the special case of $V(B, 2) \simeq B$ (where $2 = Z_2$). Instead of constructing the free Boolean algebra $B = B(P)$ and then $V(B, A)$, we will give a direct characterization of $V(B, A)$ for a free Boolean algebra B.

For any (commutative) ring A and any set P, let $A[P]$ be the polynomial ring over A with the elements of P as indeterminates. Let I be the ideal generated by the polynomials $p^2 - p$ for all p in P. Then $A[P]/I$ will be called an *idempotent polynomial ring* since it behaves like a polynomial ring except that the indeterminates are all idempotent.

CHARACTERIZATION THEOREM. The normalized valuation rings on free Boolean algebras are precisely the idempotent polynomial rings, *i.e.* for any commutative ring A (with unity) and any set P,

$$V(B(P), A) \cong A[P]/I .$$

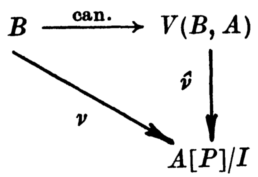
PROOF. Let B_0 be the set of idempotents in $A[P]/I$. They form a Boolean algebra with the Boolean operations of $f \vee g = f + g - fg$, $f \wedge g = fg$, and $f' = 1 - f$ for f, g in B_0. Since $P \subseteq B_0$, by the universality property of the free Boolean algebra $B = B(P)$, there exists a unique Boolean algebra homomorphism $\nu \colon B \to B_0 \subseteq A[P]/I$ which commutes with the insertion of P. Since ν is a Boolean algebra homomorphism, we have

$$\nu(z) = 0 , \qquad \nu(x \wedge y) = \nu(x) \wedge \nu(y) = \nu(x) \cdot \nu(y) ,$$

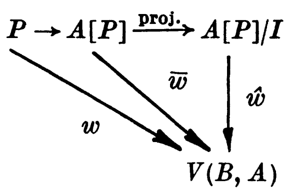
and

$$\nu(x \vee y) = \nu(x) \vee \nu(y) = \nu(x) + \nu(y) - \nu(x)\cdot\nu(y) = \nu(x) + \nu(y) - \nu(x \wedge y)$$

for any x and y in B. Hence ν is a normalized multiplicative valuation on B with values in the A-algebra $A[P]/I$, so by Theorem 1 (for normalized multiplicative valuations), there exists a unique A-algebra homomorphism $\hat{\nu}$ such that the following diagram commutes;

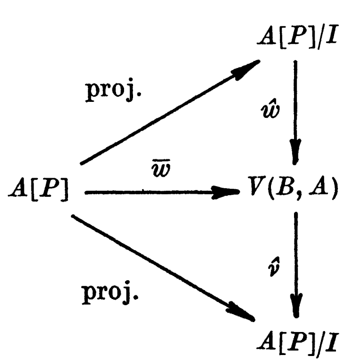


Let $w\colon P \to V(B, A)$ be the insertion of the elements of P into the A-algebra $V(B, A)$. Then by the universality property of $A[P]$ (e.g., Lang [8], p. 113), there exists a unique A-algebra homomorphism $\overline{w}$ such that the left-hand triangle in the following diagram commutes.



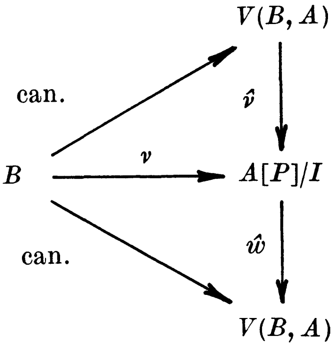
Then for any p in P, $\overline{w}(p^2 - p) = \overline{w}(p)^2 - \overline{w}(p) = p^2 - p = p - p = 0$ so $I \subseteq \ker(\overline{w})$. Hence there exists a unique A-algebra homomorphism $\hat{w}$ such that $\overline{w} = \hat{w}\cdot\text{proj}$, i.e., such that the right-hand triangle commutes.

Now consider the following diagram.



We have just seen that the upper triangle commutes. To see that the lower triangle commutes, note that $\hat{\nu}\cdot\overline{w}$ and proj are both A-algebra homomorphisms $A[P] \to A[P]/I$ which commute with the insertion of P. By the universality property of $A[P]$, there is only one such map so $\hat{\nu}\cdot\overline{w} = \text{proj}$. Since $\overline{w} = \hat{w}\cdot\text{proj}$, we have that $\text{proj} = \hat{\nu}\cdot\overline{w} = \hat{\nu}\cdot\hat{w}\cdot\text{proj}$, i.e., the outer triangle commutes. By the universality property of quotient rings, the identity is the unique A-algebra homomorphism $A[P]/I \to A[P]/I$ which commutes with the projections, so $\hat{\nu}\cdot\hat{w}$ is the identity on $A[P]/I$.

It remains to consider the following diagram.



We have seen that the upper triangle commutes. To see that the lower triangle commutes, note that when the A-algebra homomorphism $\hat{w}$ is restricted to B_0, the Boolean algebra of idempotents in $A[P]/I$, then it constitutes a Boolean algebra homomorphism into the Boolean algebra of idempotents of $V(B, A)$. Hence $\hat{w}\cdot\nu$ and can. are both Boolean algebra homomorphisms $B \to V(B, A)$ which agree on the insertion of P. By the universality property of the free Boolean algebra $B = B(P)$, there is only one such map so $\hat{w}\cdot\nu = \text{can.}$, i.e. the lower triangle commutes. Since $\nu = \hat{\nu}\cdot\text{can.}$, we have that $\text{can.} = \hat{w}\cdot\nu = \hat{w}\cdot\hat{\nu}\,\text{can.}$, i.e., the outer triangle commutes. By the universality property of $V(B, A)$, the identity is the unique A-algebra homomorphism $V(B, A) \to V(B, A)$ which commutes with the normalized multiplicative (canonical) valuation can., so $\hat{w}\cdot\hat{\nu}$ is the identity on $V(B, A)$. Hence $V(B, A) \simeq A[P]/I$.

COROLLARY. $B(P) \simeq Z_2[P]/I$.

This characterization theorem reduces much of propositional logic (e.g., the completeness theorem) to elementary polynomial algebra.

The elements of P will be construed as propositional variables, and the elements of the idempotent polynomial ring $A[P]/I \cong V(B(P), A)$ will be called *Boolean polynomials*. The Boolean operations are rendered as ring operations in the following manner: for $f, g \in V(B(P), A)$, $f \vee g = f + g - fg$, $f \wedge g = fg$, $f \supset g = u - f + fg$, and $-f = u - f$, where u is the unity in $V(B(P), A) \cong A[P]/I$. A Boolean polynomial f is said to be a *theorem* if $f = u$ and is said to be *refutable* if $f = 0$ (see Halmos [6], pp. 45-46 for a justification of these definitions in the case of $A = Z_2 = 2$). In any axiomatization of the propositional calculus, each axiom is a Boolean polynomial equal to unity, and if $f = u$ and $f \supset g = u$, then $g = u$.

A *truth-table* or *t-t valuation* is a function $v: P \to 2$. A t-t valuation induces a Boolean algebra homomorphism $Z_2[P]/I \cong B(P) \to 2$. In general, for any (non-zero) commutative ring A with unity, we can view a t-t valuation v as taking values in $2 \subseteq A$. Hence it induces an A-algebra homomorphism $\bar{v}: A[P] \to A$ which vanishes on the generators $p^2 - p$ of the ideal I. Thus we have the induced A-algebra homomorphism $\hat{v}: A[P]/I \cong V(B(P), A) \to A$. A Boolean polynomial f is said to be a *tautology* if $\hat{v}(f) = 1$, or a *contradiction* if $\hat{v}(f) = 0$, for any t-t valuation $v: P \to 2$. Since any induced $\hat{v}$ is an A-algebra homomorphism, all theorems are tautologies and all refutable polynomials are contradictions.

COMPLETENESS THEOREM FOR PROPOSITIONAL LOGIC. For any $f \in V(B(P), A) \cong A[P]/I$, if f is a tautology, then $f = u$.

PROOF. Since f is a tautology if and only if $u - f$ is a contradiction, we will prove the equivalent proposition: if f is a contradiction, then $f = 0$. Since f is actually an equivalence class of polynomials in $A[P]/I$, we consider any representative $f \in A[P]$. The proof is by induction over n, the number of propositional variables in f, which we assume for the sake of notational convenience to be $p_1, \ldots, p_n$.

Basis step: $n = 1$ so $f = f(p_1)$. For any t-t valuation $v: P \to 2$, $\bar{v}(f) = f(\bar{v}(p_1))$. Since f is a contradiction, $f(0) = 0$ so the constant term in the polynomial $f(p_1)$ is zero. Then p_1 may be factored out to obtain $f(p_1) = p_1 \cdot g(p_1)$. Now $f(1) = 1 \cdot g(1) = 0$ so $p_1 - u$ divides $g(p_1)$, i.e., for some $h(p_1)$, $g(p_1) = (p_1 - u)h(p_1)$. Hence

$$f(p_1) = p_1(p_1 - u)h(p_1) = (p_1^2 - p_1)h(p_1) \qquad \text{so } f = 0 \text{ in } A[P]/I\,.$$

Induction step: $f = f(p_1, \ldots, p_n)$ where we assume the theorem for polynomials of less than n variables. We note first that there is a polynomial in the equivalence class of f where p_n occurs with degree 1. For example, any monomial in the form $g(p_1, \dots, p_{n-1})p_n^m$ for $m \geq 2$ is equivalent modulo $p_n^2 - p_n$ to $g(p_1, \dots, p_{n-1})p_n^{m-1}$ since $gp_n^m - gp_n^{m-1} = gp_n^{m-2}(p_n^2 - p_n)$. In this manner, each occurrence of p_n in f can be reduced to an occurrence of degree 1 without changing the equivalence class. Hence we may assume that f has the form

$$f(p_n, \dots, p_n) = f_0(p_1, \dots, p_{n-1}) + f_1(p_1, \dots, p_{n-1})p_n$$

where p_n does not occur in f_0 or f_1. For any t-t valuation ν such that $\nu(p_n) = 0$, we have $\bar{\nu}(f) = \bar{\nu}(f_0) = 0$. But p_n does not occur in f_0, so $\bar{\nu}(f_0) = 0$ for all t-t valuations ν. Since f_0 is a contradiction with less than n variables, we have, by the induction hypothesis, $f_0 = 0$ in $A[P]/I$. For any ν such that $\nu(p_n) = 1$, $\bar{\nu}(f) = \bar{\nu}(f_0) + \bar{\nu}(f_1) = \bar{\nu}(f_1) = 0$. But p_n does not occur in f_1 either so it is a contradiction. Hence f_1 and thus $f = f_0 + f_1 p_n$ is zero in $A[P]/I \cong V(B(P), A)$.

In view of this approach to propositional logic, it would seem quite possible and appropriate to continue in the same spirit to generalize the algebraitization of first-order logic which has been developed by Tarski and by Halmos. The Boolean algebras used by Tarski and Halmos would be replaced by valuation rings, and their quantifiers or cylindrification operators on Boolean algebras would be generalized to similar operators on valuation rings. However, such an abstract axiomatic approach to quantification would not fulfill our goal in this paper of providing a measure-theoretic treatment of the logical quantifiers. Hence we shall herein follow the alternative course of defining universal positive measures so that the logical quantifiers can be constructed, by measure-theoretic methods, in their natural habitat (i.e. when they quantify over a set).

4. Generator rings.

Let B_0 be a Boolean algebra and let A and R be commutative rings with unity. A *measure* on B_0 is a map $\mu: B_0 \to R$ such that for all b, b_1, and b_2 in B_0;

(1) $$\mu(u) = 1 .$$

(2) $$\mu(b) + \mu(b') = 1 ,$$

and

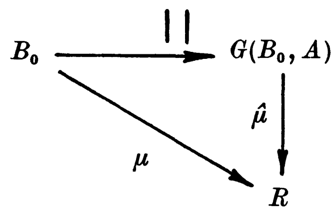
$$\mu(b_1 \vee b_2) + \mu(b_1 \wedge b_2) = \mu(b_1) + \mu(b_2) \,. \tag{3}$$

Let $A[B_0]$ be the polynomial ring generated over the ring A by taking all the elements of B_0 as indeterminates (denoted $|b|$ for $b \in B_0$), and let K be the ideal generated by the polynomials

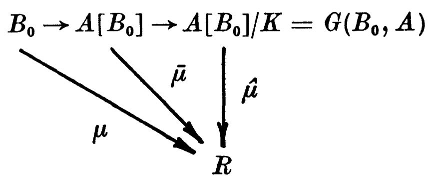
$$\{1 - |u|,\ 1 - |b| - |b'|,\ |b_1 \vee b_2| + |b_1 \wedge b_2| - |b_1| - |b_2|\colon \text{ for } b,\, b_1,\, b_2 \in B_0\} \,.$$

Then $G(B_0, A) = A[B_0]/K$ is called the *generator ring on* B_0 *over* A. The insertion of the generators $B_0 \xrightarrow{|\,|} G(B_0, A)$ is a measure on B_0 and it is universal for all measures on B_0 with values in an A-algebra.

THEOREM 2. Let R be an A-algebra and let $\mu\colon B_0 \to R$ be a measure. Then there is a unique A-algebra homomorphism $\hat{\mu}\colon G(B_0, A) \to R$ such that the following diagram commutes;



PROOF. By the universality property of polynomial rings, there is a unique A-algebra homomorphism $\bar{\mu}\colon A[B_0] \to R$ such that the left-hand triangle in the following diagram commutes.



Since μ is a measure, $\bar{\mu}$ vanishes on K, so there is a unique A-algebra homomorphism $\hat{\mu}\colon G(B_0, A) \to R$ such that the right-hand triangle commutes.

The construction of $G(B_0, A)$ from B_0 also defines a functor from the category of Boolean algebras to the category of A-algebras. The normalized valuation $B_0 \to V(B_0, A)$ is a measure on B_0 with values in an A-algebra, so there is a canonical map $\varphi_{B_0}\colon G(B_0, A) \to V(B_0, A)$ which is a surjective A-algebra homomorphism. The quotient map φ_{B_0} « interpretes » the multiplication of generators $|b_1|\cdot|b_2|$ in $G(B_0, A)$ as being their intersection $b_1 \wedge b_2$ in $V(B_0, A)$. The maps $\{\varphi_{B_0}\colon B_0$ a Boolean algebra$\}$ constitute a natural transformation $\varphi\colon G(\cdot, A) \to \to V(\cdot, A)$ from the generator ring functor to the (normalized) valuation ring functor.

If there is a positive cone (semi-ring) C_A defined in A so that A is partially ordered by the relation $a_1 \leqslant a_2$ if $a_2 - a_1 \in C_A$, then we may extend the partial ordering to $G(B_0, A)$ by defining the positive cone C as the semi-ring generated over C_A by the « absolute values » $|b|$ for b in B_0 (i.e., close $C_A \cup \{|b|\colon b \in B_0\}$ under sums and products). Then $G(B_0, A)$ is a partially ordered ring with the ordering $x_1 \leqslant x_2$ if $x_2 - - x_1 \in C$. Let $C^+ = C - \{0\}$ so that $x_1 < x_2$ iff $x_2 - x_1 \in C^+$. Hence, if A is equipped with a positive cone, then there is an induced ordering on $G(B_0, A)$ so that the universal measure $B_0 \to G(B_0, A)$ is positive.

As an example of the use of generator rings in probability theory, we consider expectation operators on rings of simple random variables. If B_0 is a Boolean algebra of events on a (sample) space X and if R is the reals, then the (normalized) valuation ring $V(B_0, R)$ is the ring of simple real random variables on X. A random variable X in $V(B_0, R)$ can always be written in the form $x = \sum r_i p_i$ where $r_i \in R$ and $\{p_i\}$ is a finite partition of B_0. Given a real-valued measure $\mu\colon B_0 \to R$ (i.e., a finitely additive probability measure), the *expectation operator with respect to* μ, $E_\mu\colon V(B_0, R) \to V(B_0, R)$, is defined by

$$E_\mu\left(\sum r_i p_i\right) = \sum r_i \mu(p_i) u\ .$$

There is a different expectation operator E_μ for every different measure μ. However, by using the generator ring $G(B_0, R)$, we may define a *universal expectation operator*

$$E\colon\ V(B_0, G(B_0, R)) \to V(B_0, G(B_0, R)) \qquad \text{by } E\left(\sum \alpha_i p_i\right) = \sum \alpha_i |p_i| u\ .$$

For any measure $\mu\colon B_0 \to R$, the particular expectation operator E_μ can then be obtained by « specializing » the universal operator in the following manner:

$$\begin{array}{ccc} V(B_0, G(B_0, R)) & \xrightarrow{\ E\ } & V(B_0, G(B_0, R)) \\ \cup\!\!| & & \big\downarrow V(B_0, \hat{\mu}) \\ V(B_0, R) & \dashrightarrow^{E_\mu} & V(B_0, R)\ . \end{array}$$

Universal expectation operators can be used to treat the quantification of unary predicates. Since certain universal conditional expectation operators will be used to treat the quantification of n-ary relations, we will first consider such operators in probability theory. If B_1 and B_2 are Boolean algebras, then their *tensor product* $B_1 \otimes B_2$ is the Boolean algebra obtained by taking the ring-theoretic tensor product (over Z_2) of B_1 and B_2 as Z_2-algebras. The tensor product $B_1 \otimes B_2$ is generated by the elements $p \otimes q$ for $p \in B_1$ and $q \in B_2$, and each element of $B_1 \otimes B_2$ can be expressed as a sum of disjoint generators. The (Boolean) sum obeys the rules $(p_1 + p_2) \otimes q = p_1 \otimes q + p_2 \otimes q$ and $p \otimes (q_1 + q_2) = p \otimes q_1 + p \otimes q_2$. Generators are multiplied « componentwise », i.e., $p_1 \otimes q_1 \cdot p_2 \otimes q_2 = p_1 p_2 \otimes q_1 q_2$, so $p_1 \otimes q_1 \leqslant p_2 \otimes q_2$ if $p_1 \leqslant p_2$ and $q_1 \leqslant q_2$. Also if z is the minimal element in each algebra, then $z(p \otimes q) = (zp) \otimes q = p \otimes (zq) =$ minimal element of $B_1 \otimes B_2$. If $\mu_1 \colon B_1 \to R$ and $\mu_2 \colon B_2 \to R$ are measures with values in a ring R, then the map $\mu_1 \times \mu_2 \colon B_1 \otimes B_2 \to R$ defined by $\mu_1 \times \mu_2(p \otimes q) = \mu_1(p) \cdot \mu_2(q)$ (and extended additively to all of $B_1 \otimes B_2$) is a measure on $B_1 \times B_2$ called the *product measure*. If B_1 and B_2 are Boolean algebras of subsets of the sets X_1 and X_2 respectively, then $B_1 \otimes B_2$ is (isomorphic to) the Boolean algebra of subsets of $X_1 \times X_2$ that is generated by the rectangular sets $p \times q$ for p in B_1 and q in B_2. In this instance, we would identify $p \otimes q$ with the subset $p \times q$ so that $B_1 \otimes B_2$ would be a Boolean subalgebra of the power set Boolean algebra $\mathfrak{P}(X_1 \times X_2)$.

We now consider the tensor power $B_0 \otimes B_0$ and the (normalized) valuation ring $V(B_0 \otimes B_0, R)$. If $\pi_1 = \{p_1, \ldots, p_m\}$ and $\pi_2 = \{q_1, \ldots, q_n\}$ are two partitions of B_0, then their *tensor product* $\pi_1 \otimes \pi_2 = \{p_i \otimes q_j\}_{i,j}$ is a partition of $B_0 \otimes B_0$. Every element in the valuation ring $V(B_0 \otimes B_0, R)$ can be taken as being defined on a partition of $B_0 \otimes B_0$, and each partition of $B_0 \otimes B_0$ is refined by a partition of the form $\pi_1 \otimes \pi_2$. Hence we will assume that each element in $V(B_0 \otimes B_0, R)$ is presented in the form

$$f = \sum_{i,j} r_{ij} p_i \otimes q_j$$

for some partitions $\pi_1 = \{p_1, \ldots, p_m\}$ and $\pi_2 = \{q_1, \ldots, q_n\}$ of B_0—so that the coefficients r_{ij} will form an $m \times n$ matrix of reals.

If B_0 is a Boolean algebra of subsets (events) of a space X, then the elements of $V(B_0 \otimes B_0, R)$ are simple random variables on the product space $X \times X$. If μ_1 and μ_2 are finitely additive probability measures on B_0, then $\mu_1\times\mu_2$ is such a measure on $B_0\otimes B_0$. Given a product measure $\mu_1\times\mu_2$, a conditional expectation operator on the valuation ring is determined by a *conditioning subalgebra* B of $B_0\otimes B_0$. We will only consider the conditioning subalgebras which are « independent » of specified coordinates. Thus $u\otimes B_0$ is the subalgebra of elements of the form $u\otimes q$ for q in B_0 and $B_0\otimes u$ is the subalgebra of elements $p\otimes u$ for p in B_0. We will consider the subalgebra $u\otimes B_0$ which ignores the first coordinate since the other coordinates would be treated similarly. The conditional expectation operator

$$E_{\mu_1\times\mu_2}(\cdot\,|u\otimes B_0):\ V(B_0\otimes B_0, R)\rightarrow V(B_0\otimes B_0, R)$$

is then defined by

$$E_{\mu_1\times\mu_2}:\Big(\sum_{i,j} r_{ij}p_i\otimes q_j|u\otimes B_0\Big)=\sum_j\Big[\sum_i r_{ij}\mu_1(p_i)\Big]u\otimes q_j$$

since each value of this linear operator is independent of the first coordinate and since it does not involve μ_2, we may simplify the notation to $E_{1,\mu_1}(\cdot)$. Hence if $f=\sum_{i,j} r_{ij}p_i\otimes q_j$, then $E_{1,\mu_1}(f)$ is the random variable whose value on $u\otimes q_j$ is the average $\sum_i r_{ij}\mu_1(p_i)$ of the j-th column in the $m\times n$ matrix of coefficients $[r_{ij}]$.

By using the generator ring $G(B_0, R)$, we may define the *universal conditional expectation operator*

$$E(\cdot\,|u\otimes B_0)=E_1(\cdot):\ V(B_0\otimes B_0, G(B_0, R))\rightarrow V(B_0\otimes B_0, G(B_0, R))$$

by

$$E_1\Big(\sum_{i,j}\alpha_{ij}p_i\otimes q_j\Big)=\sum_j\Big[\sum_i\alpha_{ij}|p_i|\Big]u\otimes q_j\,.$$

The particular operator E_{1,μ_1} is then obtained as;

$$\begin{array}{ccc} V(B_0\otimes B_0, G(B_0, R)) & \xrightarrow{E_1} & V(B_0\otimes B_0, G(B_0, R)) \\ \cup\!\mid & & \downarrow V(B_0\otimes B_0, \hat{\mu}_1) \\ V(B_0\otimes B_0, R) & \xrightarrow{E_{1,\mu_1}} & V(B_0\otimes B_0, R)\,. \end{array}$$

5. The logical quantifiers.

We will consider the standard case where there is a countably infinite set of variables indexed by the positive integers Z^+. The tensor power of a Boolean algebra B_0 over the index set Z^+ can be constructed as a direct union. Identify $\bigotimes_{i=1}^{n} B_0$ as a subalgebra of $\bigotimes_{i=1}^{n+1} B$ by identifying $p_1 \otimes \ldots \otimes p_n$ with $p_1 \otimes \ldots \otimes p_n \otimes u$ for any $p_1, \ldots, p_n$ in B_0. Then the infinite tensor power B_0^∞ is constructed as the direct union

$$B_0^\infty = \bigcup_{n \in Z^+} \bigotimes_{i=1}^{n} B_0 .$$

It is generated by the elements $\bigotimes_{n \in Z^+} p_n$ where all but a finite number of the p_n are equal to u, and each element in B_0^∞ can be expressed as a sum of disjoint generators.

Quantification over certain « rectangular » relations can be treated using universal conditional expectation generators. In treating logic, there is no need to restrict the ring of coefficients in the generator ring to the reals. Hence we will use the generator ring $G(B_0, A)$ where A is any commutative ring with unity. It is convenient for calculations if each element in the (normalized) valuation ring $V(B_0^\infty, G(B_0, A))$ is taken as being defined on some partition of B_0^∞ of the form $\bigotimes_{n \in Z^+} \pi_n$ where each π_n is a finite partition of B_0 and all but a finite number of the partitions π_n are equal to the maximal partition $\{u\}$ (so there are only a finite number of blocks in the partition $\bigotimes_n \pi_n$). For the sake of expository simplicity we will restrict attention to two or three coordinates. A typical element f in the valuation ring $V(B_0^\infty, G(B_0, A))$ has the form

$$f = \sum_{i,j,\ldots,k} \alpha_{ij\ldots k} p_i \otimes q_j \otimes \ldots \otimes r_k \otimes u \otimes \ldots .$$

The universal conditional expectation operator

$$E_1(\cdot) = E(\cdot \,|u \otimes B_0 \otimes B_0 \ldots): \; V(B_0^\infty, G(B_0, A)) \to V(B_0^\infty, G(B_0, A))$$

is then defined by

$$E_1(f) = \sum_{j,\ldots,k} \Big[\sum_i a_{ij\ldots k}|p_i|\Big] u \otimes q_i \otimes \ldots \otimes r_k \otimes u \otimes \ldots .$$

The other operators $E_n(\cdot)$ would be defined in the same manner. With these operators, as with those previously considered, it must be verified that they are well-defined, but we will skip the details.

In any valuation ring $V(B, R)$, where B is a Boolean algebra and R is a ring, if $\{b_i\}$ is a partition of B, then the *support* of an element $f = \sum_i r_i b_i$ is defined as $\operatorname{Supp}(f) = \sum_{r_i \neq 0} b_i$. The *co-support* of an element f is defined as $\operatorname{Co-Supp}(f) = \sum_{r_i = 1} b_i$. The support and the co-support of any element $f = \sum r_i b_i$ are both Boolean algebra elements $\big($i.e., $\bigvee_{r_i \neq 0} b_i$ and $\bigvee_{r_i = 1} b_i$ respectively$\big)$ in the image of the insertion $B \to V(B, R)$. The co-support operator could also be defined in terms of the support operator, i.e., $\operatorname{Co-Supp}(f) = u - \operatorname{Supp}(u - f)$.

The universal conditional expectation operators E_n are all *linear* operators which are *idempotent* in the sense that $E_n(E_n(f)) = E_n(f)$. Moreover, the operators *commute* in the sense that $E_m(E_n(f)) = E_n \cdot \cdot (E_m(f))$. The existential and universal quantification operators are defined by taking the support and co-support respectively of the conditional expectation operators. For any positive integer n, the operator $\exists_n \colon V(B_0^\infty, G(B_0, A)) \to V(B_0^\infty, G(B_0, A))$ for *existential quantification over the n-th variable* is defined by $\exists_n(f) = \operatorname{Supp}(E_n(f))$ for any f in $V(B_0^\infty, G(B_0, A))$. The operator $\forall_n$ for *universal quantification over the n-th variable* is defined by $\forall_n(f) = \operatorname{Co-Supp}(E_n(f))$. Since $\forall_n(f)$ is equal to $1 - \operatorname{Supp}(1 - E_n(f)) = 1 - \operatorname{Supp}(E_n(1 - f)) = 1 - - \exists_n(1 - f)$, where $1 = u \otimes u \otimes \ldots$, we will henceforth deal only with the existential quantifier.

We now consider a simple concrete example. Let B_0 be a Boolean algebra of subsets of X and let $p_1, q_1 \neq u, z$ be elements of B_0. Let $p_2 = p_1'$ and $q_2 = q_1'$ so that $\pi_1 = \{p_1, p_2\}$ and $\pi_2 = \{q_1, q_2\}$ are two non-trivial partitions of B_0. Let $R = (p_1 \times q_1) \cup (p_2 \times q_2) \cup (p_1 \times q_2)$ so that R is a binary relation on X that is the union of three disjoint rectangular relations. The « formula » $R(v_1, v_2)$, where v_1 and v_2 are the first two variables, would then be represented by the element

$$R(v_1, v_2) = p_1 \otimes q_1 \otimes u \otimes \ldots + p_2 \otimes q_1 \otimes u \otimes \ldots + p_1 \otimes q_2 \otimes u \otimes \ldots$$

in the valuation ring $V(B_0^\infty, G(B_0, A))$. Then we have

$$E_1(R(v_1, v_2)) = [|p_1| + |p_2|]u\otimes q_1\otimes u\otimes \ldots + |p_1|u\otimes q_2\otimes u\otimes \ldots .$$

Since $|p_1| + |p_2| = 1$ and $0 \neq |p_1| \neq 1$,

$$\exists_1 R(v_1, v_2) = \sum_{j=1}^{2} u\otimes q_j\otimes u\otimes \ldots \qquad \text{and} \qquad \forall_1 R(v_1, v_2) = u\otimes q_1\otimes u\otimes \ldots .$$

The primary limitation of the machinery developed so far is that it can only accomodate n-ary relations which (like the binary relation R) can be « paved » or « tiled » with disjoint (generalized) rectangles. The analogous problem arises in measure theory where a notion of area in R^2 or « content » in R^n can be easily defined for rectangles in R^2 or generalized rectangles in R^n. The content of certain irregularity shaped sets must then be defined as the limit of the contents of the approximating paved sets (e.g., Loomis and Sternberg [9], p. 331). We will utilize an algebraic version of this old Archimedian technique of measuring irregularity shaped objects by approximating with certain regularity shaped objects.

Our purpose is to give an abstract measure theoretic treatment of the logical quantifiers in their natural setting, i.e., when they quantify over some universe of discourse X. While the constructions will be abstract and algebraic, it will be convenient to henceforth give a set theoretical interpretation to the Boolean algebras used. Let B_0 be the power set Boolean algebra on some set X. We now wish to define the Boolean algebra B which is the Boolean algebra component of the cylindric set algebra associated with the full relational structure of all finitary relations on X (e.g., Henkin, Monk, and Tarski [7], pp. 9-10). Let X^{Z^+} be the set of sequences in X, let $R \subseteq X^n$, and let $v_{i_1}, \ldots, v_{i_n}$ be any n variables. Then the graph of the formula $R(v_{i_1}, \ldots, v_{i_n})$ is the set of sequences $x = (x_1, x_2, \ldots)$ such that the n-tuple $(x_{i_1}, \ldots, x_{i_n})$ is in R. The x_j for $j \neq i_1, \ldots, i_n$ are arbitrary. Let B be the Boolean algebra of all subsets of X^{Z^+} which, for some n, are the graphs of formulas $R(v_{i_1}, \ldots, v_{i_n})$ for some n-ary relation $R \subseteq X^n$ and some set of f variables $v_{i_1}, \ldots, v_{i_n}$. We will identify a B_0^∞ element $p\otimes q\otimes \ldots\otimes r\otimes u\otimes \ldots$ with the set of sequences $p\times q\times \ldots \times r\times u\times \ldots$ so that B_0^∞ is a subalgebra of B and $V(B_0^\infty, G(B_0, A))$ is a subring of $V(B, G(B_0, A))$. B is not a complete Boolean algebra, but each element of B is the supremum of the B_0^∞ elements below it.

Let $\{b_i\}$ be a finite partition of B and let $f = \sum \alpha_i b_i$ be an element in $V(B) = V(B, G(B_0, A))$ (where we may assume that the non-zero α_i are distinct). Then an element in $V(B_0^\infty) = V(B_0^\infty, G(B_0, A))$ of the form $\alpha \cdot p \otimes q \otimes \ldots \otimes r \otimes u \otimes \ldots$ will be called a *rectangular approximation* to f if for some i, $p \otimes q \otimes \ldots \otimes r \otimes u \otimes \ldots \leqslant b_i$ and $\alpha = \alpha_i$. For any element f in $V(B)$, let $RA(f)$ be the set of rectangular approximations to f. If we apply the expectation operator E_n to all the rectangular approximations in $RA(f)$ and then apply the support operator, the result will be a set of Boolean algebra elements from $B_0^\infty \subseteq V(B_0^\infty)$ whose supremum exists in $B \subseteq V(B)$. Hence we define the *existential quantifier* (on the n-th variable)

$$\exists_n\colon V(B) \to V(B) \quad \text{by} \quad \exists_n(f) = \operatorname{Sup}\Big(\operatorname{Supp}\big(E_n(RA(f))\big)\Big).$$

The *universal quantifier* (on the n-th variable) $\forall_n\colon V(B) \to V(B)$ is defined by

$$\forall_n(f) = \operatorname{Sup}\Big(\text{Co-Supp}\big(E_n(RA(f))\big)\Big).$$

If $f = R(v_{i_1}, \ldots, v_{i_n})$, then it is easily seen that these algebraically defined operators give the correct results from the set theoretic viewpoint.

We have utilized the Boolean algebra B which includes the graphs of formulas $R(v_{i_1}, \ldots, v_{i_n})$ for *all* finitary relations R on X. In general, a cylindric set algebra would not treat all relations on the universe X. The machinery can be easily adapted to the general case where B^* is the Boolean algebra component of any locally finite (diagonal-free) cylindric set algebra of dimension ω over the universe X (see Henkin, Monk, and Tarski [7], pp. 164-166). The only difference is that B_0^∞ is not necessarily a subalgebra of B^* so the order condition $p \otimes q \otimes \ldots \otimes r \otimes u \otimes \ldots \leqslant b_i$ in the definition of rectangular approximation would have to be formulated in B (or the subalgebra of B generated by B_0^∞ and B^*). The quantifiers $\exists_n, \forall_n\colon V(B^*) \to V(B^*)$ could then be defined as before since the existence of the required suprema in B^* is guaranteed by B^* being the Boolean algebra component of a cylinderic set algebra.

It remains to show how any finite transformation of variables can be accomplished. In the theory of polyadic algebras, substitution operators are included in the structure of a polyadic algebra. In the theory of cylindric algebras, the equality (or diagonal) relations are included in the structure of a cylindric algebra and then the result of substituting the j-th variable for the i-th variable in f is defined to be $\exists_i(f \wedge v_i = v_j)$. In a locally-finite algebra of infinite dimension, any finite transformation of variables can be obtained by an appropriate product of substitutions. Hence we only need to define $S(i/j)$: $V(B) \to V(B)$ such that $S(i/j)(f)$ is the result of substituting the j-th variable for the i-th variable in any free occurrence in f. A rectangular approximation has the general form $\alpha \cdot p \otimes q \otimes \ldots \otimes r \otimes u \otimes \ldots$ where p is the first component, q is the second component, and so forth. We will say that a rectangular approximation is *square in the i-th and j-th components* if the i-th and j-th component are the same element of B_0. Let $RA_{(i,j)}(f)$ be the set of rectangular approximations to the $V(B)$ element f which are square in the i-th and j-th components. The *substitution operator* $S(i/j)$: $V(B) \to V(B)$, for any distinct positive integers i and j, is defined by

$$S(i/j)(f) = \text{Sup}\left(\text{Supp}\left(E_i(RA_{(i,j)}(f))\right)\right) .$$

For example, if $f = R(v_1, v_2, \ldots, v_n)$, then $S(1/2)(f)$ would be the B-element which is the set of sequences $x = (x_1, x_2, \ldots)$ such that the n-tuple $(x_2, x_2, x_3, \ldots, x_n)$ was in R, i.e., the graph of the formula $R(v_2, v_2, v_3, \ldots, v_n)$.

BIBLIOGRAPHY

[1] A. Church, *Introduction to Mathematical Logic*, Princeton University Press, Princeton, 1956.

[2] L. Geissinger, *Valuations on Distributive Lattices I*, Arch. Math., **24** (1973), pp. 230-239.

[3] L. Geissinger, *Valuations on Distributive Lattices II*, Arch. Math., **24** (1973), pp. 337-345.

[4] L. Geissinger, *Valuations on Distributive Lattices III*, Arch. Math., **24** (1973), pp. 475-481.

[5] P. Halmos, *Algebraic Logic*, Chelsea Publishing C., New York, 1962.

[6] P. Halmos, *Lectures on Boolean Algebras*, Van Nostrand Co., Princeton, 1963.

[7] L. Henkin - D. Monk - A. Tarski, *Cylindric Algebras Part I*, North-Holland Publishing Co., Amesterdam, 1971.

[8] S. Lang, *Algebra*, Addison-Wesley Publishing C., Reading, Mass., 1965.

[9] L. LOOMIS - S. STERNBERG, *Advanced Calculus*, Addison-Wesley Publishing Co., Reading, Mass., 1968.
[10] G.-C. ROTA, *On the Combinatorics of the Euler Characteristic*, Studies in Pure Mathematics, edited by L. MIRSKY, Academic Press, London, (1971), pp. 221-233.
[11] G.-C. ROTA, *The Valuation Ring of a Distributive Lattice*, Proc. Univ. of Houston Lattice Theory Conf., Houston (1973), pp. 575-628.
[12] F. B. WRIGHT, *Convergence of Quantifiers and Martingales*, Illinois J. of Math., **6** (1962), pp. 296-307.

Manoscritto pervenuto in redazione il 1° dicembre 1978.