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L. CARLITZ

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L. CARLITZ (*)

1. Introduction.

The Stirling numbers of the first and second kind can be defined by

$$x(x+1)\dots(x+n-1) = \sum_{k=0}^{n} S_1(n,k)x^k \tag{1.1}$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\dots(x-k+1)\,, \tag{1.2}$$

respectively. Since $S_1(n, n-k)$ and $S(n, n-k)$ are polynomials in n of degree $2k$, it follows readily that

$$S_1(n, n-k) = \sum_{j=0}^{k-1} S'_1(k,j)\binom{n}{2k-j} \qquad (k>0) \tag{1.3}$$

and

$$S(n, n-k) = \sum_{j=0}^{k-1} S'(k,j)\binom{n}{2k-j} \qquad (k>0)\,. \tag{1.4}$$

The coefficients $S'_1(k,j)$, $S'(k,j)$ were introduced by Jordan [8, Ch. 4] and Ward [12]; the present notation is that of [2]. The numbers are closely related to the *associated* Stirling numbers of Riordan [10, Ch. 4].

(*) Indirizzo dell'A.: Department of Mathematic - Duke University - Durhan, N.C. 27706.

The Stirling numbers and the associated Stirling numbers are related in various ways. We have

$$(1.5)\qquad \begin{cases} S_1(n, n-k) = \sum\limits_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S(j+k, j)\,, \\ S(n, n-k) = \sum\limits_{j=0}^{k} \binom{k-n}{k+j}\binom{k+n}{k-j} S_1(j+k, j)\,, \end{cases}$$

and

$$(1.6)\qquad \begin{cases} S'(n, k) = \sum\limits_{j=0}^{k} (-1)^j \binom{n-j-1}{k-j} S_1'(n, j)\,, \\ S_1'(n, k) = \sum\limits_{j=0}^{k} (-1)^j \binom{n-j-1}{k-j} S'(n, j)\,. \end{cases}$$

Also

$$(1.7)\qquad \begin{cases} S_1(n, n-k) = \sum\limits_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-j} S'(k, j)\,, \\ S(n, n-k) = \sum\limits_{j=0}^{k-1} (-1)^j \binom{n+k-j-1}{2k-1} S_1'(k, j)\,, \end{cases} \qquad (k > 0)\,.$$

The first half of (1.5) is due to Schläfli [11]; the second was proved by Gould [7].

The writer [3] has defined another triangular array of numbers that is closely related to $(S_1(n, k))$ and $(S(n, k))$. Analogous to (1.3) and (1.4) we have

$$(1.8)\qquad S_1(n, n-k) = \sum_{j=1}^{k} B_1(k, j) \binom{n+j-1}{2k} \qquad (k > 0)\,,$$

and

$$(1.9)\qquad S(n, n-k) = \sum_{j=1}^{k} B(k, j) \binom{n+j-1}{2k} \qquad (k > 0)\,.$$

The coefficients $B_1(k, j)$, $B(k, j)$ are positive integers and satisfy the recurrences

$$(1.10)\qquad B_1(k, j) = jB_1(k-1, j) + (2k-j)B_1(k-1, j-1)\,, \quad (k > 1)$$

and

$$(1.11)\qquad B(k, j) = \\ = (k-j+1)B(k-1, j) + (k+j-1)B(k-1, j+1)\,, \quad (k > 1)\,,$$

respectively. Moreover

$$B_1(k, j) = B(k, k-j+1), \qquad (1 \leqslant j \leqslant k). \tag{1.12}$$

The writer [1] proved (1.5) by making use of the representations

$$\begin{cases} S_1(n, n-k) = \binom{k-n}{k} B_k^{(n)}, \\ S(n, n-k) = \binom{n}{k} B_k^{(-n+k)}, \end{cases} \tag{1.13}$$

where $B_n^{(z)}$ is the Nörlund polynomial [9, Ch. 6] defined by

$$\left(\frac{x}{e^x - 1}\right)^z = \sum_{n=0}^{\infty} B_n^{(z)} \frac{x^n}{n!}. \tag{1.14}$$

(The polynomial $B_n^{(n)}$ should not be confused with the Bernoulli polynomial $B_n(z)$.) Incidentally it follows from (1.13) that

$$S_1(-n+k, -n) = S(n, n-k). \tag{1.15}$$

In a recent paper [6] the writer showed that the above results can be generalized considerably in the following way. Let $\{f_k(z)\}$ denote a sequence of polynomials such that

$$\deg f_k(z) = k; \qquad f_k(0) = 0, \qquad (k > 0). \tag{1.16}$$

Put

$$\begin{cases} F_1(n, n-k) = \binom{k-n}{k} f_k(n), \\ F(n, n-k) = \binom{n}{k} f_k(-n+k). \end{cases} \tag{1.17}$$

Then all the above results generalize. The proof makes use of two arrays $(G_1(k, j))$, $(G(k, j))$ that generalize $(B_1(k, j))$, $(B(k, j))$, respectively. They are defined by

$$\begin{cases} F_1(n, n-k) = \sum_{j=1}^{k} G_1(k, j) \binom{n+j-1}{2k}, \\ F(n, n-k) = \sum_{j=1}^{k} G(k, j) \binom{n+j-1}{2k}, \end{cases} \tag{1.18}$$

and satisfy

$$G_1(k, j) = G(k, k-j+1)\,, \qquad (1 \leqslant j \leqslant k)\,. \tag{1.19}$$

Incidentally it follows from (1.17) that

$$F_1(-n+1, -n) = F(n, n-k)\,; \tag{1.20}$$

$F_1(x, y)$, $F(x, y)$ are defined for arbitrary x, y such that $x-y$ is equal to a nonnegative integer.

In order to generalize the orthogonality relations for the Stirling numbers, an additional condition on $\{f_k(z)\}$ is assumed, namely that

$$(\varphi(\chi))^z = \sum_{k=0}^{\infty} f_k(z)\, x^k/k! \tag{1.21}$$

for some

$$\varphi(x) = 1 + \sum_{n=1}^{\infty} c_n x^n/n!\,.$$

It is proved, using (1.21) that

$$\sum_{k=j}^{n} (-1)^{n-k} F_1(n, k) F(k, j) = \sum_{k=j}^{n} (-1)^{k-j} F(n, k) F_1(k, j) = \delta_{nj}\,. \tag{1.22}$$

In the present paper we generalize the definition (1.17) further. Let r be a fixed positive integer and define

$$\begin{cases} F_1^{(r)}(n, n-rk) = \binom{rk-n}{rk} f_k(n)\,, \\ F^{(r)}(n, n-rk) = \binom{n}{rk} f_k(rk-n)\,, \end{cases} \tag{1.21}$$

where $f_k(z)$ is a polynomial in z that satisfies (1.16). For $r=1$, (1.21) reduces to (1.17). We call a pair of polynomials

$$g_{1,k}^{(r)}(n) = F_1^{(r)}(n, n-rk)\,, \qquad g_k^{(r)}(n) = F^{(r)}(n, n-rk) \tag{1.22}$$

a *Stirling pair* of order r, or, briefly, a Stirling pair. Clearly each of the polynomials is of degree $(r+1)k$ in n. Moreover, by (1.21), $g_{1,k}^{(h)}(z)$ and $g_k^{(r)}(z)$ are defined for arbitrary (complex) z.

We shall show that the results previously obtained in the case $r=1$, when properly modified, hold for all $r \geqslant 1$. We show also that a condition similar to (1.21) suffices for orthogonality in the general case.

Finally we consider the possibility of recurrence of the type

$$F_1^{(r)}(n+1, m) = F_1^{(r)}(n, m-1) + p_r(n) F_1^{(r)}(n-r+1, m) \tag{1.23}$$

and

$$F^{(r)}(n+1, m) = F^{(r)}(n, m-1) + g_r(m) F^{(r)}(n, m+r-1)\,. \tag{1.24}$$

For $r=1$, $p_r(n)=n$, (1.23) reduces to the familiar recurrence for $S_1(n, m)$; for $r=1$, $q_r(m)=m$, (1.24) reduces to the recurrence for $S(n, m)$. Recurrences for the numbers defined by the generating functions [5]

$$1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} T_1(n, m) \frac{x^n}{n!} z^m = \left(\frac{1+x}{1-x}\right)^{z/2}, \tag{1.25}$$

$$1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} T(n, m) \frac{x^n}{n!} z^m = \exp(z \tanh x)\,, \tag{1.26}$$

are of the form (1.23) and (1.24), respectively, with $r=2$. Indeed it was the study of such arrays that motivated the generalization (1.21).

We show that (1.23) holds if and only if

$$p_r(n) = (-1)^r (r+1)\, c_r \binom{n}{r}, \tag{1.27}$$

where $f_1^{(r)}(z) = c_r z$. Similarly (1.24) holds if and only if

$$q_r(m) = -(r+1)\, c_r \binom{m+r-1}{r}. \tag{1.28}$$

Thus

$$q_r(n) = (-1)^{r-1} p_r(n+r-1)\,.$$

The two conditions (1.27), (1.28) are equivalent.

Moreover when (1.27) is satisfied we have

$$1+\sum_{n=1}^{\infty}\sum_{m=1}^{n}F_1^{(r)}(n,m)\frac{x^n}{n!}z^m=\exp\{z\psi(x)\}$$

and

$$1+\sum_{n=1}^{\infty}\sum_{m=1}^{n}F^{(r)}(n,m)\frac{x^n}{n!}z^m=\exp\{-z\omega(-x)\}\,,$$

where

$$\psi(x)=\sum_{m=0}^{\infty}\frac{a_r^m x^{rm+1}}{rm+1}\,,\qquad a_r=(-1)^r\frac{(r+1)c_r}{r!}$$

and

$$\psi(\omega(x))=\omega(\psi(x))=x\,,\qquad \omega(0)=0\,.$$

2. – We first prove the following generalized version of (1.5).

THEOREM 1. *For all integral $r\geqslant 1$, we have*

$$(2.1)\quad \begin{cases} F_1^{(r)}(n,n-k)=\displaystyle\sum_{j=0}^{rk}\binom{rk-n}{rk+j}\binom{rk+n}{rk-j}F^{(r)}(j+rk,j)\,,\\ F^{(r)}(n,r-k)=\displaystyle\sum_{j=0}^{rk}\binom{rk-n}{rk+j}\binom{rk+n}{rk-j}F_1^{(r)}(j+rk,j)\,. \end{cases}$$

PROOF. It suffices to prove the identity

$$(2.2)\qquad \binom{z}{rk}f_k(rk-z)=\sum_{j=0}^{rk}\binom{rk-z}{rk+j}\binom{rk+z}{rk-j}\binom{-j}{rk}f_k(j+rk)\,.$$

For $z=rk-n$, (2.2) reduces to the first of (2.1), for $z=n$, (2.2) reduces to the second of (2.1).

Each side of (2.2) is a polynomial in z of degree $(r+1)k$. Hence it is only necessary to show that (2.2) holds for $(r+1)k+1$ distinct values of z. For $z=t=0,1,\ldots,rk-1$, the LHS of (2.2) is equal to zero; since

$$\binom{rk-t}{rk+j}\binom{-j}{rk}=0\qquad (0\leqslant t<k;\ 0\leqslant j\leqslant k)\,,$$

it follows that (2.2) holds for these values of z. For $z = rk$ we get

$$f_k(0) = \sum_{j=0}^{rk} \binom{0}{rk}\binom{(r+1)k}{rk-j}\binom{-j}{rk} f_k(j+rk) ,$$

which is evidently correct. Finally, for $z = -t$, where $1 \leqslant t \leqslant k$, the RHS of (2.2) reduces to the single term $(j = t)$

$$\binom{rk+t}{rk+t}\binom{rk-t}{rk-t}\binom{-t}{rk} f_k(t+rk) = \binom{-t}{rk} f_k(r+rk) ,$$

so that (2.2) holds for these values of z.

This completes the proof of (2.2) and therefore of Theorem 1.

It is of some interest to ask whether the polynomials in a Stirling pair can be equal. By (1.21) and (1.22), $g_{1,k}^{(r)}(z) = g_k^{(r)}(z)$ if and only if

$$\binom{rk-z}{rk} f_k(z) = \binom{z}{rk} f_k(rk-z) . \tag{2.3}$$

Since

$$\binom{rk-z}{rk} = (-1)^{rk}\binom{z-1}{rk},$$

(2.3) reduces to

$$(-1)^{rk}(z-rk) f_k(z) = z f_k(rk-z) ,$$

so that

$$(-1)^{rk}(z-1) f_k(rkz) = z f_k\big(rk(1-z)\big) .$$

Hence we have

$$\varphi_{k-1}(1-z) = (-1)^{rk-1}\varphi_{k-1}(z) , \tag{2.4}$$

where

$$f_k(rkz) = rkz\varphi_{k-1}(z) , \qquad (k \geqslant 1) .$$

Note that $\varphi_{k-1}(z)$ is a polynomial of degree $k-1$.

We may state

THEOREM 2. *The polynomials $g_{1,k}^{(r)}(z)$, $g_k^{(r)}(z)$ in the Stirling pair*

$$g_{1,k}^{(r)} = \binom{rk-z}{rk} f_k(z) , \qquad g_k^{(r)}(z) = \binom{z}{rk} f_k(rk-z)$$

are equal if and only if

$$f_k(rkz) = rkz\varphi_{k-1}(z)\,, \qquad (k \geqslant 1)\,,$$

where $\varphi_{k-1}(z)$ *is a polynomial of degree* $k-1$ *that satisfies* (2.4).

For r odd, the condition (2.4), or, what is the same,

$$\varphi_k(1-z) = (-1)^k \varphi_k(z) \tag{2.5}$$

is a familiar one. An equivalent condition is

$$\bar{\varphi}_k(-z) = (-1)^k \bar{\varphi}_k(z)\,, \tag{2.6}$$

where $\bar{\varphi}_k(z) = \varphi(\frac{1}{2} + z)$.

The Bernoulli and Euler polynomials $B_n(z)$, $E_n(z)$ defined by

$$\frac{xe^{xz}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(z) \frac{x^n}{n!} \tag{2.7}$$

and

$$\frac{2e^{xz}}{e^x + 1} = \sum_{n=0}^{\infty} E_n(z) \frac{x^n}{n!}\,, \tag{2.8}$$

respectively, are well known examples of polynomials satisfying (2.5). It therefore follows from Theorem 2 that, for r odd,

$$g_{1,k}^{(r)}(z) = g_k^{(r)}(z) = \binom{rk-z}{rk} \frac{z}{rk} B_{k-1}\left(\frac{z}{rk}\right) = \binom{z}{rk} \frac{rk-z}{rk} B_{k-1}\left(\frac{rk-z}{rk}\right) \tag{2.9}$$

and

$$g_{1,k}^{(r)}(z) = g_k^{(r)}(z) = \binom{rk-z}{rk} \frac{z}{rk} E_{k-1}\left(\frac{z}{rk}\right) = \binom{z}{rk} \frac{rk-z}{rk} E_{k-1}\left(\frac{rk-z}{rk}\right) \tag{2.10}$$

are instances of Stirling pairs consisting of equal polynomials.

Another example with $f_k(z) = \binom{z/r}{k}$ is

$$g_{1,k}^{(r)}(z) = g_k^{(r)}(z) = \binom{rk-z}{rk}\binom{z/r}{k} = \binom{z}{rk}\binom{(rk-z)/r}{k}. \tag{2.11}$$

For even r, (2.4) becomes

$$\varphi_{k-1}(1-z) = -\varphi_{k-1}(z)\,. \tag{2.12}$$

For $k = 1$, this implies $\varphi_0(z) \equiv 0$, so that $f_1(z) \equiv 0$. Hence, for even r, $g^{(r)}_{1,k}(z)$ and $g^{(r)}_k(z)$ cannot be equal for all k. However if we require only that $g^{(r)}_{1,2k}(z) = g^{(r)}_{2k}(z)$ then the previous discussion (for odd r) applies.

Clearly, by (1.21), $F^{(r)}_1(x, y)$, $F^{(r)}(x, y)$ are defined for arbitrary x, y such that $x - y = rk$, where k is any nonnegative integer. Indeed

$$\begin{cases} F^{(r)}_1(x, y) = \dbinom{-y}{rk} f_k(x)\,, \\ F^{(r)}(x, y) = \dbinom{x}{rk} f_k(-y)\,. \end{cases} \tag{2.13}$$

Moreover it follows from (2.13) that

$$F^{(r)}(x, y) = F^{(r)}_1(-y, -x)\,, \qquad (x - y = rk)\,. \tag{2.14}$$

3. – We now consider

$$\begin{cases} F^{(r)}_1(n, n - rk) = \displaystyle\sum_{j=1}^{k} G^{(r)}_1(k, j) \binom{n + j - 1}{(r+1)k}, \\ F^{(r)}(n, n - rk) = \displaystyle\sum_{j=1}^{k} G^{(r)}(k, j) \binom{n + j - 1}{(r+1)k}, \end{cases} \tag{3.1}$$

where $G^{(r)}_1(k, j)$, $G^{(r)}(k, j)$ are independent of n. That such representations hold follows from (1.21); for the general situation see [4].

Inverting each of (3.1) we get

$$\begin{cases} G^{(r)}_1(k, k - j + 1) = \\ \qquad = \displaystyle\sum_{t=0}^{j} (-1)^t \binom{(r+1)k + 1}{t} F^{(r)}_1(rk + j - t, j - t)\,, \\ G^{(r)}(k, k - j + 1) = \\ \qquad = \displaystyle\sum_{t=0}^{j} (-1)^t \binom{(r+1)k + 1}{t} F^{(r)}(rk + j - t, j - t)\,. \end{cases} \tag{3.2}$$

We shall now show that

$$(3.3)\qquad (-1)^{(r+1)k}G_1^{(r)}(k,j) = G^{(r)}(k, k-j+1)\,, \qquad (1\leqslant j\leqslant k)\,.$$

It follows from (1.21) and (3.1) that

$$(3.4)\qquad \binom{n}{rk} f_k(rk-n) = \sum_{j=1}^{k} G^{(r)}(k,j)\binom{n+j-1}{(r+1)k}$$

and

$$(3.5)\qquad \binom{rk-n}{rk} f_k(n) = \sum_{j=1}^{k} G_1^{(r)}(k,j)\binom{n+j-1}{(r+1)k}.$$

Both (3.4) and (3.5) are polynomial identities in n. Thus in (3.5) we may replace n by $rk-n$. This gives

$$\binom{n}{rk} f_k(rk-n) = \sum_{j=1}^{k} G_1^{(r)}(k,j)\binom{rk-n+j-1}{(r+1)k}.$$

Since

$$\binom{rk-n+j-1}{(r+1)k} = (-1)^{(r+1)k}\binom{n+k-j}{(r+1)k},$$

we get

$$\binom{n}{rk} f_k(rk-n) = (-1)^{(r+1)k}\sum_{j=1}^{k} G_1^{(r)}(k,j)\binom{n+k-j}{(r+1)k}$$

and therefore

$$(3.6)\qquad \binom{n}{rk} f_k(rk-n) = (-1)^{(r+1)k}\sum_{j=1}^{k-j+1} G_1^{(r)}(k,j)\binom{n+j-1}{(r+1)k}.$$

Comparison of (3.6) with (3.4) yields (3.3).
We may state

THEOREM 3. *The coefficients* $G_1^{(r)}(k,j)$, $G^{(r)}(k,j)$ *defined by* (3.1) *satisfy the symmetry relation*

$$(3.7)\qquad (-1)^{(r+1)k}G_1^{(r)}(k,j) = G^{(r)}(k, k-j+1)\,, \qquad (1\leqslant j\leqslant k)\,.$$

Theorem 3 can be used to give another proof of Theorem 1. However we shall not take the space to do so.

4. – The generalized versions of (1.3) and (1.4) are

$$F_1^{(r)}(n, n-rk) = \sum_{j=0}^{k-1} F_1^{(r)'}(k, j)\binom{n}{(r+1)k-j} \tag{4.1}$$

and

$$F^{(r)}(n, n-rk) = = \sum_{j=0}^{k-1} F^{(r)'}(k, j)\binom{n}{(r+1)k-j}, \tag{4.2}$$

respectively.

To invert (4.1) multiply both sides by x^{n-rk} and sum over h. Thus

$$\sum_{n=0}^{\infty} F_1^{(r)}(n+rk, n)x^n = \sum_{j=0}^{k-1} F_1^{(r)'}(k, j)x^{k-j} \sum_{n=(r+1)k-j}^{\infty} \cdot$$

$$\cdot \binom{n}{(r+1)k-j} x^{n-(r+1)k+j} = \sum_{j=0}^{k-1} F_1^{(r)'}(k, j)x^{k-j}(1-x)^{-(r+1)k+j-1} =$$

$$= \sum_{j=1}^{k} F_1^{(r)'}(k, k-j)x^j(1-x)^{-rk-j-1}$$

so that

$$\sum_{j=1}^{k} F_1^{(r)'}(k, k-j)x^j(1-x)^{-j} = (1-x)^{rk+1} \sum_{n=0}^{\infty} F_1^{(r)}(n+rk, n)x^n .$$

Put

$$z = \frac{x}{1-x}, \qquad x = \frac{z}{1+z}, \qquad 1-x = \frac{1}{1+z}$$

and we get

$$\sum_{j=1}^{k} F_1^{(r)'}(k, k-j)z^j = \sum_{n=0}^{\infty} F_1^{(r)}(n+rk, n)z^n(1+z)^{-n-rk-1} .$$

Expanding the right member and equating coefficients, we get

$$F_1^{(r)'}(k, k-j) = \sum_{t=0}^{j} (-1)^{j-t}\binom{j+rk}{j-t} F_1^{(r)}(t+rk, t) . \tag{4.3}$$

Similarly we have

$$F^{(r)'}(k, k-j) = \sum_{t=0}^{j} (-1)^{j-t}\binom{j+rk}{j-t} F^{(r)}(t+rk, t) . \tag{4.4}$$

By (4.3) and (3.1) we have

$$F_1^{(r)'}(k, k-j) = \sum_{t=0}^{j} (-1)^{j-t} \binom{j+rk}{j-t} \sum_{s=1}^{k} G_1^{(r)}(k, s) \binom{t+rk+s-1}{(r+1)k} =$$
$$= \sum_{s=1}^{k} G_1^{(r)}(k, s) \sum_{t=0}^{j} (-1)^{j-t} \binom{j+rk}{j-t} \binom{t+rk+s-1}{(r+1)k}.$$

By Vandermonde's theorem the inner sum is equal to $\binom{s-1}{k-j}$, so that

$$F_1^{(r)'}(k, j) = \sum_{s=j+1}^{k} \binom{s-1}{j} G_1^{(r)}(k, s) . \tag{4.5}$$

Similarly

$$F^{(r)'}(k, j) = \sum_{s=j+1}^{k} \binom{s-1}{j} G^{(r)}(k, s) . \tag{4.6}$$

The inverse formulas are

$$G_1^{(r)}(k, t) = \sum_{s=t-1}^{k-1} (-1)^{s-t+1} \binom{s}{t-1} F_1^{(r)'}(k, s) \tag{4.7}$$

and

$$G^{(r)}(k, t) = \sum_{s=t-1}^{k-1} (-1)^{s-t+1} \binom{s}{t-1} F^{(r)'}(k, s) . \tag{4.8}$$

Next, by (3.7) and (4.8), we have

$$F_1^{(r)}(n, n-rk) = \sum_{t=1}^{k} G_1^{(r)}(k, t) \binom{n+t-1}{(r+1)k} =$$
$$= \sum_{t=1}^{k} G_1^{(r)}(k, k-t+1) \binom{n+k-t}{(r+1)k} = (-1)^{(r+1)k} \sum_{t=1}^{k} G^{(r)}(k, t) \binom{n+k-t}{(r+1)k} =$$
$$= (-1)^{(r+1)k} \sum_{t=1}^{k} \binom{n+k-t}{(r+1)k} \sum_{s=t-1}^{k-1} (-1)^{s-t+1} \binom{s}{t-1} F^{(r)'}(k, s) =$$
$$= (-1)^{(r+1)k} \sum_{s=0}^{k-1} F^{(r)'}(k, s) \sum_{t=1}^{s+1} (-1)^{s-t+1} \binom{s}{t-1} \binom{n+k-t}{(r+1)k}.$$

The inner sum is equal to

$$(-1)^s \binom{n+k-s-1}{(r+1)k-s},$$

so that

$$(4.9)\qquad F_1^{(r)}(n, n-rk) = \sum_{j=1}^{k} (-1)^{rk-j} \binom{n+j-1}{rk+j} F^{(r)'}(k, k-j) .$$

The companion formula is

$$(4.10)\qquad F^{(r)}(n, n-rk) = \sum_{j=1}^{k} (-1)^{rk-j} \binom{n+j-1}{rk+j} F_1^{(r)'}(k, k-j) .$$

Again, by (4.5) and (4.8),

$$\begin{aligned} F_1^{(r)'}(n, k) &= \sum_{s=k+1}^{n} \binom{s-1}{k} G^{(r)}(n, s) = \\ &= (-1)^{(r+1)n} \sum_{s=k+1}^{n} \binom{s-1}{k} G^{(r)}(n, n-s+1) = \\ &= (-1)^{(r+1)n} \sum_{s=1}^{n-k} \binom{n-s}{k} G^{(r)}(n, s) = \\ &= (-1)^{(r+1)n} \sum_{s=1}^{n-k} \binom{n-s}{k} \sum_{t=s-1}^{n-1} (-1)^{t-s+1} \binom{t}{s-1} F^{(r)'}(n, t) = \\ &= (-1)^{(r+1)n} \sum_{t=0}^{n-1} F^{(r)'}(n, t) \sum_{s=1}^{t+1} (-1)^{t-s+1} \binom{t}{s-1} \binom{n-s}{k} . \end{aligned}$$

The inner sum is equal to

$$\sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} \binom{n-s-1}{k} = (-1)^{t} \binom{n-t-1}{k-t}$$

and therefore

$$(4.11)\qquad F_1^{(r)'}(n, k) = (-1)^{(r+1)n} \sum_{t=0}^{n-1} (-1)^{t} \binom{n-t-1}{k-t} F^{(r)'}(n, t) .$$

Similarly

$$(4.12)\qquad F^{(r)'}(n, k) = (-1)^{(r+1)n} \sum_{t=0}^{n-1} (-1)^{t} \binom{n-t-1}{k-t} F_1^{(r)'}(n, t) .$$

To sum up the results of § 4 we state

THEOREM 4. *The coefficients* $F_1^{(r')}(n, n-rk)$, $F^{(r)'}(n, n-rk)$ *defined by* (4.1) *and* (4.2) *satisfy* (4.5), (4.6), (4.9), (4.10), (4.11) *and* (4.12).

5. – For the results obtained above it sufficed to assume that the $\{f_k(z)\}$ were a sequence of polynomials in z satisfying

$$\deg f_k(z) = k\,; \qquad f_k(0) = 0\,, \qquad (k \geq 1)\,. \tag{5.1}$$

In order to obtain orthogonality relations more is needed. We-shall make use of a sufficient condition that is convenient for applications.

Let

$$\varphi^{(r)}(x) = 1 + \sum_{n=1}^{\infty} c_n x^{rn}/(rn)! \tag{5.2}$$

denote a function that is analytic in the neighborhood of $x = 0$ and such that $\varphi(0) = 1$. Put

$$(\varphi(x))^z = \sum_{k=0}^{\infty} f_k^{(r)}(z) x^{rk}/(rk)! \tag{5.3}$$

It is easily verified that the $(f_k^{(r)}(z)\}$ are polynomials in z that satisfy (5.1). The Nörlund polynomials $B_n^{(z)}$ are given by $\varphi(x) = x/(e^x - 1)$ and $r = 1$.

We remark that a sequence of polynomials $\{f_k^{(r)}(z)\}$ satisfies (5.3) if and only if it satisfies.

$$\sum_{j=0}^{k} \binom{rk}{rj} f_j^{(r)}(y) f_{k-j}^{(r)}(z) = f_k^{(r)}(y+z) \qquad (k = 0, 1, 2, \ldots)\,. \tag{5.4}$$

It follows from (5.3) that

$$z\varphi^{z-1}(x)\varphi'(x) = \sum_{k=0}^{\infty} f_k^{(r)}(z) x^{rk-1}/(rk-1)!\,,$$

$$(y+z)\varphi^{y+z-1}(x)\varphi'(x) = \sum_{k=0}^{\infty} f_k^{(r)}(y+z) x^{rk-1}/(rk-1)!\,.$$

Since

$$(y+z)\varphi^{y+z-1}(x)\varphi'(x) = \frac{y+z}{z}\varphi^y(x)\cdot z\varphi^{z-1}(x)\varphi'(x)\,,$$

we get

$$f_k^{(r)}(y+z) = \frac{z}{y+z}\sum_{j=1}^{k}\binom{rk-1}{rj-1} f_{k-j}^{(r)}(y) f_j^{(r)}(z)\,. \tag{5.5}$$

THEOREM 5. *Let*

$$(5.6)\qquad \begin{cases} F_1^{(r)}(n, n-rk) = \binom{rk-n}{rk} f_k^{(r)}(n)\,, \\ F^{(r)}(n, n-rk) = \binom{n}{rk} f_k^{(r)}(rk-n)\,, \end{cases}$$

where the $\{f_k^{(r)}(z)\}$ *are defined by* (5.2) *and* (5.3). *Then we have*

$$(5.7)\qquad \sum_{k=0}^{j} (-1)^{rk} F_1^{(r)}(n, n-rk) F^{(r)}(n-rk, n-rj) = \\ = \sum_{k=0}^{j} (-1)^{r(j-k)} F^{(r)}(n, n-rk) F_1^{(r)}(n-rk, n-rj) = \delta_{j,0}\,.$$

PROOF. - Put

$$(5.8)\qquad H(n, j) = \sum_{k=0}^{j} (-1)^{rk} F_1^{(r)}(n, n-rk) F^{(r)}(n-rk, n-rj)\,.$$

By (5.6),

$$(5.9)\qquad H(n, j) = \sum_{k=0}^{j} (-1)^{rk} \binom{rk-n}{rk} f_k^{(r)}(n) \cdot \binom{n-rk}{r(j-k)} f_{j-k}^{(r)}(rj-n) = \\ = \sum_{k=0}^{j} \binom{n-1}{rk} \binom{n-rk}{r(j-k)} f_k^{(r)}(n) f_{j-k}^{(r)}(rj-n) = \\ = \frac{1}{n} \binom{n}{rj} \sum_{k=0}^{j} (n-rk) \binom{rj}{rk} f_k^{(r)}(n) f_{j-k}^{(r)}(rj-n)\,.$$

By (5.4) and (5.5), for $j > 0$,

$$\sum_{k=0}^{j} (n-rk) \binom{rj}{rk} f_k^{(r)}(n) f_{j-k}^{(r)}(rj-n) = = n \sum_{k=0}^{j} \binom{rj}{rk} f_k^{(r)}(n) f_{j-k}^{(r)}(rj-n) - \\ - rj \sum_{k=1}^{j} \binom{rj-1}{rk-1} f_k^{(r)}(n) f_{j-k}^{(r)}(rj-n = n f_j^{(r)}(rj) - rj \cdot \frac{n}{rj} f_j^{(r)}(rj) = 0\,.$$

Thus

$$(5.10)\qquad H(n, j) = 0\,, \qquad (j > 0)\,.$$

For $j = 0$, it is evident from (5.8) that

$$H(n, 0) = 1 . \tag{5.11}$$

This proves the first half of (5.7). The second half then follows as a corollary.

6. – We recall that the Stirling numbers $S_1(n, k)$, $S(n, k)$ satisfy the respective recurrences

$$\begin{cases} S_1(n+1, k) = S_1(n, k-1) + nS_1(n, k) , \\ S(n+1, k) \;\; = S(n, k-1) + ks(n, k) . \end{cases} \tag{6.1}$$

Also it is proved that in [5] that

$$T_1(n+1, k) = T_1(n, k-1) + n(n-1)T_1(n-1, k) \tag{6.2}$$

and

$$T(n+1, k) = T(n, k-1) - k(k+1)T(n, k+1) , \tag{6.3}$$

where

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} T_1(n, k) \frac{x^n}{n!} z^k = \left(\frac{1+x}{1-x}\right)^{z/2} \tag{6.4}$$

and

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} T(n, k) \frac{x^n}{n!} z^k = \exp(z \operatorname{Tanh} x) \; ; \tag{6.5}$$

$\big(T_1(n, k)\big)$ and $\big(T(n, k)\big)$ are reciprocal arrays.

Another example from [5] is

$$U(n+2, k) = U(n, k-2) + k^2 U(n, k) , \tag{6.6}$$

where

$$1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} U(n, k) \frac{x^n}{n!} z^k = \exp(z \sinh x) . \tag{6.7}$$

Thus it is clear that, for $r > 1$, there are apparently numerous possible recurrences for $F_1^{(r)}(n, k)$ and $F^{(r)}(n, k)$. The instances (6.2), (6.3) and (6.6) illustrate the case $r = 2$.

To begin with, we consider the possibility of recurrences of the type

$$(6.8)\quad F_1^{(r)}(n+1, n-rk+1) = \\ = F_1^{(r)}(n, n-rk) + p_r(n) F_1^{(r)}(n-r+1, n-rk+1),$$

where $p_r(n)$ is independent of k.

By the first of (5.6), the recurrence (6.8) becomes

$$(6.9)\quad \binom{rk-n-1}{rk} f_k^{(r)}(n+1) = \\ = \binom{rk-n}{rk} f_k(n) + p_r(n) \binom{rk-n-1}{r(k-1)} f_{k-1}(n-r+1).$$

We now take $k=1$ in (6.9). Since $f_0(z)=1$, $f_1(z)=c_r z$, where c_r is independent of z, we get

$$(-1)^r p_r(n) = c_r(n+1)\binom{n}{r} - c_r n \binom{n-1}{r} = \\ = (r+1)c_r\left\{\binom{n+1}{r+1} - \binom{n}{r+1}\right\}.$$

Hence

$$(6.10)\qquad p_r(n) = (-1)^r (r+1) c_r \binom{n}{r}.$$

Thus we have proved that (6.10) is a *necessary* condition for the existence of the recurrence (6.8).

Now let

$$(6.11)\qquad G_1^{(r)} \equiv G_1^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} F_1^{(r)}(n, m) \frac{x^n}{n!} z^m.$$

Since, by (6.8) and (6.10),

$$(6.12)\qquad F_1^{(r)}(n+1, m) = \\ = F_1^{(r)}(n, m-1) + (-1)^r (r+1) c_r \binom{n}{r} F_1^{(r)}(n-r+1, m),$$

it follows from (6.11) that

$$D_x G_1^{(r)} = zG_1^{(r)} + (-1)^r \frac{(r+1)c_r}{r!} \sum_{n,m} F_1^{(r)}(n-r+1, m) \frac{x^n z^m}{(n-r)!} =$$
$$= zG_1^{(r)} + (-1)^r \frac{(r+1)c_r}{r!} x^r D_x G_1^{(r)} ,$$

where $D_x \equiv \partial/\partial x$. Thus we have

$$D_x G_1^{(r)} = \frac{z}{1-a_r x^r}, \qquad a_r = (-1)^r \frac{(r+1)c_r}{r!} . \tag{6.13}$$

For example, for $r = 2$, $a_r = 1$, (6.13) reduces to

$$D_x G_1^{(2)} = \frac{z}{1-x^2},$$

which yields

$$G_1^{(2)}(x, z) = \left(\frac{1+x}{1-x}\right)^{z/2} \tag{6.14}$$

in agreement with (6.4).

For the general case (6.13), we have

$$G_1^{(r)}(x, z) = \exp\left\{z \sum_{m=0}^{\infty} \frac{a_r^m x^{mr+1}}{mr+1}\right\} . \tag{6.15}$$

Conversely, given (6.15) we get the recurrence (6.8) with $p_r(n)$ satisfying (6.10). Thus (6.10) is both necessary and sufficient for the recurrence (6.8).

Note that substitution of (6.10) in (6.9) gives

$$f_k^{(r)}(n+1) = \frac{n-rk}{n} f_k^{(r)}(n)\, c_r \binom{rk}{r} f_{k-1}^{(r)}(n-r+1) \qquad (n > 0) . \tag{6.16}$$

By (6.11) and (5.6) we have

$$D_x G_1^{(r)}(x, z) = \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} F_1^{(r)}(n+1, m) \frac{x^n}{n!} z^m =$$
$$= \sum_{m,k=0}^{\infty} (-1)^{rk} f_k^{(r)}(m+rk+1) \frac{x^{m+rk} z^m}{m!(rk)!} .$$

Hence (6.15) becomes

$$\sum_{m,k=0}^{\infty}(-1)^{rk}f_k^{(r)}(m+rk+1)\frac{x^{m+rk}z^m}{m!(rk)!}=D_x\{\exp(z\psi(x))\},$$

so that

(6.17) $$1+\sum_{m=1}^{\infty}\frac{x^m z^m}{m!}\sum_{k=0}^{\infty}(-1)^{rk}\frac{f_k^{(r)}(m+rk+1)}{m+rk+1}\frac{x^{rk}}{(rk)!}=\exp(z\psi(x)),$$

where

(6.18) $$\psi(x)=\sum_{k=0}^{\infty}\frac{a_r^k x^{rk+1}}{rk+1}.$$

Similarly we have

$$G^{(r)}(x,z)=1+\sum_{n=1}^{\infty}\sum_{m=1}^{n}F^{(r)}(n,m)\frac{x^n}{n!}z^m=\sum_{m,k=0}^{\infty}\binom{m+rk}{rk}f_k(-m)\frac{x^{m+rk}z^m}{m!(rk)!}$$

and so

(6.19) $$G^{(r)}(x,z)=\sum_{m=0}^{\infty}\frac{x^m z^m}{m!}\sum_{k=0}^{\infty}f_k(-m)\frac{x^{rk}}{(rk)!}.$$

7. – Parallel to (6.8) we consider the possibility of a recurrence of the type

(7.1) $$F^{(r)}(n+1,n-rk+1)=$$
$$=F^{(r)}(n,n-rk)+q_r(n-rk+1)F^{(r)}(n,n-r(k-1)).$$

By the second of (5.6), (7.1) becomes

$$\binom{n+1}{rk}f_k^{(r)}(rk-n-1)=$$
$$=\binom{n}{kr}f_k^{(r)}(rk-n)+q_r(n-rk+1)\binom{n}{r(k-1)}f_{k-1}(r(k-1)-n).$$

For $k=1$, this reduces to

$$\binom{n+1}{r}f_1^{(r)}(-n)=\binom{n}{r}f_1^{(r)}(r-n)+q_r(n-r+1).$$

Hence

$$q_r(n-r+1) = -c_r n\binom{n+1}{r} + c_r(n-r)\binom{n}{r},$$

so that

$$q_r(n-r+1) = -(r+1)c_r\binom{n}{r}. \tag{7.2}$$

Thus (7.2) is a necessary condition for the existence of the recurrence (7.1).

Comparison of (7.2) with (6.10) gives

$$q_r(n-r+1) = (-1)^{r-1}p_r(n). \tag{7.3}$$

By (7.1) and (7.2) we have

$$F^{(r)}(n+1, m) = \tag{7.4}$$
$$= F^{(r)}(n, m-1) - (r+1)c_r\binom{m+r-1}{r}F^{(r)}(n, m+r-1).$$

As above put

$$G^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty}\sum_{m=1}^{n} F^{(r)}(n, m)\frac{x^n}{n!}z^m. \tag{7.5}$$

Then

$$D_x G^{(r)}(x, z) = \sum_{n=0}^{\infty}\sum_{m=1}^{n+1} F^{(r)}(n+1, m)\frac{x^n}{n!}z^m =$$
$$= \sum_{n=0}^{\infty}\sum_{m=1}^{n+1}\left\{F^{(r)}(n, m-1) - (r+1)c_r\binom{m+r-1}{r}F^{(r)}(n, m+r-1)\right\}\cdot$$
$$\cdot\frac{x^n}{n!}z^m = zG^{(r)}(x, z) - (r+1)c_r z\sum_{n=1}^{\infty}\sum_{m=1}^{n-r+1}\binom{m+r-1}{r}\cdot$$
$$\cdot F^{(r)}(n, m+r-1)\frac{x^n}{n!}z^{m-1}.$$

Since the double sum on the extreme right is equal to

$$\sum_{n=r}^{\infty}\sum_{m=r}^{n}\binom{m}{r}F^{(r)}(n, m)\frac{x^n}{n!}z^{m-r} = \frac{1}{r!}D_z^r G^{(r)}(x, z),$$

we get the partial differential equation

$$(7.6)\qquad D_x G^{(r)}(x, z) = zG^{(r)}(x, z) + (-1)^{r-1} a_r z D_z^r G^{(r)}(x, z) ,$$

where, as in (6.13),

$$(7.7)\qquad a_r = (-1)^r \frac{(r+1)c_r}{r!} .$$

Let $\omega(x)$ denote the inverse of $\psi(x)$ that vanishes at the origin:

$$(7.8)\qquad \psi(\omega(x)) = x = \omega(\psi(x)) ,$$

where as above

$$(7.9)\qquad \psi(x) = \sum_{m=0}^{\infty} \frac{a_r^m x^{rm+1}}{rm+1} .$$

Let $u = \psi(x)$, $x = \omega(u)$. Since

$$\psi'(x) = \frac{1}{1 - a_r x^r}$$

and $\psi'(x)\omega'(u) = 1$, it follows that

$$(7.10)\qquad \omega'(u) = 1 - a_r \omega^r(u) .$$

Now put

$$(7.11)\qquad H(x, z) = \exp\{z\omega(x)\} .$$

This implies an expansion of the form

$$(7.12)\qquad H(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} c(n, m) \frac{x^n}{n!} z^m .$$

Differentiation of (7.11) gives

$$D_x H(x, z) = z\omega'(x) H(x, z) ,$$

$$D_z^r H(x, z) = \omega^r(x) H(x, z) ,$$

Hence, by (7.10),

$$D_x H(x, z) = zH(x, z) - a_r z D_z^r H(x, z) . \tag{7.13}$$

In (7.6) replace x by $-x$, z by $-z$. Then

$$D_x G^{(r)}(-x, -z) = zG^{(r)}(-z, -z) - a_r z D_z^r H(x, z) . \tag{7.14}$$

Thus $H(x, z)$ and $G^{(r)}(-x, -z)$ satisfy the same partial differential equation.

Next, since by (7.12),

$$\begin{aligned} D_x H(x, z) &= \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} c(n+1, m) \frac{x^n}{n!} z^m , \\ D_z^r H(x, z) &= r! \sum_{n=r}^{\infty} \sum_{m=r}^{n} \binom{m}{r} c(n, m) \frac{x^n}{n!} z^{m-r} , \\ z D_z^r H(x, z) &= r! \sum_{n=r}^{\infty} \binom{m+r-1}{r} c(n, m+r-1) \frac{x^n}{n!} z^m . \end{aligned}$$

substitution in (7.13) yields the recurrence

$$\begin{aligned} & c(n+1, m) = \\ & = c(n, m-1) + (-1)^{r-1}(r+1) c_r \binom{m+r-1}{r} c(n, m+r-1) . \end{aligned} \tag{7.15}$$

As for $F^{(r)}(n+1, m)$, by (7.4) we have

$$\begin{aligned} (-1)^{n-m+1} F^{(r)}(n+1, m) &= (-1)^{n-m+1} F^{(r)}(n, m-1) + \\ &+ (-1)^{r-1}(r+1) c_r \cdot (-1)^{n-m-r+1} F^{(r)}(n, m+r-1) . \end{aligned} \tag{7.16}$$

Since $\omega(x)$ is of the form

$$\omega(x) = \sum_{m=0}^{\infty} \frac{b_m x^{mr+1}}{mr+1} , \qquad b_0 = 1 ,$$

it follows from (7.11) that

$$c(n, m) = 0 , \qquad (n \not\equiv m \pmod{r}) . \tag{7.17}$$

Also it is clear from (7.15) that

$$c(n, n) = 0 , \qquad (n = 0, 1, 2, ...) . \tag{7.18}$$

We conclude that

$$c(n, m) = (-1)^{n-m} F^{(r)}(n, m)$$

and therefore

$$G^{(r)}(-x, -z) = \exp\{z(\omega(x))\} . \tag{7.19}$$

Combining the results of §§ 6, 7 we state the following

THEOREM 6. *The function* $F_1^{(r)}(n, m)$ *satisfies a recurrence of the form*

$$F_1^{(r)}(n+1, m) = F_1^{(r)}(n, m-1) + p_r(n) F_1^{(r)}(n-r+1, m) \tag{7.20}$$

if and only if

$$p_r(n) = (-1)^r (r+1) c_r \binom{n}{r}, \tag{7.21}$$

where $f_1^{(r)}(z) = c_r z$. *The function* $F^{(r)}(n, m)$ *satisfies a recurrence of the form*

$$F^{(r)}(n+1, m) = F^{(r)}(n, m-1) + q_r(m) F^{(r)}(n, m+r-1)$$

if and only if

$$q_r(m) = (-1)^{r-1} p_r(m+r-1) , \tag{7.22}$$

where $p_r(m)$ *satisfies* (7.21).

Moreover (7.20) *and* (7.21) *are satisfied if and only if*

$$G_1^{(r)}(x, z) \equiv 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} F_1^{(r)}(n, m) \frac{x^r}{n!} z^m = \exp\{z\psi(x)\} ,$$

where

$$\psi(x) = \sum_{m=0}^{\infty} \frac{a_r^m x^{rm+1}}{rm+1} , \qquad a_r = (-1)^r \frac{(r+1) c_r}{r!} .$$

It then follows that

$$G^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{m=1}^{n} F^{(r)}(n, m) \frac{x^r}{n!} z^m = \exp\{-z\omega(-x)\},$$

where $\omega(x)$ *is the inverse of* $\psi(x)$ *that vanishes at the origin:*

$$\psi(\omega(x)) = \omega(\psi(x)) = x .$$

8. - Put

$$\exp\{zf(x)\} = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} B_{nk} \frac{x^n}{n!} z^k , \tag{8.1}$$

where

$$f(x) = \sum_{n=1}^{\infty} b_n \frac{x^n}{n!}, \qquad b_1 = 1 . \tag{8.2}$$

Let $g(x)$ denote the inverse of $f(x)$:

$$g(x) = \sum_{n=1}^{\infty} c_n \frac{x^n}{n!}, \qquad c_1 = 1 \tag{8.3}$$

and put

$$\exp\{zf(x)\} = 1 + \sum_{n=1}^{\infty} C_{nk} \frac{x^n}{n!} z^k . \tag{8.4}$$

It is proved in [5] that (B_{nk}) and (C_{nk}) are reciprocal arrays:

$$\sum_{k=j}^{n} B_{nk} C_{kj} = \sum_{k=j}^{n} C_{nk} B_{kj} = \delta_{nj} . \tag{8.5}$$

We now apply this result to

$$G_1^{(r)}(x, z) = \exp\{z\psi(x)\},$$

and

$$G^{(r)}(x, z) = \exp\{-z\omega(-x)\} .$$

It follows at once from

$$G_1^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} F_1^{(r)}(n, k) \frac{x^n}{n!} z^k$$

and

$$G^{(r)}(x, z) = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} F^{(r)}(n, k) \frac{x^n}{n!} z^k$$

that

$$(8.6) \qquad \sum_{k=1}^{n} (-1)^{n-k} F_1^{(r)}(n, k) F^{(r)}(k, j) = \sum_{k=j}^{n} (-1)^{k-j} F^{(r)}(n, k) F_1^{(r)}(k, j) = \delta_{nj} .$$

We may state

THEOREM 7. *Let*

$$G_1^{(r)}(x, z) = \exp \{z\psi(x)\}$$

or, equivalently,

$$G^{(r)}(x, z) = \exp \{-z\omega(-x)\} ,$$

where

$$\psi(x) = \sum_{m=0}^{\infty} \frac{a_r^m x^{rm+1}}{rm+1} , \qquad a_r = (-1)^r \frac{(r+1) c_r}{r!}$$

and $\psi(\omega(x)) = \omega(\psi(x)) = x$, $\omega(0) = 0$. *Then* $(F_1^{(r)}(n, k))$, $(F^{(r)}(n, k))$ *satisfy the orthogonality relations* (8.6).

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