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The Search for Fixed Points under Perturbations.

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Introduction.

In what follows S is a bounded, open, convex subset of E^n, $F: \bar{S} \to S$ a C^2 mapping; K is the fixed point set of F. We shall actually assume that F is defined on a neighborhood of $\bar{S}$, with values in S. Fix $\xi^0 \in \partial S$ and, following [3], consider the set of those x's such that the half-line from $F(x)$ through x intersects ∂S at ξ^0. In the case ξ^0 is a regular value of the mapping H defined below, there exists a differential equation

$$\dot{x} = u(x)$$

such that the solution of the Cauchy problem with $x(0) = \xi^0$ exists on $[0, \omega)$, its path is contained in the above mentioned set, and $\lim_{t\to\infty} d(x(t), K) = 0$.

In this paper we investigate what happens when we perturb this set, allowing ξ to vary in a neighborhood of ξ^0. We remark that, although the set we are interested in, i.e. those x's allined with ξ, is completely defined, it is not so neither for the mapping H nor for the differential equations. Both of these depend on the way the new, fictitious boundaries through ξ are defined and on the regularity properties of the functions describing them.

Under genericity assumptions for F, we can prove the following. Let ξ^0 be a regular value for H, so that theıe exists a differential equa-

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tion with a solution starting at ξ_0 and leading to the fixed point set. Then for every ξ in a neighborhood of ξ^0 there exist a differential equation and a solution leading from ξ to the fixed point set. Moreover solutions of these differential equations converge uniformly on compacta to the soution of the original differential equation.

Notations and basic assumptions.

We assume that the mapping F is defined on the closure of a bounded, open set Σ containing $\bar{S}$ with values in S, so that $F(\bar{\Sigma})$ has a positive distance 2δ from ∂S. We assume that a ball about S of radius δ is contained in Σ. Further we suppose that the boundary of S is locally sufficiently smooth i.e. that any given ξ^0 belongs to an open neighborhood $\mathcal{U} \subset \Sigma$ such that: defining $\varphi \colon \partial S \cap \mathcal{U} \to \mathbb{R}$ by $\varphi(x) = 0$, φ can be extended to $\mathcal{U}$ as a C^3 mapping into $\mathbb{R}$ with a nowhere vanishing gradient and such that $\langle \operatorname{grad} \varphi(x), \operatorname{grad} \varphi(x') \rangle \geqslant 0$ for x and x' in $\partial S \cap \mathcal{U}$. $N(x)$ is the unique outward oriented, unit, normal vector to a given surface through x.

Let us set $\bar{S} = S^0$; for $r > 0$, $S^r = \{y \in E^n \colon d(y, S) \leqslant r\}$; for $r < 0$, $S^r = \{y \in S \colon d(y, \complement(S)) \geqslant -r\}$. It follows from the assumptions that, for all sufficiently small r, S^r is a non empty, closed convex body. For $x \in \Sigma \setminus K$ let $L(x)$ be the half-line from $F(x)$ through x. Set $H(x) = L(x) \cap \partial S$ and, for $|r| < \delta$, $H^r(x) = L(x) \cap \partial S^r$. We also set $f(x) = F(x) - x$ and $g(x) = f(x)/\|f(x)\|$. The norm $\|A\|$ of a matrix A is the operator norm. The Jacobian matrix of h is $D(h)$. The unit ball is denoted by B.

As in [5] most results will depend on the following genericity hypothesis:

Hypothesis (GH). When $x \in K$, $D(f)$ at x is non singular.

§ 1. – In this section we study the set of critical values of our mapping. Theorem 1 establishes that, generically, it is a compact subset of a full neighborhood of our initial point ξ^0.

Lemma 1. *For every pair (r, s) with $|r|, |s| < \delta$, for every $x \in \Sigma \setminus K$, $\|H^r(x) - H^s(x)\| < |r - s| \operatorname{diam}(\Sigma)/\delta$.*

Proof. Set $n = \min\{r, s\}$, $m = \max\{r, s\}$ and $D = \max\big\{\|F(x) - H^r(x)\|, \|F(x) - H^s(x)\|\big\}$. Since the ball $B[F(x), \delta]$ is contained in the convex set S^n, the ball about $H^n(x)$ of radius $\|H^r(x) - H^s(x)\|\delta/D$

is contained in S^m. Hence $|r-s| \geqslant \delta\|H^r(x)-H^s(x)\|/D$, i.e. $\|H^r(x)-H^s(x)\| \leqslant |r-s| \operatorname{diam}(\Sigma)/\delta$.

Whenever defined, for $x \in \partial S$, we set $P\colon (x, r) \mapsto x + rN(x)$.

LEMMA 2. Let $\mathcal{O}$ be an open set whose closure is in $\mathcal{U}$. There exist $\varrho^*\colon B[\mathcal{O}, \varrho^*] \subset \mathcal{U}$ and ϱ, $\varrho \leqslant \frac{1}{2}\varrho^*$, such that: i) P is injective on $(B[\mathcal{O}, \varrho^*] \cap \partial S) \times (-\varrho, \varrho)$ and ii) $d(y, \mathcal{O} \cap \partial S) < \varrho$ implies there exist r, $|r| < \varrho$, and $x \in B[\mathcal{O}, \varrho^*] \cap \partial S\colon y = P(x, r)$. Moreover the mapping $y \mapsto x$ is lipschitzean.

PROOF. Ad i). Since ∂S in C^3 in $\mathcal{U}$, the matrix $D(N)$ is bounded in norm by some L on $B[\mathcal{O}, \varrho^*]$, so that the mapping $x \mapsto N(x)$ is lipschitzean with Lipschitz constant L. Set $\varrho = \min\{L^{-1}, \frac{1}{2}\varrho^*\}$. Assume there exist (x, r) and (x_1, r_1), with $r \geqslant r_1$, such that $y = x + rN(x) = x_1 + r_1 N(x_1)$. The case both r and r_1 non negative is well known [2].

Assume $r_1 < 0$, $r > 0$. By assumption

$$0 \leqslant \langle N(x), N(x_1)\rangle = -(r_1)^{-1}\langle N(x), x_1 - (x_1 + r_1 N(x_1))\rangle$$

so that $\langle N(x), (x_1 + r_1 N(x_1)) - x_1\rangle \leqslant 0$ and, since the tangent plane at x supports S, $\langle N(x), x_1 - x\rangle \leqslant 0$. By adding we have $\langle N(x), (x_1 + r_1 N(x_1)) - x\rangle \leqslant 0$. Hence the points $x_1 + r_1 N(x_1)$ and $x + rN(x)$ are at the opposite sides of the tangent plane.

For the case $r_1 \leqslant r \leqslant 0$, consider the ball centered at y with radius $|r|$: an easy computation shows that $\|N(x) - N(x_1)\| = |r|^{-1}\|x - r(r_1)^{-1}x_1\|$. Also

$$\|x - r(r_1)^{-1}x_1\|^2 = \|x - x_1\|^2 + \|x_1 - r(r_1)^{-1}x_1\|^2 + 2\langle x - x_1, x_1 - r(r_1)^{-1}x_1\rangle .$$

Since the tangent plane to S at x_1 is of support, x and $r(r_1)^{-1}x_1$ are at the opposite sides and $\langle x - x_1, x - r(r_1)^{-1}x_1\rangle \geqslant 0$.

Hence $\|x - r(r_1)^{-1}x_1\| \geqslant \|x - x_1\|$. Finally

$$L\|x_1 - x\| \geqslant \|N(x) - N(x_1)\| = |r|^{-1}\|r(r_1)^{-1}x_1 - x\| > (\varrho)^{-1}\|x_1 - x\| \geqslant L\|x_1 - x\| ,$$

a contradiction.

Ad ii). Let $x \in \partial S$ and r be such that $d(y, \mathcal{O} \cap \partial S) = d(y, x) = r\ (< \varrho)$. Then $x \in (B[\mathcal{O}, \varrho^*] \cap \partial S)$, and $N(x)$ is well defined. The case $y \notin S$ is well known [2]. Assume $y \in S$. We remark that the ball centered at y with radius r is fully contained in $\bar{S}$. Then at x the normal to the ball coincides with $N(x)$ and, as before, $y = P(x, -r)$.

Now consider y_1, y_2 and the corresponding (x_1, r_1), (x_2, r_2). We limit our considerations to the case $r_1 \leqslant r_2 \leqslant 0$, the other cases being treated analogously. Set $y_2' = x_2 + r_1 N(x_2)$. Then

$$\|x_1 - x_2\| \leqslant \|y_1 - y_2'\| + L|r_1|\|x_1 - x_2\| \leqslant$$
$$\leqslant \|y_1 - y_2\| + r_2 - r_1 + L|r_1|\|x_1 - x_2\| .$$

Since $r_2 - r_1 \leqslant \|y_1 - y_2\|$ it follows $\|x_1 - x_2\| \leqslant 2(1 - L|r_1|)^{-1}\|y_1 - y_2\|$.

LEMMA 3. Set $\mathcal{V} = \{y = x + rN(x), x \in \mathcal{O} \cap \partial S \text{ and } |r| < \varrho\}$. Then for $r \in J = (-\varrho, \varrho)$, i) $\partial S^r \cap \mathcal{V} = P(\partial S \cap \mathcal{V}, r)$, ii) $\partial S^r \cap \mathcal{V}$ is a C^2 $(n-1)$-surface and iii) $\mathcal{V}$ is open.

PROOF. Ad i). We have $(\partial S^r \cap \mathcal{V}) \subset P(\partial S \cap \mathcal{V}, r)$. In fact let y: $d(y, \partial S) = |r|$ and $y = x' + r' N(x')$ with $x' \in \mathcal{O}$. Then clearly $|r| \leqslant |r'|$ so that $r \in J$. Let $x \in \partial S$ be such that $d(y, x) = d(y, \partial S)$. Then $d(x, \mathcal{O} \cap \partial S) \leqslant d(x, x') \leqslant 2|r'| \leqslant \varrho^*$ and by i) of Lemma 2, $x = x'$ and $r = r'$. The converse implication is proved in exactly the same way.

Ad ii). Since $\varphi(\cdot)$ is $C^3(\mathcal{U})$, $N(\cdot)$ and $P_r(\cdot) = P(\cdot, r)$ are $C^2(\mathcal{U})$. Also the norm of $D(N)$ is bounded by $L < 1/\varrho$ so that $D(P_r) = I + rD(N)$ is a linear homeomorphism. It follows then that P_r^{-1} is C^2 and that $\varphi^r = \varphi \circ P_r^{-1}$ is C^2. It is then easy to show that $\partial S^r \cap \mathcal{V} = \{y\colon \varphi^r(y) = 0\}$, thus proving the claim.

Ad iii). It follows from point ii) of Lemma 2.

We consider a sequence $a(m) \to r \in J$ as $m \to \infty$. We set $H^* = H^r$, $H^m = H^{a(m)}$ and we denote by h_j^*, h_j^m the j-th component of H^*, H^m.

LEMMA 4. The sequence $\{\partial h_j^m / \partial x_i\}$ converges to $\partial h_j^* / \partial x_i$ uniformly on compact subset of $(H^*)^{-1}(\mathcal{V} \cap \partial S^r)$.

PROOF. Let $C \subset (H^*)^{-1}(\mathcal{V} \cap \partial S^r)$ be compact, $P^0 = (x_1^0, \ldots, x_n^0)^T$ in C and set $P(x_i) = (x_1^0, \ldots, x_i, \ldots, x_n^0)^T$, for x_i such that $P(x_i) \in (H^*)^{-1}(\mathcal{V} \cap \partial S^r)$. The half-line from $F(P)$ through P, $\xi = F(P) + t(P - F(P))$, $t \geqslant 0$, can be reparametrized as

$$\xi = F(P) + \frac{\xi_i - F_i(P)}{x_i - F_i(P)} (P - F(P)) .$$

By setting $\alpha(x_i) = (\alpha_1(x_i), \dots, \alpha_n(x_i))^T$ with

$$\alpha_j(x_i) = \frac{x_j^0 - F_j(P)}{x_i - F_i(P)},$$

when $j \neq i$ and $\alpha_i(x_i) = 1$, we can write also

$$\xi = F(P) + \alpha(x_i)(\xi - F(P)).$$

Let $\varphi^{a(m)}(x) = 0$ $[\varphi^* = 0]$ be the equations of $\partial S^{a(m)} \cap \mathcal{U}$ $[\partial S^r \cap \mathcal{U}]$, and consider the system

$$(1) \qquad \begin{cases} \varphi^{a(m)}(\xi_1, \dots, \xi_n) = 0, \\ \xi_j - F_j(P) - \alpha_j(x_i)(\xi_i - F_i(P)) = 0, \qquad j \neq i, \end{cases}$$

of n equations in the $(n+1)$ unknowns $x_i, \xi_1, \dots, \xi_i, \dots, \xi_n$. By the uniform convergence of the H^m to H^* provided by Lemma 1, this system has a solution for all sufficiently large m.

Set $(\bar{\xi}_1^m, \dots, \bar{\xi}_n^m)^T = L(P^0) \cap (\partial S^m \cap \mathcal{U}) = H^m(P^0)$. The vector $Q = (x_0^i, \bar{\xi}_1^m, \dots, \bar{\xi}_n^m)$ is a solution to the above system. By developing along the elements of the first row, and taking into account that at Q, $\alpha_i(x_i) = 1$, the determinant of the Jacobian matrix of the left hand side of (1) with respect to $(\xi_1, \dots, \xi_n)$, computed at Q, is found to be

$$\text{Det} = (-1)^{i+1} \sum_{j=1}^{n} \left(\frac{\partial \varphi^m}{\partial \xi_j}\right) \alpha_j(x_i^0).$$

We claim that $\text{Det} \neq 0$, i.e. that

$$\langle \text{grad}\, \varphi^m(H^m(P_0)), \alpha(x_i^0) \rangle \neq 0.$$

Otherwise, by multiplying the vector $\alpha(x_i^0)$ by $x_i^0 - F_i^0(P^0)$, grad φ^m would be orthogonal to the vector $P^0 - F(P^0)$, in $H^m(P^0)$. This is a contradiction since $F(P^0)$ is internal to S^m and grad φ^m is a supporting functional.

The implicit function theorem yields the existence of a vector $\Xi^m = (\xi_1^m, \dots, \xi_n^m)$, function of x_i, whose derivatives satisfi

$$(2) \qquad \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{\partial \varphi^m}{\partial \xi_k} \frac{\partial \xi_k^m}{\partial x_i} = 0, \\ \displaystyle\frac{\partial \xi_j^m}{\partial x_i} - \frac{\partial F_j}{\partial x_i} - \frac{\partial \alpha_j}{\partial x_i}(\xi_i^m - F_i(P)) - \alpha_j(x_i)\left(\frac{\partial \xi_i^m}{\partial x_i} - \frac{\partial F_i}{\partial x_i}\right) = 0. \end{cases}$$

System (2) can be solved to give

$$\frac{\partial \varphi^m}{\partial \xi_i}\frac{\partial \xi_i^m}{\partial x_i}+\sum_{k\neq i}\frac{\partial \varphi^m}{\partial \xi_k}\left[\frac{\partial F_k}{\partial x_i}+\frac{\partial \alpha_k}{\partial x_i}(\xi_i^m - F_i(P))+\alpha_k(x_i)\left(\frac{\partial \xi_i^m}{\partial x_i}-\frac{\partial F_i}{\partial x_i}\right)\right]=0$$

that can be written as

$$\sum_{k=1}^{n}\frac{\partial \varphi^m}{\partial \xi_k}\left[\frac{\partial F_k}{\partial x_i}+\frac{\partial \alpha_k}{\partial x_i}(\xi_i^m - F_i(P))+\alpha_k(x_i)\left(\frac{\partial \xi_i^m}{\partial x_i}-\frac{\partial F_i}{\partial x_i}\right)\right]=0$$

Finally

$$(3)\qquad \frac{\partial \xi_i^m}{\partial x_i}=-\frac{\sum_{k=1}^{n}\frac{\partial \varphi^m}{\partial \xi_k}\left[\frac{\partial F_k}{\partial x_i}+\frac{\partial \alpha_k}{\partial x_i}(\xi_i^m - F_i(P))-\alpha_k(x_i)\frac{\partial F_i}{\partial x_i}\right]}{\sum_{k=1}^{n}\frac{\partial \varphi^m}{\partial \xi_k}\alpha_k(x_i)}\,.$$

We interested in the above expression for $P = P^0$. At that point

$$\xi_k^m = \bar{\xi}_k^m = h_k^m(P^0)$$

and

$$\frac{\partial \xi_i^m}{\partial x_i}=\frac{\partial h_i^m}{\partial x_i}\,;\qquad \frac{\partial \varphi^m}{\partial \xi_k}=\frac{\partial \varphi^m}{\partial x_k}\left(H^m(P^0)\right)\,.$$

Let R be the intersection of the line through $H^m(P^0)$ parallel to $N(H^m(P^0))$ with ∂S^r. By construction

$$\frac{\partial \varphi^m}{\partial x_k}\left(H^m(P^0)\right)=\frac{\partial \varphi^*}{\partial x_k}(R)$$

and, as consequence of Lemma 1, it converges uniformly to $(\partial\varphi^*/\partial x_k)(H^*(P^0))$. Moreover

$$\sum_{k=1}^{n}\frac{\partial \varphi^*}{\partial x_k}\left(H^*(P^0)\right)\alpha_k(x)_i^0$$

is bounded away from zero on C, so that the right hand side of (3) converges uniformly to $\partial h_i^*/\partial x_i$.

It is left to prove the same for $\partial h_j^m/\partial x_i$. System (2) yields the above derivative as a linear function of $\partial h_i^m/\partial x_i$ and of h_i^m with bounded coefficients independent of m. Hence uniform convergence holds for $\partial h_j^m/\partial x_i$.

Following Sard [4] we call a point x regular for the mapping H if $D(H)$ at x has maximal rank. An image v is called a regular value if $H^{-1}(v)$ consists of regular points, a critical value otherwise. It is known that, for every r, Z^r, the set of critical value of H^r, is of $(n-1)$-measure zero in ∂S^r. Next Theorem 1 states that the critical set is generically a compact zero dimensional subset of $\mathcal{V}$. To prove it we need a further Lemma.

LEMMA 5. Under assumption (GH), there exists ε such that for every $r \in J$, the set of critical points of H^r is at a distance at least ε from K.

PROOF. Let $\eta > 0$ be such that whenever $e_1, \ldots, e_{n-1}$ are orthonormal vectors and $u_1, \ldots, u_{n-1}$ are bounded in norm by 1, then the vectors $\eta e_i - u_i$, $i = 1, \ldots, n-1$, are linearly independent.

Fix $x^0 \in K$. By (GH), $D(f)$ at x^0 has maximal rank. Since $x \mapsto D(f)$ is continuous, $x \mapsto D(g)$ is both continuous and of maximal rank at x^0 [5] and $f(x^0) = 0$, there exist ε and ζ such that $\|x - x^0\| < \varepsilon$ implies

$$D(f)B \supset \zeta B\,, \qquad \text{rank } D(g) = n-1$$

and

$$\|f(x)\| \leqslant \zeta\delta/\eta\,.$$

We claim that for every $r \in J$, $x \in B[x^0, \varepsilon]$, implies that x is not a critical point of H^r.

We have $H^r(x) = x - \lambda_r(x) g(x)$ and:

$$D(H^r)v = v - \lambda_r(x) D(g) v - \langle \lambda_r', v\rangle g(x)\,,$$

$$D(g)v \;= \big(D(f)v - (f/\|f\|^2)\langle f, D(f)v\rangle\big)/\|f\|\,.$$

Let w be a vector of norm ζ, orthogonal to $f(x)$. Then $w = D(f)v$ for some $v \in B$. Hence

$$\lambda_r D(g)v = \big(\lambda_r D(f)v\big)/\|f\| = \lambda_r w/\|f\|$$

and also

$$\|\lambda_r D(g)v\| \geqslant \lambda_r \zeta\eta/\zeta\delta \geqslant \lambda_r \eta/\lambda_r = \eta$$

i.e. $\lambda_r D(g)v$ contains a $(n-1)$-dimensional ball of radius η in $f^\perp$.

Denote by Π the projection on $\operatorname{Im}(D(g))$; let $e_1, \ldots, e_{n-1}$ be an orthonormal basis in $\operatorname{Im}(D(g))$ and let $v_1, \ldots, v_{n-1}$ be of norm bounded by 1 and such that

$$\lambda_r D(g) v_i = \eta e_i .$$

By our choice of η, the vectors $(\Pi v_i - \lambda_r D(g) v_i)$, $i = 1, \ldots, n-1$, are linearly independent and, being orthogonal to g, so are the vectors

$$(\Pi - \lambda_r D(g)) v_i + \langle g - \lambda_r', v_i \rangle g .$$

The preceding expression is $D(H^r) v_i$. Hence rank $D(H^r) \geqslant n-1$, i.e. x is not critical.

To each $x^0 \in K$ we have associated a positive ε. Since K is compact, an easy argument proves the Lemma.

THEOREM 1. Set $\mathcal{N} = \{y \in \mathcal{V} \colon \exists r \in J \colon y \in Z^r\}$. Under assumption (GH), $\mathcal{N}$ is a relatively compact zero dimensional subset of $\mathcal{V}$.

PROOF. Let $y^* \in \partial S^r \setminus Z^r$ and assume there exist $y^m \in Z^{a(m)}$, with $y^m \to y^*$. Clearly $a(m) \to r$. Let $y^m = H^m(x^m)$, x^m a critical point of H^m, and, using compactness and the statement of Lemma 5, assume that $x^m \to x^* \notin K$. From the uniform convergence of the H^m to H^* it follows that $H^*(x^*) = y^*$. Since x^* is not a critical point, the Jacobian of H^* computed at x^* is such that for some $\zeta > 0$,

$$D(H^*) B \supset 4\zeta B .$$

By continuity, computing $D(H^*)$ at any point x sufficiently close to x^*,

$$D(H^*) B \supset 2\zeta B .$$

Finally the uniform convergence of $D(H^m)$ provided by Lemma 4 gives that, for m large, at the same points,

$$D(H^m) B \supset \zeta B .$$

Hence, for m large, x^m is not a critical point of H^m, a contradiction. Then some ball about y^* does not contain critical points, proving the first claim.

$\mathcal{N}$ is a measurable subset of $\mathcal{V}$. For every $r \in J$, Z^r is of $(n-1)$-dimensional measure zero. By Fubini's theorem $\mathcal{N}$ has n-dimensional measure zero.

§ 2. – The differential equation mentioned in the introduction is defined below, following [3]. Theorem 2 of this section is the convergence result for the solutions of the perturbed problems.

We are going to define continuous functions u^r on the open sets $\mathcal{W}^r = (H^r)^{-1}(\mathcal{V} \cap \partial S^r)$. It is proved in [3] that solutions to

$$\dot{x}^r = u^r(x^r), \qquad x^r(0) = \xi^r \in \partial S^r \cap \mathcal{V}$$

are solutions to

$$H^r(x^r(t)) = \xi^r$$

i.e. to

$$D\big(H^r(x^r(t))\big)\frac{dx^r}{dt} = 0, \qquad x^r(0) = \xi^r.$$

Consider ξ^0: there exists an index i: the i-th component of the normal to ∂S at ξ^0 is not zero. By continuity the same is true for ξ in some $\mathcal{O} \cap \partial S$ (we identify this $\mathcal{O}$ and the induced $\mathcal{V}$ with those of Lemma 3). Then by construction, it holds true for the normals to $\partial S^r \cap \mathcal{V}$.

Set u_j^r, the j-th component of u^r, to be the cofactor of the element on the i-th row and j-th column of $D(H^r)$ (so that $D(H^r)u^r = 0$).

THEOREM 2. Let ξ^0 be a regular value of H. Let the solution to $\dot{x} = u(x)$, $x(0) = \xi^0$ exist on $[0, \omega)$. Then, under assumption (GH), for every $T < \omega$, for every $\varepsilon > 0$, there exists δ such that: whenever $\|\xi - \xi^0\| < \delta$, the solution to

$$\dot{x}^\xi = u^\xi(x^\xi), \qquad x^\xi(0) = \xi$$

exists on $[0, T)$ and $\|x^\xi - x\| < \varepsilon$.

PROOF. Set $K_n = \big\{y\colon d\big(y, \complement(H^{-1}(V))\big) < 1/n\big\}$. On $[0, T+\eta]$, the solution $x(\cdot)$ exists and has positive distance from $\complement(H^{-1}(\mathcal{V}))$, i.e. it belongs to K_ν for some ν. Assume δ does not exist. Then there exists a sequence of regular values $\xi(m) \to \xi^0$ for which the conclusion of the theorem does not hold. However, by Lemma 4, for m large, the functions $u^{\xi(m)}$ are defined on K_ν and converge to u. This contradicts the basic convergence theorem [1].

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