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The λ-Inductive Topology on Abelian p-Groups (*).

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1. Introduction.

Let G be an abelian p-group. The p-adic topology on G has the family of subgroups $\{p^n G\}_{n\in\mathbb{N}}$ as a basis of neighborhoods of 0. The Hausdorff completion $L_\omega(G) = \varprojlim G/p^n G$ of G in this topology is not a p-group if G is unbounded; the maximal torsion subgroup of $L_\omega(G)$, which is a p-group, is denoted by $T_\omega(G)$ and is called the torsion completion of G; G is said torsion complete if the canonical homomorphism $\delta\colon G \to L_\omega(G)$ sends G isomorphically upon $T_\omega(G)$. The class of torsion complete p-groups is one of the best known in the theory of abelian p-groups (see [4, Chap. XI]).

B. Charles made in 1967 the important observation that the torsion completion $T_\omega(G)$ of G can be realized as the completion of G in a suitable topology, which is the inductive topology; a basis of neighborhoods of 0 for this topology is the family of large subgroups, introduced by Pierce [11], which are those fully invariant subgroups L of G such that $G = L + B$ for every basic subgroup B of G; for the proof of this fact see [2] or [4, 70.2]. Large subgroups have a nice description: in fact Pierce proved ([11]; see also [4, 67.2]) that, if G is unbounded, every large subgroup L of G can be realized as $L = G(\boldsymbol{u})$, where $\boldsymbol{u} = (r_n)_{n\in\mathbb{N}}$ is an increasing sequence of integers, and $G(\boldsymbol{u}) = \{x \in G \mid h_G(p^n x) \geq r_n, \forall n \in \mathbb{N}\}$ (h_G denotes the height in G); note that

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the subgroups $G(\boldsymbol{u})$ described above are a natural generalization of the subgroups $p^n G$ $(n \in \mathbf{N})$.

The generalization of the p-adic topology from the ordinal ω to an arbitrary limit ordinal λ is the λ-adic topology, which has as a basis of neighborhoods of 0 the family of subgroups $\{p^\sigma G\}_{\sigma<\lambda}$; this topology has been studied by Mines [10]. The Hausdorff completion of G in this topology and its torsion part are denoted respectively by $L_\lambda(G)$ and $T_\lambda(G)$; if λ is cofinal with ω (we shall write $\lambda = \mathrm{cf}\,\omega$), then, if $\delta_\lambda\colon G \to L_\lambda(G)$ is the canonical homomorphism, $L_\lambda(G)/\delta_\lambda(G)$ is divisible and the topology of the completion on $L_\lambda(G)$ coincides with the λ-adic topology; if λ is not cofinal with ω (we shall write $\lambda \neq \mathrm{cf}\,\omega$), then $L_\lambda(G) = T_\lambda(G)$ and in general $L_\lambda(G)/\delta_\lambda(G)$ is not divisible, and the two above topologies do not coincide. The generalization of torsion complete groups is given by those p-groups G such that δ_λ sends G isomorphically upon $T_\lambda(G)$; these groups coincide, if $\lambda = \mathrm{cf}\,\omega$, with the p^λ-high injective groups, which have been studied by Megibben [9] and Dubois [3].

It is natural to try to generalize the inductive topology from the ordinal ω to an arbitrary limit ordinal λ; recall that, given an increasing sequence of ordinals and symbols ∞: $\boldsymbol{u} = (\sigma_n)_{n\in\mathbf{N}}$ $(\sigma < \infty$ for each ordinal σ and $\infty < \infty)$, the subgroup of G

$$G(\boldsymbol{u}) = \{x \in G \mid h_G(p^n x) \geqslant \sigma_n,\ \forall n \in \mathbf{N}\}$$

is a fully invariant subfroup of G. We define the *λ-inductive topology* on G by means of a basis $\mathfrak{B}_\lambda$ of neighborhoods of 0: $\mathfrak{B}_\lambda$ consists of those subgroups $G(\boldsymbol{u})$ defined above such that $\sigma_n < \lambda$ for every $n \in N$. In section 2 we shall study some properties of the λ-inductive topology and we shall prove the analogue of Charle's result, i.e. that the completion $I_\lambda(G)$ of G in the λ-inductive topology is canonically isomorphic to $T_\lambda(G)$; it follows that, if $\lambda = \mathrm{cf}\,\omega$, a reduced p-group is p^λ-high injective if and only if it is complete in the λ-inductive topology.

A more natural definition of λ-inductive topology would get as a basis of neighborhoods of 0 the family of λ-large subgroups of G, i.e. those fully invariant subgroups of G such that $G = L + B$ for every λ-basic subgroup B of G; this definition makes sense only for those p-groups which have λ-basic subgroups; in the non trivial cases, these groups are, as Wallace proved in [12], the C_λ-groups with $\lambda = \mathrm{cf}\,\omega$ (for the definitions of λ-basic subgroup and C_λ-group see Megibben [8] and Wallace [12]). In section 3 we shall prove that for a C_λ-group, with $\lambda = \mathrm{cf}\,\omega$, the λ-inductive topology has as a basis of neighborhoods of 0 the family of λ-large subgroups, by generalizing the result of Pierce which characterize large subgroups.

2. The λ-inductive topology.

All groups considered are abelian groups. Notation and terminology do not depart from those in [4]. In particular, if G is a reduced p-group, $l(G)$ denotes the length of G, i.e. the minimal ordinal σ such that $p^\sigma G = 0$; to every element $x \in G$ is associated an increasing sequence of ordinals and ∞'s: $H(x) = \{h_G(p^n x) | n \in \mathrm{N}\}$, which is said the indicator of x. Let λ be a limit ordinal; let $\mathcal{B}_\lambda$ be the family of subgroups $G(\boldsymbol{u})$ of G, where $\boldsymbol{u} = (\sigma_n)_{n \in \mathrm{N}}$, such that $\sigma_n < \lambda$ for every $n \in \mathrm{N}$; if $\boldsymbol{u}$ and $\boldsymbol{v}$ are two such sequences of ordinals, the equality $G(\boldsymbol{u}) \cap G(\boldsymbol{v}) = G(\boldsymbol{u} \vee \boldsymbol{v})$, where $\boldsymbol{u} \vee \boldsymbol{v}$ denotes the pointwise supremum, ensures that $\mathcal{B}_\lambda$ is a basis of neighborhoods of 0 for a topology, which is said the λ-inductive topology. We shall consider now some properties of the λ-inductive topology.

1) *Every $G(\boldsymbol{u}) \in \mathcal{B}_\lambda$ is closed in the λ-adic topology.*

It follows from the easily proved equality: $G(\boldsymbol{u}) = \bigcap_{\tau < \lambda} (p^\tau G + G(\boldsymbol{u}))$.

2) *G is discrete in the λ-inductive topology if and only if $l(G) < \lambda$ and it is Hausdorff if and only if $l(G) \leqslant \lambda$.*

G is discrete if and only if there exists $G(\boldsymbol{u}) \in \mathcal{B}_\lambda$ such that $G(\boldsymbol{u}) = 0$; this implies that $p_{\sigma_0} G[p] = 0$, where σ_0 is the first ordinal of $\boldsymbol{u}$, hence $p^{\sigma_0} G = 0$ and $l(G) \leqslant \sigma_0 < \lambda$. The viceversa is obvious. The second claim follows from the following property.

3) $p^\lambda G = \bigcap \{G(\boldsymbol{u}) | G(\boldsymbol{u}) \in \mathcal{B}_\lambda\}$.

If $\boldsymbol{u} = (\sigma_n)_{n \in \mathrm{N}}$ with $\sigma_n < \lambda$ for every $n \in \mathrm{N}$, then obviously $p^\lambda G \subseteq G(\boldsymbol{u})$, hence $p^\lambda G \subseteq \bigcap \{G(\boldsymbol{u}) | G(\boldsymbol{u}) \in \mathcal{B}_\lambda\}$; the opposite inclusion follows from the fact that, if $\sigma < \lambda$, then $p^\sigma G \in \mathcal{B}_\lambda$: in fact $p^\sigma G = G(\sigma, \sigma + 1, \sigma + 2, \ldots)$.

4) *The λ-adic topology is coarser then the λ-inductive topology; the two topologies coincide if $\lambda \neq cf\omega$.*

We have already seen that, if $\sigma < \lambda$, then $p^\sigma G \in \mathcal{B}_\lambda$; if $\lambda \neq cf\,\omega$, given $G(\boldsymbol{u}) \in \mathcal{B}_\lambda$ let $\sigma = \sup \{\sigma_n | n \in \mathrm{N}\}$; then $\sigma < \lambda$ and $G(\boldsymbol{u}) \supseteq p^\sigma G$ imply that $G(\boldsymbol{u})$ is open in the λ-adic topology.

The above property, with the following theorem 2.1, clarify the role of the cofinality with ω of the ordinal λ in the λ-adic topology. Let now A be a subgroup of the reduced p-group G. We say that A satisfies the λ-th *Pierce's condition* if, for every $n \in \mathbb{N}$, there exists an ordinal $\tau_n < \lambda$ such that

$$p^{\tau_n} G[p^n] \subseteq A \,. \tag{1}$$

5) *A subgroup A of G is open in the λ-inductive topology if and only if it satisfies the λ-th Pierce's condition.*

If A is open there exists $G(\boldsymbol{u}) \in \mathfrak{B}_\lambda$, $\boldsymbol{u} = (\sigma_n)_{n\in\mathbb{N}}$, such that $G(\boldsymbol{u}) \subseteq A$; then A satisfies (1): just take $\tau_n = \sigma_n$. Viceversa, let $\sigma_n = \tau_{n+1} + n$ for each $n \in \mathbb{N}$. If $\boldsymbol{u} = (\sigma_n)_{n\in\mathbb{N}}$ and $x \in G(\boldsymbol{u}) \cap G[p]$, then $px = 0$ and $h_G(x) \geqslant \sigma_0 = \tau_1$ imply $x \in A$; now we suppose by induction that, given $k \geqslant 1$, $G(\boldsymbol{u}) \cap G[p^{k-1}] \subseteq A$ and let $p^k x = 0$, $x \in G(\boldsymbol{u})$; then $h_G(p^{k-1}x) \geqslant \sigma_{k-1} = \tau_k + k - 1$, therefore there exists $y \in G$ such that $p^{k-1}x = p^{k-1}y$ and $h_G(y) \geqslant \tau_k$; it follows that $y \in A$ and $h_G(p^n(x-y)) \geqslant \inf\{\sigma_n, \tau_k + n\} = \sigma_n$ for $n = 0, 1, \ldots, k-2$; hence $x - y \in G(\boldsymbol{u}) \cap G[p^{k-1}] \subseteq A$ and $x \in A$.

6) *The λ-inductive topology on $G/p^\lambda G$ coincides with the quotient topology of the λ-inductive topology on G.*

It follows easily from the equality $G(\boldsymbol{u})/p^\lambda G = (G/p^\lambda G)(\boldsymbol{u})$, which holds if $\boldsymbol{u} = (\sigma_n)_{n\in\mathbb{N}}$ with $\sigma_n < \lambda$ for every $n \in \mathbb{N}$.

Let now $I_\lambda(G)$ be the Hausdorff completion of G in the λ-inductive topology; if $l(G) < \lambda$, then $I_\lambda(G) = G$; if $l(G) > \lambda$, from property 6 it follows that $I_\lambda(G) = I_\lambda(G/p^\lambda G)$; therefore we can suppose $l(G) = \lambda$. Recall that $I_\lambda(G) = \varprojlim_{G(\boldsymbol{u}) \in \mathfrak{B}_\lambda} G/G(\boldsymbol{u})$ and $L_\lambda(G) = \varprojlim_{\sigma < \lambda} G/p^\sigma G$; but $p^\sigma G \in \mathfrak{B}_\lambda$ if $\sigma < \lambda$, therefore we can define a homomorphism

$$\varphi \colon I_\lambda(G) \to L_\lambda(G)$$

in the following way: if $(g_{\boldsymbol{u}} + G(\boldsymbol{u}))_{\boldsymbol{u}} \in I_\lambda(G)$, we consider only the cosets modulo the subgroups $p^\sigma G$, where $\sigma < \lambda$, denote them by $g_\sigma + p^\sigma G$ and put

$$\varphi\big((g_{\boldsymbol{u}} + G(\boldsymbol{u}))_{\boldsymbol{u}}\big) = (g_\sigma + p^\sigma G)_\sigma ; \tag{2}$$

with this notation we can prove the following theorem.

THEOREM 2.1. *The homomorphism φ sends $I_\lambda(G)$ isomorphically upon $T_\lambda(G)$.*

PROOF. We prove first that, if in (2) $g_\sigma \in p^\sigma G$ for each $\sigma < \lambda$ and if $G(\boldsymbol{u}) \in \mathfrak{B}_\lambda$, then $g_{\boldsymbol{u}} \in G(\boldsymbol{u})$, hence φ is injective. Let τ be an arbitrary ordinal less then λ and $\boldsymbol{v} = p^\tau G = (\tau, \tau+1, \tau+2, \ldots)$; put $\boldsymbol{w} = \boldsymbol{u} \vee \boldsymbol{v}$; then $g_{\boldsymbol{w}} - g_{\boldsymbol{v}} \in p^\tau G$, therefore $g_{\boldsymbol{w}} \in p^\tau G$; it follows that $g_{\boldsymbol{u}} \in p^\tau G + G(\boldsymbol{u})$, for every $\tau < \lambda$; by property 1) $g_{\boldsymbol{u}} \in G(\boldsymbol{u})$. We shall prove now that $I_\lambda(G)$ is a p-group; let exist, by way of contradiction, $(g_{\boldsymbol{u}} + G(\boldsymbol{u}))_{\boldsymbol{u}} \in I_\lambda(G)$ such that, for a suitable increasing sequence of indexes: $\boldsymbol{u}_1 < \boldsymbol{u}_2 < \boldsymbol{u}_3 < \ldots$ the orders p^{m_k} of $g_{\boldsymbol{u}_k} + G(\boldsymbol{u}_k)$ give an increasing sequence of positive integers; we can suppose that $m_{k+1} \geqslant m_k + k$ for every $k \in \mathrm{N}$. Let $\boldsymbol{u}_k$ be fixed; then $p^{m_k-1} g_{\boldsymbol{u}_k} + G(\boldsymbol{u}_k)$ has order p and, if $G(\boldsymbol{u}) \subseteq G(\boldsymbol{u}_k)$, then $p^{m_k-1} g_{\boldsymbol{u}} + G(\boldsymbol{u}_k)$ has also order p. Then $p^{m_k-1} g_{\boldsymbol{u}_k} \notin G(u_k)$, therefore, by property 1), there exists $\tau_k < \lambda$ such that

$$\tau_k = \sup \{h_G(p^{m_k-1} g_{\boldsymbol{u}}) | \boldsymbol{u} \geq \boldsymbol{u}_k,\, G(u) \in \mathfrak{B}_\lambda\}\,.$$

For every $k \in \mathrm{N}$ let ϱ_k be ordinals such that $\tau_{k+1} < \varrho_k < \lambda$ and $\varrho_k < \varrho_{k+1}$; consider now $G(\boldsymbol{v}) \in \mathfrak{B}_\lambda$ where $\boldsymbol{v} = (\varrho_n)_{n\in\mathrm{N}}$. If $g_{\boldsymbol{v}} + G(\boldsymbol{v})$ has order p^n, then $p^n g_{\boldsymbol{u}} \in G(\boldsymbol{v})$ for every $\boldsymbol{u} \geqslant \boldsymbol{v}$ with $G(\boldsymbol{u}) \in \mathfrak{B}_\lambda$. Let $r \in \mathrm{N}$ be such that $m_r \geqslant n$. If $\boldsymbol{u} \geqslant \boldsymbol{v} \vee \boldsymbol{u}_{k+1}$ and $G(\boldsymbol{u}) \in \mathfrak{B}_\lambda$, then $p^{m_r} g_{\boldsymbol{u}} \in G(\boldsymbol{v})$ implies

$$h_G(p^{m_{r+1}-1} g_{\boldsymbol{u}}) \geqslant \varrho_{m_{r+1}-m_r-1};$$

but $m_{r+1} - m_r - 1 \geqslant r$ implies $\varrho_{m_{r+1}-m_r-1} \geqslant \varrho_r > \tau_{r+1}$ and $\boldsymbol{u} \geqslant \boldsymbol{u}_{r+1}$ implies

$$h_G(p^{m_{r+1}-1} g_{\boldsymbol{u}}) \leqslant \tau_{r+1}$$

which is absurd. From the above proof we get the inclusion $\varphi(I_\lambda(G)) \subseteq T_\lambda(G)$; let now $(g'_\sigma + p^\sigma G)_\sigma \in T_\lambda(G)$ be an element of order m; if $G(\boldsymbol{u}) \in \mathfrak{B}_\lambda$ let $g_{\boldsymbol{u}} = g'_{\sigma_m}$; it is easy to see that $\varphi\big((g_{\boldsymbol{u}} + G(\boldsymbol{u}))_{\boldsymbol{u}}\big) = (g'_\sigma + p^\sigma G_\sigma)_\sigma$ and that $(g_{\boldsymbol{u}} + G(\boldsymbol{u}))_{\boldsymbol{u}} \in I_\lambda(G)$, therefore the image of φ is the whole of $T_\lambda(G)$.

An obvious consequence of theorem 2.1 is the following

COROLLARY 2.2. *A reduced p-group G is complete in the λ-inductive topology if and only if $p^\lambda G = 0$ and $\delta_\lambda(G) = T_\lambda(G)$.*

By property 4) we are interested in the λ-inductive topology on a reduced p-group only if $\lambda = \mathrm{cf}\,\omega$; in this case the λ-adic topology on $I_\lambda(G) \cong T_\lambda(G)$ coincides with the topology of the completion of G endowed with the λ-adic topology, hence the subgroups $p^\sigma I_\lambda(G)$ with $\sigma < \lambda$, are open in the topology of the completion of G endowed with the λ-inductive topology; it follows that, if $G(\boldsymbol{u}) \in \mathfrak{B}_\lambda$, then $I_\lambda(G)(\boldsymbol{u})$ is closed in the above topology, hence it contains the closure of $\delta_\lambda(G(\boldsymbol{u}))$ which is open; therefore $I_\lambda(G)(\boldsymbol{u})$ is open in the topology of the completion of G endowed with the λ-inductive topology. A subbase of neighborhoods of 0 for this topology is given by the subgroups $I_\lambda(G) \cap \operatorname{Ker} \pi_{\boldsymbol{u}}$, where $G(\boldsymbol{u}) \in \mathfrak{B}_\lambda$ and

$$\pi_{\boldsymbol{u}}: \prod_{G(\boldsymbol{v}) \in \mathfrak{B}_\lambda} G/G(\boldsymbol{v}) \to G/G(\boldsymbol{u}) ,$$

is the canonical surjection; we shall prove that $I_\lambda(G)(\boldsymbol{u})$ is contained in $I_\lambda(G) \cap \operatorname{Ker} \pi_{\boldsymbol{u}}$; let $(g_{\boldsymbol{v}} + G(\boldsymbol{v}))_{\boldsymbol{v}} \in I_\lambda(G)(\boldsymbol{u}) \subseteq \prod_{G(\boldsymbol{v}) \in \mathfrak{B}_\lambda} (G/G(\boldsymbol{v}))(\boldsymbol{u})$, let r be the order of $g_{\boldsymbol{u}}$ and, if $\boldsymbol{u} = (\sigma_n)_{n \in \mathbf{N}}$, put $\boldsymbol{w} = (\sigma_r, \sigma_{r+1}, \sigma_{r+2}, \dots)$. If $\sigma < \sigma_r$ we have; $p^\sigma(G/G(\boldsymbol{w})) = p^\sigma G/G(\boldsymbol{w})$, therefore, for $i = 0, 1, \dots, \dots, r-1$, the following inequalities hold: $h_G(p^i g_{\boldsymbol{w}}) \geqslant \sigma_i$. But $\boldsymbol{w} \geqslant \boldsymbol{u}$ implies $g_{\boldsymbol{w}} - g_{\boldsymbol{u}} \in G(\boldsymbol{u})$, hence $h_G(p^i g_{\boldsymbol{u}}) \geqslant \sigma_i$ for $i = 0, 1, \dots, r-1$, which means that $g_{\boldsymbol{u}} \in G(\boldsymbol{u})$. Thus we have proved the following

THEOREM 2.3. *Let G be a p-group and λ a limit ordinal cofinal with ω. Then the topology of the completion (of G endowed with the λ-inductive topology) on $I_\lambda(G)$ coincides with its own λ-inductive topology.*

3. λ-large subgroups of C_λ-groups.

A p-group G is a C_λ-group, where λ is a limit ordinal, if $G/p^\sigma G$ is totally projective for every ordinal $\sigma < \lambda$. The C_λ-groups have been introduced by Megibben [8] with the restriction on λ to be countable; Wallace studied [12] the general case; in particular, he proved that a reduced p-group G of length $\geqslant \lambda$ contains a proper λ-basic subgroup B if and only if G is a C_λ-group and $\lambda = \operatorname{cf} \omega$ (recall that B is a λ-basic subgroup of G if it is totally projective of length $\leqslant \lambda$ and isotype and dense in the λ-adic topology in G). We are interested on groups with proper λ-basic subgroups, therefore from now on G will denote a C_λ-group with $\lambda = \operatorname{cf} \omega$. A subgroup L of G is said λ-large if it is fully invariant and $G = L + B$ for every λ-basic subgroup B of G; this definition has been introduced by Linton (see [6]). If $l(G) < \lambda$, then the unique λ-basic subgroup of G is G itself and every fully invariant subgroup is λ-large; to exclude this trivial case, we suppose that $l(G) \geqslant \lambda$. If L is λ-large in G, then $p^n L$ is also λ-large for every $n \in \mathbb{N}$.

LEMMA 3.1. *Let L be a λ-large subgroup of the C_λ-group G, with $\lambda = \operatorname{cf} \omega$. Then $p^\lambda G$ is contained in L.*

PROOF. We can suppose $l(G) > \lambda$. Let $x \in p^\lambda G$; if $G = L + B$ for a λ-basic subgroup B of G, then $x = b + y$, with $b \in B$, $y \in L$. There exists a direct summand B' of B such that $b \in B'$ and $l(B') < \lambda$. By [1, 2.3] we have: $G = B' \oplus G'$; if π is the projection of G upon G' with respect to this decomposition, then $\pi(x) = \pi(y) \in L$ because L is fully invariant, and $(1 - \pi)(x) = 0$, because $h_G(x) = h_B(x) \geqslant \lambda$; it follows that $x = \pi(y) \in L$.

We will need in the following some results on C_λ-groups.

LEMMA 3.2. *Let G be a C_λ-group, $\lambda = \operatorname{cf} \omega$; let $x \in G$ such that $\langle x \rangle \cap p^\lambda G = 0$; then there exists a direct summand X of G containing x such that $l(X) < \lambda$.*

PROOF. The lemma is proved in [1, 2.7] if $l(G) = \lambda$; if $l(G) > \lambda$ the proof is quite similar and it is omitted.

The fact that every C_λ-group of length $\leqslant \lambda$ is fully transitive, which is proved by Le Borgne [5] (see also Linton [6]), is generalized in the following

LEMMA 3.3. *Let G be a C_λ-group, $\lambda = \operatorname{cf} \omega$. Let $x, y \in G$ such that $H(x) \leqslant H(y)$ and $\langle x \rangle \cap p^\lambda G = 0 = \langle y \rangle \cap p^\lambda G$. Then there exists an endomorphism $f \in \operatorname{End}(G)$ such that $f(x) = y$.*

PROOF. From lemma 3.2 one easily deduces a decomposition $G = X \oplus G'$, with $x, y \in X$ and $l(X) < \lambda$; since X is totally projective, hence fully transitive, the claim is now obvious.

LEMMA 3.4. *Let C be a C_λ-group, $\lambda = \operatorname{cf} \omega$, and L a fully invariant subgroup of G such that $p^\lambda G \subseteq p^\omega L$. If $\sigma_n = \inf \{h_G(p^n x) | x \in L\} < \lambda$ for every $n \in \mathbb{N}$, there exists, for each $r \in \mathbb{N}$, an element $g \in L$ such that $h_G(p^i g) = \sigma_i$ for $i = 0, 1, \ldots, r$.*

PROOF. We induct on r. If $r = 0$ the claim is obvious; suppose $r \geqslant 1$; there exists an element $g' \in L$ such that

$$H(g') = (\sigma_0, \sigma_1, \ldots, \sigma_{r-1}, \tau_r, \tau_{r+1}, \ldots)$$

where we can suppose $\tau_r > \sigma_r$, otherwise g' is the wanted element. There exists an element $t \in L$ such that $h_G(p^r t) = \sigma_r$. Let $k(t)$ be the minimal non negative integer such that $h_G(p^i t) = \sigma_i$ if $k(t) \leqslant i \leqslant r$. We choose $t \in L$ such that $k(t)$ is minimal; let us observe that $k(t) \leqslant \leqslant r - 1$ (otherwise consider $t + g'$). Our goal is to prove that $k(t) = 0$. Among the elements t with $k(t)$ minimal, choose one of minimal order $m + 1$ $(m \geqslant r)$; then $h_G(p^m t) < \lambda$ otherwise $p^m t \in p^\lambda G \subseteq p^\omega L$, hence $p^m t = p^{m+1} t'$ for a $t' \in L$; in this case $p^m(t - pt') = 0$ and $h_G(p^i(t - pt')) = \sigma_i$ if $k(t) \leqslant i \leqslant r$, because $h_G(p^i t) = \sigma_i$ and $h_G(p^{i+1} t') \geqslant \geqslant \sigma_{i+1} > \sigma_i$. Thus, without loss of generality, we can assume, if $k = k(t)$:

$$H(t) = (\sigma_0', \sigma_1', \ldots, \sigma_{k-1}', \sigma_k, \sigma_{k+1}, \ldots, \sigma_r, \sigma_{r+1}', \ldots, \sigma_m', \infty, \ldots)$$

where $\sigma_{k-1}' > \sigma_{k-1}$, $\sigma_m' < \lambda$ and $k \leqslant r - 1$; suppose, by way of contradiction, that $k > 0$. There exist elements $b \in L$ such that $h_G(p^{k-1} b) = \sigma_{k-1}$ and $h_G(p^r b) = \sigma_r$ (consider for instance $t + g'$); choose b among these elements such that $k(b)$ is minimal; it is enough to prove that $k(b) \leqslant k - 1$. Let, by absurd, $k(b) \geqslant k$ and put $k(b) = k'$. Then $h_G(p^{k'} b) = \sigma_{k'}$ and $h_G(p^{k'-1} b) > \sigma_{k'-1}$ imply that $\sigma_{k'} > \sigma_{k'-1} + 1$; therefore there exists an element $w \in G$ such that $p^{k'} t = p^{k'} w$ and $h_G(p^{k'-1} w) > \sigma_{k'-1}$; of course we can choose w in such a way that $H(w) \geqslant H(t)$; because of $\langle t \rangle \cap p^\lambda G = 0 = \langle w \rangle \cap p^\lambda G$, we can apply lemma 3.3 and deduce that $w \in L$. Then we have: $p^{k'}(t - w) = 0$ and $h_G(p^{k'-1}(t - w)) = \sigma_{k'-1}$. Consider now the element $b - t + w \in L$; then $k(b - t + w) < k(b) = k'$, because $h_G(p^{k'-1}(b - t + w)) = \sigma_{k'-1}$, which contradicts the choice of b.

We can now prove the main result of this section, which generalizes the characterization of large subgroups given by Pierce [11].

THEOREM 3.5. *Let G be a C_λ-group, $\lambda = \operatorname{cf} \omega$, with $l(G) \geqslant \lambda$. Let L be a fully invariant subgroup of G. Then the following are equivalent*:

1) *$L = G(\boldsymbol{u})$, where $\boldsymbol{u} = (\sigma_n)_{n \in \mathbb{N}}$ is an increasing sequence of ordinals and $\sigma_n < \lambda$ for every $n \in \mathbb{N}$.*

2) *L satisfies the λ-th Pierce's condition.*

3) *L is λ-large.*

PROOF. 1) $\rightarrow$ 2) See property 5) of section 2.

2) $\rightarrow$ 3) Let B be a λ-basic subgroup of G and let $g \in G$ be an element of order $k \in \mathbb{N}$; there exists an ordinal $\sigma < \lambda$ such that $p^\sigma G[p^k] \subseteq L$;

but $G = p^\sigma G + B$, hence $g = g' + b$ with $g' \in p^\sigma G$, $b \in B$; then $0 = p^k g = p^k g' - p^k b$, and $p^k b \in p^{\sigma+k} G \cap B = p^{\sigma+k} B$; therefore there exists a $b' \in p^\sigma B$ such that $p^k b = p^k b'$; thus we obtain: $g = (g' + b') + (b - b')$ where $b - b' \in B$ and $p^k(b - b') = 0$; it follows that $g' + b' \in p^\sigma G[p^k] \subseteq L$ and $g \in L + B$.

3) $\to$ 1) If $l(G) = \lambda$, then G is fully transitive, therefore every fully invariant subgroup L is equal to $G(\boldsymbol{u})$, for a suitable increasing sequence $\boldsymbol{u} = (\sigma_n)_{n \in \mathrm{N}}$ of ordinals and ∞'s. If $\sigma_n \geqslant \lambda$ for some n, then $p^n L \subseteq p^\lambda G = 0$, which is absurd, because $p^n L$ is λ-large while 0 is not λ-large. Suppose now that $l(G) > \lambda$. For every $n \in \mathrm{N}$ we put:

$$\sigma_n = \inf \{h_G(p^n x) | x \in L\};$$

obviously $\sigma_n < \sigma_{n+1}$ and $\sigma_n < \lambda$ for each n, otherwise $p^n L = p^\lambda G$, which is absurd because $p^\lambda G$ is not λ-large; the inclusion $L \subseteq G(\boldsymbol{u})$, where $\boldsymbol{u} = (\sigma_n)_{n \in \mathrm{N}}$ is obvious; we suppose by absurd that this inclusion is proper; let y be an element of minimal order among the elements of $G(\boldsymbol{u}) \setminus L$; if $H(y) = (\tau_0, \tau_1, \ldots, \tau_k, \infty, \infty, \ldots)$ with $\tau_k < \infty$, then $\tau_k < \lambda$: in fact, if $\tau_k \geqslant \lambda$, then $p^k y \in p^\lambda G$, which is contained in $\bigcap_{n \in \mathrm{N}} p^n L = p^\omega L$, by lemma 3.1; therefore there exists an element $z \in L$ such that $p^k(y - z) = 0$ and $y - z \in G(\boldsymbol{u}) \setminus L$, contradicting the choice of y. Let now x be an element of minimal order among the elements of L such that $h_G(p^i x) = \sigma_i$ for $i = 0, 1, \ldots, k$; such an element does exist, as lemma 2.4 shows; let $H(x) = (\sigma_0, \sigma_1, \ldots, \sigma_k, \sigma'_{k+1}, \ldots, \sigma'_n, \infty, \ldots)$, with $\sigma'_n < \infty$; then $\sigma'_n < \lambda$, otherwise, fixed an integer $m \geqslant 1$, there exists and element $a \in L$ such that $p^n x = p^{n+m} a$, which implies that $p^n(x - p^m a) = 0$; but if $i = 0, 1, \ldots, k$, then $h_G(p^i(x - p^m a)) = \inf \{\sigma_i, h_G(p^{m+i} a)\} = \sigma_i$, contradicting the choice of x. We are now in the hypotheses of lemma 3.3, therefore there exists an endomorphism f of G such that $f(x) = y$; but $x \in L$ and L is fully invariant, hence $y \in L$ and $L = G(\boldsymbol{u})$.

Theorem 3.5 enables us to improve theorems 4 and 5 of [6] to C_λ-groups of arbitrary length.

COROLLARY 3.6. *Let G be a C_λ-group, $\lambda = \mathrm{cf}\,\omega$, and $L = G(\boldsymbol{u})$ a λ-large subgroup, where $\boldsymbol{u} = (\sigma_n)_{n \in \mathrm{N}}$ and $\sigma_n < \lambda$ for each $n \in \mathrm{N}$. Then G/L is totally projective and, if $\sigma = \sup \sigma_n$ and $\lambda = \sigma + \tau$, then L is a $C_{\omega+\tau}$-group.*

PROOF. If B is a λ-basic subgroup of G, the first fact follows from [6, Thm. 1] and the following isomorphism

$$G/L = G/G(\boldsymbol{u}) = G(\boldsymbol{u}) + B/G(\boldsymbol{u}) \cong B/G(\boldsymbol{u}) \cap B = B/B(\boldsymbol{u}) .$$

It is easy to show that $p^{\omega}L = p^{\sigma}G$; then the second fact follows from the easily proved equalities, which hold for every ordinal ϱ

$$L/p^{\omega+\varrho}L = G(\boldsymbol{u})/p^{\omega+\varrho}G(\boldsymbol{u}) = G(\boldsymbol{u})/p^{\sigma+\varrho}G = (G/p^{\sigma+\varrho}G)(\boldsymbol{u}) .$$

COROLLARY 3.7. *Let G be a C_λ-group, $\lambda = \mathrm{cf}\,\omega$, and L a λ-large subgroup of G. Then L is totally projective if and only if G is totally projective.*

PROOF. The sufficiency follows from [6, Thm. 1]. The necessity follows from [7, Thm. 3], because G/L is totally projective by corollary 3.6.

Let us finally remark that, in the class of C_λ-groups, $\lambda = \mathrm{cf}\,\omega$, two groups complete in the λ-inductive topology, i.e. two reduced p^λ-high injective groups, are isomorphic if and only if they have the same Ulm-Kaplansky invariants. This result has been proved by Megibben [8, Thm. 7] in the case of λ countable and by Crawley [1] in the general case; it has only to be pointed out that, if $\lambda = \mathrm{cf}\,\omega$, the C_λ-groups torsion complete defined by Crawley are essentially the same as the reduced p^λ-high injective C_λ-groups.

Added in proof. While this paper was submitted for publication, appeared R. C. LINTON's paper: *λ-large subgroups of C_λ-groups*, in Pacific J. Math., **78**, no. 2 (1978), in which the equivalence of 1) and 3) of our theorem 3.5 and corollary 3.6 are proved. However his proof is quite different and is based on topological methods.

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