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## Partial Hölder continuity of the gradient of solutions of some nonlinear elliptic systems


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# Partial Hölder Continuity of the Gradient of Solutions of Some Nonlinear Elliptic Systems.

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## Introduction.

The problem which we shall study in this paper is suggested by the following considerations.

Let $\Omega$ be a bounded open set of $R^n$, $N$ an integer $\geqslant 1$, $H^{k,p}(\Omega, R^N)$ and $H_0^{k,p}(\Omega, R^N)$ the usual Sobolev spaces of vector-valued functions $u\colon \Omega \to R^N$. We denote with $(\ |\ )$ and $\|\cdot\|$ the scalar product and the norm in $R^N$ and set $Du = (D_1 u|\dots|D_n u)$, $p = (p^1|\dots|p^n)$ with $p^h \in R^N$.

Let $a^i(x, u, p)$, $i = 0, 1, \dots, n$, be continuous mappings $\bar{\Omega}\times R^N \times \times R^{nN} \to R^N$ and we suppose that $\forall (x, u, p) \in \bar{\Omega}\times R^N \times R^{nN}$ with $\|u\| \leqslant K$ we have

$$(1)\qquad \|a^i(x, u, p)\| \leqslant M(K)\Big(1 + \sum_{h=1}^n \|p^h\|\Big), \qquad i = 1, \dots, n,$$

$$(2)\qquad \|a^0(x, u, p)\| \leqslant M(K)\Big(1 + \sum_{h=1}^n \|p^h\|^2\Big).$$

Let $u \in H^{1,2} \cap C^{0,\gamma}(\Omega, R^N)$, $0 < \gamma < 1$, a solution of the system

$$(3)\qquad \int_\Omega \sum_{i=1}^n (a^i(x, u, Du)|D_i\varphi)\, dx = \int_\Omega (a^0(x, u, Du)|\varphi)\, dx, \quad \forall \varphi \in C_0^\infty(\Omega, R^N).$$

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It is well known that $u \in H^{2,2}_{\mathrm{loc}}(\Omega, R^N)$ under these further conditions: $\forall (x, u, p) \in \bar{\Omega} \times R^N \times R^{nN}$ with $\|u\| \leqslant K$

$$a^i \in C^1(\bar{\Omega} \times R^N \times R^{nN}), \qquad i = 0, \ldots, n, \tag{4}$$

$$\left\| \frac{\partial a^i}{\partial p^j_k} \right\| \leqslant M(K), \qquad i = 1, \ldots, n, \tag{5}$$

$$\left\| \frac{\partial a^i}{\partial u_k} \right\| + \left\| \frac{\partial a^i}{\partial x_r} \right\| \leqslant M(K)\left(1 + \sum_{h=1}^{n} \|p^h\|\right), \qquad i = 1, \ldots, n, \tag{6}$$

$$\left\| \frac{\partial a^0}{\partial p^j_k} \right\| \leqslant M(K)\left(1 + \sum_{h=1}^{n} \|p^h\|\right), \tag{7}$$

$$\left\| \frac{\partial a^0}{\partial u_k} \right\| + \left\| \frac{\partial a^0}{\partial x_r} \right\| \leqslant M(K)\left(1 + \sum_{h=1}^{n} \|p^h\|^2\right), \tag{8}$$

$$\sum_{ij=1}^{n} \sum_{hk=1}^{N} \frac{\partial a^i_h(x, u, p)}{\partial p^j_k} \xi^h_i \xi^k_j \geqslant \nu(K) \sum_{i=1}^{n} \sum_{h=1}^{N} (\xi^h_i)^2, \qquad \forall \xi^1, \ldots, \xi^N \in R^n. \tag{9}$$

Let us suppose that these conditions are true and set the problem of the higher regularity (1), in the Sobolev spaces and in the Hölder continuous spaces, of the solutions of (3). This problem is connected as is known to the Hölder regularity of the derivatives $D_i u$.

In fact if in (3) we assume $\varphi = D_s v$, with $1 \leqslant s \leqslant n$ and $v \in C^\infty_0(\Omega, R^N)$, we have that $u$ is a solution of the system

$$\int_\Omega \sum_{ij=1}^{n} (A_{ij}(x, u, Du) D_{js} u | D_i v)\, dx = \int_\Omega \sum_{i=1}^{n} (F_{is}(x, u, Du) | D_i v)\, dx, \tag{10}$$
$$\forall v \in C^\infty_0(\Omega, R^N) \text{ and } s = 1, \ldots, n.$$

where $A_{ij}$ are $N \times N$ matrices and $F_{is}$ are vectors of $R^N$ defined in the following way

$$A^{hk}_{ij} = \frac{\partial a^i_h}{\partial p^j_k}, \qquad F_{is} = -\left(\frac{\partial a^i}{\partial x_s} + \sum_{k=1}^{N} \frac{\partial a^i}{\partial u_k} p^s_k + \delta_{is} a^0\right). \tag{11}$$

At this point we can write the system (10) as a strongly elliptic system of second order in the vector $U = Du$, with coefficients which depend

(1) That is regularity in $H^{k,2}_{\mathrm{loc}}$ with $k > 2$ and in $C^{h,\gamma}$ with $h \geqslant 1$.

on $x$, $u$, $U$, or as a strongly elliptic system of fourth order in the vector $u$. For what we will prove later the two writings are equivalent, therefore we will use the second writing which is more usual.

In (10) we take $v = D_s\varphi$ with $\varphi \in C_0^\infty(\Omega, R^N)$ and we add with respect to $s$; we obtain that $u$ is a solution of the system of fourth order

$$(12)\qquad \begin{aligned} & u \in H^{2,2}_{\mathrm{loc}} \cap C^{0,\gamma}(\Omega, R^N)\,, \\ & \int_\Omega \sum_{ijrs=1}^{n} (B_{ir,js} D_{js} u | D_{ir}\varphi)\, dx = \int_\Omega \sum_{ir=1}^{n} (F_{ir}|D_{ir}\varphi)\, dx\,, \\ & \forall \varphi \in C_0^\infty(\Omega, R^N)\,, \end{aligned}$$

where

$$(13)\qquad B_{ir,js} = \delta_{rs} A_{ij}$$

are $N \times N$ matrices, $F_{ir}$ are the vectors defined in (11) and their growth follows from the hypotheses (2), (5), (6), in particular $\forall (x, u, p) \in \in \bar{\Omega} \times R^N \times R^{nN}$ with $\|u\| \leqslant K$

$$\begin{aligned} & \|B_{ir,js}\| \leqslant M(K)\,, \\ & \|F_{ir}\| \quad \leqslant M(K)\Big(1 + \sum_{h=1}^{n} \|p^h\|^2\Big)\,. \end{aligned}$$

Furthermore the system (12) is strongly elliptic in the sense that

$$(14)\qquad \sum_{ijrs=1}^{n} (B_{ir,js}\xi^{js}|\xi^{ir}) \geqslant \nu(K) \sum_{ir=1}^{n} \|\xi^{ir}\|^2$$

$\forall$ system of vectors $\xi^{ir} \in R^N$ and $\forall (x, u, p) \in \bar{\Omega} \times R^N \times R^{nN}$ with $\|u\| \leqslant K$.

As, by hypothesis, $u \in C^{0,\gamma}(\bar{\Omega}, R^N)$, if we prove that also $D_s u \in \in C^{0,\gamma}(\Omega, R^N)$ the higher regularity of the solutions of system (12) follows from the regularity of the solutions of a linear system of the fourth order with regular coefficients.

In this paper we will study just this problem: The Hölder continuous regularity of the derivatives of solutions $u$ of systems of type (12).

We shall prove for the $D_i u$ a result of partial regularity as it is natural to expect ([4], [5], [6], [11]).

We note that the system (12) is not of the type of those studied in [6] because the vectors $F_{ir}$ have a quadratic growth in the $D_i u$.

To conclude we observe also that up stream of the problem studied in this paper there is the problem of knowing if the solutions $u \in H^{1,2} \cap \cap L^\infty(\Omega, R^N)$ of the system (3) are also (partial) Hölder continuous and this in the hypotheses (4) ... (9) ([2]) which guarantee that, if $u$ is Hölder continuous, then $u \in H^{2,2}_{\text{loc}}$. This problem is studied in [9] in the special case of diagonal systems and in [4] for the case in which

$$a_h^i(x, u, p) = \sum_{j=1}^{n} \sum_{k=1}^{N} A_{ij}^{hk}(x, u) p_k^j \tag{15}$$

but it is open in the general case (if $N > 1$) ([3]).

In [4] also the partial Hölder continuity of $D_i u$ is obtained supposing that $A_{ij}^{hk}$ (and therefore $a_h^i$) are only Hölder continuous in $(x, u)$. This result is due to the very great regularity of the dependence on $p$ which we have in case (15).

## 1. Statement of the problem.

Let $A_{\alpha\beta}(x, u, p)$, $|\alpha| = |\beta| = 2$, be $N \times N$ matrices and $f_\beta(x, u, p)$, $|\beta| = 2$, vectors of $R^N$ defined in $\bar{\Omega} \times R^N \times R^{nN}$ with these properties: In any set $\bar{\Omega} \times \{\|u\| \leqslant K\} \times R^{nN}$

(1.1) $A_{\alpha\beta}$ are uniformly continuous and bounded: $\|A_{\alpha\beta}\| \leqslant M(K)$.

(1.2) $f_\beta$ are continuous and

$$\|f_\beta(x, u, p)\| \leqslant M(K)\left\{1 + \sum_{h=1}^{n} \|p^h\|^2\right\}.$$

(1.3) Is satisfied the strong ellipticity condition

$$\sum_{|\alpha|=|\beta|=2} (A_{\alpha\beta}\xi^\alpha | \xi^\beta) \geqslant \nu(K) \sum_{|\alpha|=2} \|\xi^\alpha\|^2$$

for any system $\{\xi^\alpha\}_{|\alpha|=2}$ of vectors of $R^N$.

([2]) Furthermore, eventually, the additional condition on $\sup_\Omega \|u(x)\|$ (see (0.7), (0.8) of [4]).

([3]) For $N = 1$ see for instance [10].

Let $u$ be a solution of the system

$$u \in H^{2,2} \cap C^{0,\gamma}(\bar{\Omega}, R^N) ,$$

$$(1.4) \qquad \int_{\Omega} \sum_{|\alpha|=|\beta|=2} (A_{\alpha\beta}(x, u, Du) D^{\alpha} u | D^{\beta}\varphi)\, dx = \\ = \int_{\Omega} \sum_{|\beta|=2} (f_{\beta}(x, u, Du) | D^{\beta}\varphi)\, dx , \qquad \forall \varphi \in C_0^{\infty}(\Omega, R^N) .$$

In particular $u$ is bounded. Denoting by $K$ the $\sup_{\Omega} \|u(x)\|$, we shall omit, for simplicity, in what follows to point out the dependence on $K$ of the constants with appear in (1.1), (1.2), (1.3).

We shall prove (section 3) that there exists an open set $\Omega_0 \subset \Omega$ such that

$$D_i u \in C^{0,\delta}(\Omega_0, R^N) , \qquad \forall \delta \in (0, 1)$$

and

$$H_{n-q}(\Omega - \Omega_0) = 0 , \qquad \text{for a } q > 2$$

where $H_{n-q}$ is the $(n-q)$-dimensional Hausdorff measure.

## 2. Some lemmas.

We denote with $B(R)$ the general open ball of radius $R$ contained in $\Omega$ and by $u_R$ the average on $B(R)$ of $u\colon \Omega \to R^N$.

From Theorem 3.III of [3] it follows that if $u \in H^{2,p} \cap C^{0,\gamma}(\bar{\Omega}, R^N)$, $p \geqslant 1$ and $\gamma \in (0, 1)$, then $\forall B(R) \subset \Omega$

$$(2.1) \qquad D_i u - (D_i u)_R \in L^q_{\text{deb}}(B(R), R^N) , \qquad i = 1, \ldots, n ,$$

where

$$(2.2) \qquad q = \frac{2pn}{n - p\gamma}$$

and $\forall t > 0$ we have the inequality [4]

$$\text{(2.3)} \qquad \operatorname{meas}\{x \in B(R)\colon \|D_i u - (D_i u)_R\| > t\} \leqslant \leqslant c_1(p, q) \frac{[u]^{q/2}_{\gamma,\bar\Omega}}{t^q} \left( \int_{B(R)} \sum_{|\alpha|=2} \|D^\alpha u\|^p \, dx \right)^{q/2p}.$$

It follows, as is known, that $D_i u \in L^s(B(R), R^N)$, $\forall 1 \leqslant s < q$ and we easily get the estimate [5]

$$\text{(2.4)} \qquad \int_{B(R)} \|D_i u - (D_i u)_R\|^s \, dx \leqslant \leqslant c_2(s, p, [u]_{\gamma,\bar\Omega}) \, R^{n(1-s/q)} \left( \int_{B(R)} \sum_{|\alpha|=2} \|D^\alpha u\|^p \, dx \right)^{s/2p}$$

where [6]

$$\text{(2.5)} \qquad c_2 = c_1^{s/q} \omega_n^{1-s/q} \frac{q}{q-s} [u]^{s/2}_{\gamma,\bar\Omega}.$$

From this result ($p = 2$) it follows for the moment that if $u$ is a solution of the system (1.4) then the derivatives $D_i u$ belong to $L^4_{\text{loc}}(\Omega, R^N)$ therefore, in virtue of (1.2), $f_\beta(x, u, Du) \in L^2_{\text{loc}}(\Omega, R^N)$.

Also the two lemmas follow which we now prove and which have a considerable interest for what follows.

LEMMA 2.I. *If $v \in H^{2,2} \cap C^{0,\gamma}(\bar\Omega, R^N)$, for every ball $B(R)$ with $R \leqslant 1$ and for every $p$ satisfying*

$$\text{(2.6)} \qquad \frac{2n}{n+2\gamma} < p < 2$$

*we have*

$$\text{(2.7)} \qquad \sum_{i=1}^n \int_{B(R)} \|D_i v\|^4 \, dx \leqslant \frac{c_3(p, [v]_{\gamma,\bar\Omega})}{R^{n(2/p-1)}} \left( \int_{B(R)} \left\{ \sum_{i=1}^n \|D_i v\|^{2p} + \sum_{|\alpha|=2} \|D^\alpha v\|^p \right\} dx \right)^{2/p}.$$

(4) $[u]_{\gamma,\bar\Omega} = \sup\limits_{x,y\in\bar\Omega} \dfrac{\|u(x)-u(y)\|}{\|x-y\|^\gamma}$.

(5) $\displaystyle\int_A \|v\|^s \, dx = s \int_0^{+\infty} t^{s-1} \operatorname{meas}\{x \in A\colon \|v(x)\| > t\} \, dt.$

(6) $\omega_n$ is the measure of the ball of radius 1 in $R^n$.

PROOF. Since

$$H^{2,2} \cap C^{0,\gamma}(\bar{\Omega}, R^N) \subset H^{2,p} \cap C^{0,\gamma}(\bar{\Omega}, R^N) \quad \text{and} \quad 2pn/(n-p\gamma) > 4\,,$$

we have from (2.4)

$$\int\limits_{B(R)} \|D_i v\|^4\,dx \leqslant c(n)\Big\{\int\limits_{B(R)} \|D_i v - (D_i v)_R\|^4\,dx + R^n \|(D_i v)_R\|^4\Big\} \leqslant$$
$$\leqslant c(n)\Big\{c_2(4, p, [v]_{\gamma,\bar{\Omega}})\,R^{n+2\gamma-(2n/p)}\Big(\int\limits_{B(R)} \sum_{|\alpha|=2} \|D^\alpha v\|^p\,dx\Big)^{2/p} +$$
$$+ R^{n(1-2/p)}\Big(\int\limits_{B(R)} \|D_i v\|^{2p}\,dx\Big)^{2/p}\Big\}$$

and hence the thesis follows provided $R \leqslant 1$, $n + 2\gamma - 2n/p > 0$, $n(2/p - 1) > 0$.

LEMMA 2.II. *If* $v \in H^{2,2} \cap C^{0,\gamma}(\bar{\Omega}, R^N)$, *for every pair of concentric balls* $B(\varrho) \subset B(R) \subset \Omega$ *we have the inequality*

$$(2.8) \qquad \sum_{i=1}^n \int\limits_{B(R)} \|D_i v\|^4 \leqslant$$
$$\leqslant c(n)\Big(\frac{\varrho}{R}\Big)^n \sum_{i=1}^n \int\limits_{B(R)} \|D_i v\|^4\,dx + c_4([v]_{\gamma,\bar{\Omega}})\,R^{2\gamma}\int\limits_{B(R)} \sum_{|\alpha|=2} \|D^\alpha v\|^2\,dx\,.$$

PROOF. Clearly we have that

$$(2.9) \qquad \int\limits_{B(\varrho)} \|D_i v\|^4\,dx \leqslant c(n)\Big\{\int\limits_{B(\varrho)} \|D_i v - (D_i v)_R\|^4\,dx + \varrho^n \|(D_i v)_R\|^4\Big\} \leqslant$$
$$\leqslant c(n)\Big\{\int\limits_{B(R)} \|D_i v - (D_i v)_R\|^4\,dx + \Big(\frac{\varrho}{R}\Big)^n \int\limits_{B(R)} \|D_i v\|^4\,dx\Big\}$$

and from (2.4) where we assume $p = 2$ and $s = 4$

$$(2.10) \qquad \int\limits_{B(R)} \|D_i v - (D_i v)_R\|^4\,dx \leqslant c_2(4, 2, [v]_{\gamma,\bar{\Omega}})\,R^{2\gamma}\int\limits_{B(R)} \sum_{|\alpha|=2} \|D^\alpha v\|^2\,dx\,.$$

From (2.9), (2.10) the thesis follows.

LEMMA 2.III. *Let $F(t)$ and $\phi(t)$ be nonnegative functions, $\phi$ nondecreasing, defined on $(0, R]$. Let $B, \alpha, \beta$ be positive constants with $\beta < \alpha$ and we suppose that* $\forall 0 < \varrho < \sigma \leqslant R$

$$F(\varrho) \leqslant B\left(\frac{\varrho}{\sigma}\right)^{\alpha} F(\sigma) + \sigma^{\beta}\phi(\sigma) . \tag{2.11}$$

*Then* $\forall \varepsilon \in (0, \alpha - \beta]$ *and* $\forall 0 < \varrho < R$

$$F(\varrho) \leqslant BF(R)\left(\frac{\varrho}{R}\right)^{\alpha-\varepsilon} + K(B)\varrho^{\beta}\phi(R) \tag{2.12}$$

*where*

$$K(t) = \frac{(1+t)^{2\alpha/\varepsilon}}{(1+t)^{(\alpha-\beta)/\varepsilon} - t} . \tag{2.13}$$

This is a trivial consequence of Lemma 6.II of [2] since $\phi(t)$ is nondecreasing.

More in general we get this lemma.

LEMMA 2.IV. *Let $\varphi(t)$, $F(t)$, $\phi(t)$ be nonnegative functions, $\phi(t)$ non decreasing, defined in $(0, R]$. Let $A, \alpha, \beta$ be positive constants with $\beta < \alpha$, let $B \geqslant 0$ and we suppose that* $\forall 0 < \varrho \leqslant \sigma \leqslant R$

$$\varphi(\varrho) \leqslant A\left(\frac{\varrho}{\sigma}\right)^{\alpha} \varphi(\sigma) + F(\sigma) , \tag{2.14}$$

$$F(\varrho) \leqslant B\left(\frac{\varrho}{\sigma}\right)^{\alpha} F(\sigma) + \sigma^{\beta}\phi(\sigma) \tag{2.15}$$

*then* $\forall \varepsilon \in (0, \alpha - \beta]$ *and* $\forall 0 < \varrho < R$

$$\varphi(\varrho) \leqslant \{A\varphi(R) + BNF(R)\}\left(\frac{\varrho}{R}\right)^{\alpha-\varepsilon} + K(A)K(B)\varrho^{\beta}\phi(R) \tag{2.16}$$

*where $N = (1 + A)^{2\alpha/\varepsilon}$ and $K(t)$ is defined in* (2.13).

PROOF. Having fixed $\varepsilon \in (0, \alpha - \beta]$, let $\tau \in (0, 1)$ such that

$$(1 + A)\tau^{\varepsilon} = 1 . \tag{2.17}$$

The (2.16) is obviously true if $\tau R \leqslant \varrho < R$. So we suppose

$$R\tau^{h+1} \leqslant \varrho < R\tau^{h} \qquad h \text{ integer } \geqslant 1\,. \tag{2.18}$$

From (2.14) by induction we get $\forall h \geqslant 1$

$$\varphi(\tau^h R) \leqslant (A\tau^\alpha)^h \varphi(R) + (A\tau^\alpha)^{h-1} \sum_{i=0}^{h-1} F(\tau^i R)(A\tau^\alpha)^{-i}\,. \tag{2.19}$$

On the other hand, by Lemma 2.III (7), $\forall i \geqslant 0$

$$F(\tau^i R) \leqslant B\tau^{i(\alpha-\varepsilon)} F(R) + K(B)(\tau^i R)^\beta \phi(R) \tag{2.20}$$

and hence (8)

$$\sum_{i=0}^{h-1} F(\tau^i R)(A\tau^\alpha)^{-i} \leqslant BF(R) \sum_{i=0}^{h-1} (A\tau^\varepsilon)^{-i} + K(B)R^\beta \phi(R) \sum_{i=0}^{h-1} (A\tau^{\alpha-\beta})^{-i}\,. \tag{2.21}$$

From (2.19), (2.21) it follows that $\forall h \geqslant 1$

$$\varphi(\tau^h R) \leqslant (A\tau^\alpha)^h \varphi(R) + \frac{BF(R)}{1 - A\tau^\varepsilon}\, \tau^{(\alpha-\varepsilon)(h-1)} + K(B)R^\beta \phi(R) \frac{\tau^{\beta(h-1)}}{1 - A\tau^{\alpha-\beta}}\,. \tag{2.22}$$

But from (2.14), (2.20)

$$\begin{aligned} \varphi(\varrho) &\leqslant A\left(\frac{\varrho}{R}\right)^\alpha \tau^{-h\alpha} \varphi(\tau^h R) + F(\tau^h R) \leqslant \\ &\leqslant A\left(\frac{\varrho}{R}\right)^\alpha \tau^{-h\alpha} \varphi(\tau^h R) + B\tau^{h(\alpha-\varepsilon)} F(R) + K(B)\,\tau^{h\beta} R^\beta \phi(R)\,. \end{aligned} \tag{2.23}$$

Therefore from (2.22), (2.23), because of the choice we have made of $\tau$ (9),

$$\varphi(\varrho) \leqslant A^{h+1} \left(\frac{\varrho}{R}\right)^\alpha \varphi(R) + \frac{BF(R)}{\tau^{2\alpha}}\, \tau^{(\alpha-\varepsilon)(h+1)} + K(B)K(A)\phi(R)R^\beta \tau^{\beta(h+1)}\,.$$

(7) Or, obviously, from the hypotheses if $i = 0$ or $B = 0$.

(8) $A\tau^{\alpha-\beta} \leqslant A\tau^\varepsilon < 1$.

(9) $(1 + A)t^\varepsilon = 1 \Rightarrow \dfrac{A\tau^{2\varepsilon}}{1 - A\tau^\varepsilon} + \tau^{\alpha+\varepsilon} < 1$

$$\tau \leqslant 1 \Rightarrow \frac{1}{\tau^\beta}\left[\frac{A}{\tau^\beta - A\tau^\alpha} + 1\right] \leqslant \frac{1}{\tau^\alpha(\tau^\beta - A\tau^\alpha)} = K(A)\,.$$

From this the thesis follows provided

$$A^h\left(\frac{\varrho}{R}\right)^\varepsilon < 1 .$$

Let $A_{\alpha\beta}$ be, $|\alpha| = |\beta| = 2$, $N\times N$ constant matrices which satisfy the condition (1.3) and let $f_\beta(x)$ be, $|\beta| = 2$, vectors $\in L^2(B(R), R^N)$.

LEMMA 2.V. *If* $v \in H^{2,2}(B(R), R^N)$ *is a solution of the system*

$$\int\limits_{B(R)} \sum_{|\alpha|=|\beta|=2} (A_{\alpha\beta} D^\alpha v | D^\beta \varphi)\, dx = \int\limits_{B(R)} \sum_{|\beta|=2} (f_\beta(x) | D^\beta \varphi)\, dx , \tag{2.24}$$
$$\forall \varphi \in C_0^\infty(B(R), R^N)$$

*then* $\forall 0 < \varrho \leqslant R$ *we have*

$$\int\limits_{B(\varrho)} \sum_{|\alpha|=2} \|D^\alpha v\|^2\, dx \leqslant c_5(\nu)\left(\frac{\varrho}{R}\right)^n \int\limits_{B(R)} \sum_{|\alpha|=2} \|D^\alpha v\|^2\, dx + \int\limits_{B(R)} \sum_{|\beta|=2} \|f_\beta(x)\|^2\, dx . \tag{2.25}$$

For this result see [2] section 7 and 8 [10].

## 3. The theorem of partial regularity.

Let us begin by proving a regularity result in $L^q$ which will be useful in the following but which is interesting in itself.

THEOREM 3.I. *If* $u$ *is a solution of the system* (1.4) *there exists* $q > 2$ *such that*

$$\sum_{i=1}^n \|D_i u\|^2 + \sum_{|\alpha|=2} \|D^\alpha u\| \in L^q_{\text{loc}}(\Omega) \tag{3.1}$$

[10] In [2] the case of elliptic equations of second order is considered but the method of proof is in no way tied to this particular situation therefore the proof, in our case, would be a useless repetition.

The result has been used in these years in very general situations therefore I consider it known in mathematical literature.

*and for every* $B(2R) \Subset \Omega$, *with* $R \leqslant 1$,

$$(3.2) \qquad \frac{1}{R^n} \int_{B(R)} \Big\{ 1 + \sum_{i=1}^n \|D_i u\|^{2q} + \sum_{|\alpha|=2} \|D^\alpha u\|^q \Big\} dx \leqslant$$

$$\leqslant c_6 \left( \frac{1}{R^n} \int_{B(2R)} \Big\{ 1 + \sum_{i=1}^n \|D_i u\|^4 + \sum_{|\alpha|=2} \|D^\alpha u\|^2 \Big\} dx \right)^{q/2}$$

*where* $c_6 = c_6(\nu, M, [u]_{\gamma,\bar{\Omega}})$.

PROOF. Let $R \leqslant 1$, $B(R) \subset B(2R) \Subset \Omega$ and let

$$\frac{2n}{n+2\gamma} < p < 2 .$$

Since $f_\beta(x, u, Du) \in L^2_{\mathrm{loc}}(\Omega, R^N)$ the (1.4) is true for every $\varphi \in \in H_0^{2,2}(B(2R), R^N)$. Let $\theta \in C_0^\infty(B(2R))$, $0 \leqslant \theta \leqslant 1$, $\theta = 1$ in $B(R)$, $|D^\beta \theta| \leqslant c/R^{|\beta|}$. In (1.4) we assume $\varphi = \theta^4(u - P)$, where $P = (P_1, \ldots, P_N)$ is a polynomial-vector of degree $\leqslant 1$. With standard calculation we get the Cacciappoli inequality

$$(3.3) \qquad \int_{B(R)} \sum_{|\alpha|=2} \|D^\alpha u\|^2 dx \leqslant$$

$$\leqslant c(\nu, M) \Big\{ \sum_{|\gamma| \leqslant 1} \frac{1}{R^{2(2-|\gamma|)}} \int_{B(2R)} \|D^\gamma(u-P)\|^2 dx + \int_{B(2R)} \sum_{|\beta|=2} \|f_\beta(x, u, Du)\|^2 dx \Big\} .$$

In (3.3) we take as $P$ the (unique) polynomial-vector of degree $\leqslant 1$ such that

$$\int_{B(2R)} D^\gamma(u-P)\, dx = 0 , \qquad \forall \gamma\colon |\gamma| \leqslant 1 .$$

Then by Poincaré and Hölder inequalities $(|\gamma| \leqslant 1)$

$$(3.4) \qquad \int_{B(2R)} \|D^\gamma(u-P)\|^2 dx \leqslant c(n) R^{2(1-|\gamma|)} \Big\{ \int_{B(2R)} \sum_{|\alpha|=2} \|D^\alpha u\|^{2n/(n+2)} dx \Big\}^{(n+2)/n} \leqslant$$

$$\leqslant c(n) R^{2(2-|\gamma|)+n(1-2/p)} \Big\{ \int_{B(2R)} \sum_{|\alpha|=2} \|D^\alpha u\|^p dx \Big\}^{2/p} .$$

By (3.3), (3.4) we have that

$$(3.5)\qquad \int_{B(R)} \sum_{|\alpha|=2} \|D^\alpha u\|^2\, dx \leqslant c(\nu, M) R^{n(1-2/p)} \left\{ \int_{B(2R)} \sum_{|\alpha|=2} \|D^\alpha u\|^p\, dx \right\}^{2/p} + \\ + c(\nu, M) \int_{B(2R)} \sum_{|\beta|=2} \|f_\beta\|^2\, dx\,.$$

From (3.5), (1.2) and Lemma 2.I we get that

$$(3.6)\qquad \frac{1}{R^n} \int_{B(R)} \left\{ 1 + \sum_{i=1}^n \|D_i u\|^4 + \sum_{|\alpha|=2} \|D^\alpha u\|^2 \right\} dx \leqslant \\ \leqslant c([u]_{\gamma,\bar{\Omega}}) \left( \frac{1}{R^n} \int_{B(2R)} \left\{ 1 + \sum_{i=1}^n \|D_i u\|^{2p} + \sum_{|\alpha|=2} \|D^\alpha u\|^p \right\} dx \right)^{2/p}.$$

The thesis follows from this estimate and from the Proposition 5.1 of [8].

Let $B(x_0, R) \subset \Omega$; we pose

$$(3.7)\qquad F(x_0, R) = \int_{B(x_0,R)} \left( 1 + \sum_{i=1}^n \|D_i u\|^4 \right) dx\,,$$

$$(3.8)\qquad \phi(x_0, R) = \int_{B(x_0,R)} \left( 1 + \sum_{i=1}^n \|D_i u\|^4 + \sum_{|\alpha|=2} \|D^\alpha u\|^2 \right) dx\,.$$

By the hypothesis (1.1) there exists a function $\omega(t)$ defined and continuous on $t \geqslant 0$, bounded, increasing, concave with $\omega(0) = 0$ such that $\forall x, y \in \bar{\Omega}$, $\forall u, v \in R^N$ with $\|u\| \leqslant K$, $\|v\| \leqslant K$ and $\forall p, p_* \in R^{nN}$

$$(3.9)\qquad \sum_{|\alpha|=|\beta|=2} \|A_{\alpha\beta}(x, u, p) - A_{\alpha\beta}(y, v, p_*)\| \leqslant \\ \leqslant \omega\left( \|x-y\|^2 + \|u-v\|^2 + \sum_{h=1}^n \|p^h - p_*^h\|^2 \right).$$

If $x_0 \in \Omega$ we pose $d(x_0) = \operatorname{dist}(x_0, \partial\Omega)$.

THEOREM 3.II. *If $u$ is a solution of the system* (1.4) $\forall x_0 \in \Omega$, $\forall 0 < \varrho < R \leqslant \min\{2, d(x_0)\}$ *and* $\forall \varepsilon \in (0, n-2\gamma]$ *we have the inequality*

$$\phi(x_0, \varrho) \leqslant c_{11}\phi(x_0, R)\left\{\left(\frac{\varrho}{R}\right)^{n-\varepsilon} + R^{2\gamma} + \left[\omega\left(c_8\frac{\phi(x_0, R)}{R^{n-2}}\right)\right]^{1-2/q}\right. \tag{3.10}$$

*where* $c_{11} = c_{11}(\nu, M, [u]_{\gamma,\overline{\Omega}})$ *and* $q > 2$ *is the exponent which figures in* (3.1).

PROOF. Let $B(2R) = B(x_0, 2R) \subset \Omega$. We pose $A^0_{\alpha\beta} = A_{\alpha\beta}(x_0, u_R, (Du)_R)$ and split $u$ restricted to $B(R)$ into the sum $v + w$ where $w$ is the solution of Dirichlet's problem

$$\begin{aligned} &w \in H_0^{2,2}(B(R), R^N)\,, \\ &\int\limits_{B(R)} \sum_{|\alpha|=|\beta|=2} (A^0_{\alpha\beta} D^\alpha w | D^\beta \varphi)\, dx = \\ &\qquad = \int\limits_{B(R)} \sum_{\substack{|\alpha|=2\\|\beta|=2}} ([A^0_{\alpha\beta} - A_{\alpha\beta}(x, u, Du)] D^\alpha u | D^\beta\varphi)\, dx\,, \\ &\forall \varphi \in H_0^{2,2}(B(R), R^N)\,, \end{aligned} \tag{3.12}$$

and $v$ is solution of the system

$$\begin{aligned} &v \in H^{2,2}(B(R), R^N)\,, \\ &\int\limits_{B(R)} \sum_{|\alpha|=|\beta|=2} (A^0_{\alpha\beta} D^\alpha v | D^\beta\varphi)\, dx = \int\limits_{B(R)} \sum_{|\beta|=2} (f_\beta(x, u, Du) | D^\beta\varphi)\, dx\,, \\ &\forall\varphi \in C_0^\infty(B(R), R^N)\,. \end{aligned} \tag{3.13}$$

By Lemma 2.V and by (1.2) we have that $\forall 0 < \varrho \leqslant \sigma \leqslant R$

$$\int\limits_{B(\varrho)} \sum_{|\alpha|=2} \|D^\alpha v\|^2\, dx \leqslant c_5(\nu)\left(\frac{\varrho}{\sigma}\right)^n \int\limits_{B(\sigma)} \sum_{|\alpha|=2} \|D^\alpha v\|^2\, dx + MF(x_0, \sigma)\,. \tag{3.14}$$

But by Lemma 2.II, for every $0 < \varrho \leqslant \sigma \leqslant R$

$$F(x_0, \varrho) \leqslant c(n)\left(\frac{\varrho}{\sigma}\right)^n F(x_0, \sigma) + c_4\sigma^{2\gamma}\phi(x_0, \sigma)\,. \tag{3.15}$$

Then, by Lemma 2.IV, from (3.14) follows

$$\forall\varepsilon\in(0, n-2\gamma] \text{ and } \forall 0<\varrho<R\,,$$

$$\int\limits_{B(\varrho)}\sum_{|\alpha|=2}\|D^\alpha v\|^2\,dx\leqslant c_5(\nu)\left(\frac{\varrho}{R}\right)^{n-\varepsilon}\int\limits_{B(R)}\sum_{|\alpha|=2}\|D^\alpha v\|^2\,dx + {}$$
$$+\,c_7(\nu, M, [u]_{\gamma,\bar\Omega})\,\phi(x_0, R)\left\{\left(\frac{\varrho}{R}\right)^{n-\varepsilon}+\varrho^{2\gamma}\right\}. \tag{3.16}$$

For estimating the second derivatives of $w$ we now proceed as in [4] (Lemma 2.2):

From (3.12), taken $\varphi = w$, we get

$$\int\limits_{B(R)}\sum_{|\alpha|=2}\|D^\alpha w\|^2\,dx\leqslant c(\nu)\int\limits_{B(R)}\sum_{|\alpha|=|\beta|=2}\|A^0_{\alpha\beta}-A_{\alpha\beta}(x, u, Du)\|^2\cdot\|D^\alpha u\|^2\,dx\leqslant$$
$$\leqslant c(\nu, M)\left[\int\limits_{B(R)}\sum_{|\alpha|=2}\|D^\alpha u\|^q\,dx\right]^{2/q}\cdot$$
$$\cdot\left[\int\limits_{B(R)}\omega\left(R^2+\|u-u_R\|^2+\sum_{i=1}^n\|D_i u-(D_i u)_R\|^2\right)dx\right]^{1-2/q}$$

where $q>2$ is the exponent which figures in Theorem 3.I. By (3.2)

$$\int\limits_{B(R)}\sum_{|\alpha|=2}\|D^\alpha w\|^2\,dx\leqslant c_7(\nu, M, [u]_{\gamma,\bar\Omega})\,\phi(x_0, 2R)\cdot$$
$$\cdot\left[\omega_R\left(R^2+\|u-u_R\|^2+\sum_{i=1}^n\|D_i u-(D_i u)_R\|^2\right)\right]^{1-2/q}. \tag{3.17}$$

But $\omega$ is concave and hence

$$[\omega(g)]_R\leqslant\omega(g_R)\,. \tag{3.18}$$

On the other hand by the Poincaré and Hölder inequalities

$$\frac{1}{\operatorname{meas} B(R)}\int\limits_{B(R)}\left\{\|u-u_R\|^2+\sum_{i=1}^n\|D_i u-(D_i u)_R\|^2\right\}dx\leqslant$$
$$\leqslant c(n)\,\frac{\phi(x_0, R)}{R^{n-2}}\,. \tag{3.19}$$

Then, as $\omega$ is increasing,

$$(3.20)\qquad \int_{B(R)} \sum_{|\alpha|=2} \|D^\alpha w\|^2\, dx \leqslant c_7 \phi(x_0, 2R)\cdot \left[\omega\left(c_8(n)\frac{\phi(x_0, 2R)}{(2R)^{n-2}}\right)\right]^{1-2/q}.$$

From (3.16), (3.20) follows, $\forall 0<\varrho<R$,

$$(3.21)\qquad \int_{B(\varrho)} \sum_{|\alpha|=2} \|D^\alpha u\|^2\, dx \leqslant 2\int_{B(\varrho)} \sum_{|\alpha|=2} \|D^\alpha v\|^2\, dx + 2\int_{B(R)} \sum_{|\alpha|=2} \|D^\alpha w\|^2\, dx \leqslant$$

$$\leqslant c_9\left(\frac{\varrho}{2R}\right)^{n-\varepsilon}\phi(x_0, 2R) + c_{10}\phi(x_0, 2R)\left\{(2R)^{2\gamma} + \left[\omega\left(c_8\frac{\phi(x_0, 2R)}{(2R)^{n-2}}\right)\right]^{1-2/q}\right\}$$

where the constants $c_9$ and $c_{10}$ depend on $\nu$, $M$, $[u]_{\gamma,\bar{\Omega}}$. The (3.21) is trivial for $R\leqslant\varrho<2R$ and from (3.21) the thesis follows because (see (3.15))

$$\phi(x_0, \varrho) = \int_{B(\varrho)} \sum_{|\alpha|=2} \|D^\alpha u\|^2\, dx + F(x_0, \varrho) \leqslant$$

$$\leqslant \int_{B(\varrho)} \sum_{|\alpha|=2} \|D^\alpha u\|^2\, dx + c(n)\left(\frac{\varrho}{2R}\right)^n \phi(x_0, 2R) + c_4 R^{2\gamma}\phi(x_0, 2R).$$

The estimate (3.10) allows us to obtain the partial Hölder continuity on $\Omega$ of the derivatives $D_i u$ with the same method of [4].

Let

$$(3.22)\qquad \Omega_0 = \left\{x\in\Omega\colon \liminf_{R\to 0} \frac{\phi(x, R)}{R^{n-2}} = 0\right\}$$

$H_{n-2}(\Omega - \Omega_0) = 0$ (see [7]) also as (11)

$$\frac{\phi(x_0, R)}{R^{n-2}} \leqslant c\left[\frac{1}{R^{n-q}} \int_{B(x_0,R)} \left\{1 + \sum_{i=1}^n \|D_i u\|^{2q} + \sum_{|\alpha|=2} \|D^\alpha u\|^q\right\} dx\right]^{2/q}$$

by [7] we have

$$(3.23)\qquad H_{n-q}(\Omega - \Omega_0) = 0.$$

(11) $q$ is the exponent which appears in theorem 3.1.

Having set

$$\chi(x_0, R) = R^{2\gamma} + \left[\omega\left(c_8 \frac{\phi(x_0, R)}{R^{n-2}}\right)\right]^{1-2/q}$$

if $x_0 \in \Omega_0$ also

$$\liminf_{R\to 0} \chi(x_0, R) = 0 \tag{3.24}$$

and $x \to \chi(x, R)$ is continuous in $x_0$ for any fixed $R$. It follows that ([4], Theor. 2.1).

LEMMA 3.I. *For every $x_0 \in \Omega_0$ there exists $R < \min\{1, d(x_0)/2\}$ and there exists a ball $B(x_0, r)$ with $r + R < d(x_0)$ such that $\forall \delta \in (0, 1)$, $\forall y \in B(x_0, r)$ and for every $0 < \varrho < R$*

$$\phi(y, \varrho) \leqslant c_{12}\left(\frac{\varrho}{R}\right)^{n-2+2\delta} \phi(y, R) \tag{3.25}$$

*where*

$$c_{12} = [2c_{11}]^{(n-2+2\delta)/(2-2\delta-\varepsilon)} \qquad (0 < \varepsilon < 2 - 2\delta)\ (^{12})\,. \tag{3.26}$$

In particular $B(x_0, r) \subset \Omega_0$ and $\Omega_0$ is an open set.

We give the proof for the reader's convenience. It is the same as in [4], Theorem 2.1 (final part).

PROOF. Having fixed $\delta \in (0, 1)$, we take $\varepsilon$ sufficiently small in such way that

$$2 - 2\delta - \varepsilon > 0$$

We fix $\tau \in (0, 1)$ in this way

$$2c_{11}\tau^{2-2\delta-\varepsilon} = 1\,.$$

If $x_0 \in \Omega_0$ by (3.24) $\exists R < \min\{1, d(x_0)/2\}$ such that

$$\chi(x_0, R) < \tau^{n-\varepsilon}\,. \tag{3.27}$$

(12) If $n \geqslant 3$ then $1 - \delta \leqslant n - 2\gamma$ and we can assume $\varepsilon = 1 - \delta$. Thus we have more simply

$$c_{12} = [2c_{11}]^{(n-2+2\delta)/1-\delta}$$

Because $x \to \chi(x, R)$ is continuous in $x_0$, $\exists B(x_0, r)$ with $r + R < d(x_0)$ such that

$$\chi(y, R) < \tau^{n-\varepsilon}, \qquad \forall y \in B(x_0, r). \tag{3.28}$$

Having done this from (3.10) it follows for every $y \in B(x_0, r)$

$$\phi(y, \tau R) \leqslant 2c_{11}\tau^{n-\varepsilon}\phi(y, R) = \tau^{n-2+2\delta}\phi(y, R) \tag{3.29}$$

hence

$$\frac{\phi(y, \tau R)}{(\tau R)^{n-2}} < \frac{\phi(y, R)}{R^{n-2}}$$

hence, as $\omega$ is increasing,

$$\chi(y, \tau R) < \chi(y, R) < \tau^{n-\varepsilon}. \tag{3.30}$$

Therefore from (3.10)

$$\phi(y, \tau^2 R) < \tau^{n-2+2\delta}\phi(y, \tau R) < \tau^{2(n-2+2\delta)}\phi(y, R) \tag{3.31}$$

in particular

$$\frac{\phi(y, \tau^2 R)}{(\tau^2 R)^{n-2}} < \frac{\phi(y, R)}{R^{n-2}}$$

and hence

$$\chi(y, \tau^2 R) < \chi(y, R) < \tau^{n-\varepsilon}. \tag{3.32}$$

By induction we get $\forall h \geqslant 1$

$$\phi(y, \tau^h R) \leqslant \tau^{h(n-2+2\delta)}\phi(y, R). \tag{3.33}$$

From this, in a standard way, it follows for every $0 < \varrho < R$

$$\phi(y, \varrho) \leqslant \tau^{2-n-2\delta}\left(\frac{\varrho}{R}\right)^{n-2+2\delta}\phi(y, R). \tag{3.34}$$

Then we can prove the following theorem of partial regularity.

THEOREM 3.III. *If $u$ is a solution of the system* (1.4) *there exists an open set $\Omega_0 \subset \Omega$ such that*

$$H_{n-q}(\Omega - \Omega_0) = 0 , \tag{3.35}$$

$$D_i u \in C^{0,\delta}(\Omega_0, R^N) , \qquad \forall \delta \in (0, 1) . \tag{3.36}$$

PROOF. Let $\Omega_0$ be the open set defined in (3.22). Let $\delta \in (0, 1)$ and $x_0 \in \Omega_0$; there exists $R < \min\{1, d(x_0)/2\}$ which depends on $x_0$ but not on $\delta$ and there exists a ball $B(x_0, r) \subset \Omega_0$, $r + R < d(x_0)$, such that $\forall y \in B(x_0, r)$ and $\forall 0 < \varrho < R$ the estimate (3.25) holds; in particular

$$\int_{B(y,\varrho)} \sum_{|\alpha|=2} \|D^\alpha u\|^2 dx \leqslant c_{12}\, \varrho^{n-2+2\delta} R^{2-n-2\delta} \phi(x_0, R + r) . \tag{3.37}$$

By the Poincaré inequality

$$\sum_{i=1}^{n} \int_{B(y,\varrho)} \|D_i u - (D_i u)_\varrho\|^2 dx \leqslant c(n)\, c_{12}\, \varrho^{n+2\delta} R^{2-n-2\delta} \phi(x_0, R + r) . \tag{3.38}$$

From this, by [1], it follows that $D_i u \in C^{0,\delta}(\overline{B(x_0, r)}, R^N)$, $i = i, \ldots, n$, and the theorem is proved.

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