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Normal Subgroups and Projectivities of Finite Groups.

FEDERICO MENEGAZZO (*)

Let G be a group; by a projectivity $\sigma: G \rightarrow G^\sigma$ we shall mean an isomorphism of the lattice $\mathfrak{L}(G)$ of subgroups of G onto the lattice $\mathfrak{L}(G^\sigma)$ of subgroups of the group G^σ . If H is a normal subgroup of G , then H^σ needs not be normal in G^σ ; H^σ is a Dedekind subgroup of G^σ , i.e. it satisfies the modular identities $(H^\sigma \vee U) \wedge V = H^\sigma \vee (U \wedge V)$ for every pair U, V of subgroups of G^σ such that $H^\sigma \leq V$, and $(U \vee H^\sigma) \wedge V = U \vee (H^\sigma \wedge V)$ for every pair U, V of subgroups of G^σ such that $U \leq V$. Assuming that G is finite, R. Schmidt has shown in [3] that if $S^\sigma = (H^\sigma)^{\sigma^\sigma}$ is the normal closure of H^σ in G^σ and $T^\sigma = (H^\sigma)_{G^\sigma}$ is the core of H^σ in G^σ , then S and T are normal subgroups of G and S/T is supersolubly embedded in G/T . In the above notation, we prove that if $|G|$ is odd, then H/T is abelian.

1. - LEMMA. Let D be a Dedekind subgroup of the group G . Then $D_\sigma = \bigwedge \{D^x \mid x \in \mathfrak{J}(D)\}$, where $\mathfrak{J}(D) = \{x \in G \mid \langle x \rangle / \langle x \rangle \wedge D \mid \text{ is either } \infty \text{ or a prime power}\}$.

PROOF. We have to show that, for every $g \in G$, $D^\sigma \geq \bigwedge \{D^x \mid x \in \mathfrak{J}(D)\}$. If $g \in \mathfrak{J}(D)$ this is obvious; so, assume $|\langle g \rangle / \langle g \rangle \wedge D| = n$ is finite, $n = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ is the prime decomposition of n , $n_i = n/p_i^{\alpha_i}$, and r_i, s_i are integers such that $1 = p_i^{\alpha_i} r_i + n_i s_i$ ($i = 1, \dots, t$). Since $g^{n_i s_i} \in \mathfrak{J}(D)$, we have

$$\bigwedge \{D^x \mid x \in \mathfrak{J}(D)\} \leq \bigwedge_{i=1}^t D^{\sigma^{n_i s_i}} \leq \bigwedge_{i=1}^t (D \vee \langle g^{p_i^{\alpha_i}} \rangle)^{\sigma^{n_i s_i}} = \bigwedge_{i=1}^t (D \vee \langle g^{p_i^{\alpha_i}} \rangle)^\sigma = D^\sigma,$$

q.e.d.

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COROLLARY. If D is a Dedekind subgroup of G , then $D_G = \bigwedge_{x \in \mathcal{J}(D)} D_{\langle D, x \rangle}$.

THEOREM. Let G be a finite p -group, $p \neq 2$, $\sigma: G \rightarrow G^\sigma$ an index-preserving projectivity, H a normal subgroup of G such that G/H is cyclic and $H_{G^\sigma}^\sigma = 1$. Then H is abelian.

REMARK. The proof does not work for $p = 2$; but no actual counterexample is known to the author.

PROOF. We proceed by induction on $|H|$. If $|\Omega_1(H)| = p$, then H is cyclic; if $\Omega_1(H) = H$, then for every $x_1 \in G^\sigma$ $H^{\sigma x_1 \sigma^{-1}} \wedge H$ is quasinormal, hence normal in H with cyclic factor group: $H' \leq \bigwedge_{x_1 \in G^\sigma} (H^{\sigma x_1 \sigma^{-1}} \wedge H) = 1$ follows. So, we may assume that $|\Omega_1(H)| > p$ and that the exponent p^r of H is greater than p ; assume also that $G = \langle a \rangle H$. We have $\langle a \rangle \wedge H = 1$; by [1] $\Omega_1(G) = \Omega_1(\langle a \rangle) \Omega_1(H)$ and $H^\sigma \Omega_1(G^\sigma) / \Omega_1(G^\sigma)$ has trivial core in $G^\sigma / \Omega_1(G^\sigma)$; by induction $H / \Omega_1(H) \cong H \Omega_1(G) / \Omega_1(G)$ is abelian. We now choose $a_1 \in G^\sigma$ such that $\langle a_1 \rangle = \langle a \rangle^\sigma$; $H^\sigma \cap \mathcal{C}_{G^\sigma}(a_1) = 1$, so the centralizer of a_1 in $\Omega_1(G^\sigma) = \Omega_1(\langle a_1 \rangle) \times \Omega_1(H^\sigma)$ has order p ; it follows that $|\mathcal{C}_{\Omega_1(H)}(a)| = p$, and there is a basis e_0, e_1, \dots, e_m (m is ≥ 3) of $\Omega_1(G)$ such that

$$\begin{aligned} \langle e_0 \rangle &= \Omega_1(\langle a \rangle), & \langle e_1, \dots, e_m \rangle &= \Omega_1(H), \\ [e_0, a] &= [e_1, a] = 1, & e_i^a &= e_{i-1} e_i \quad \text{for } i > 1. \end{aligned}$$

In particular, every non trivial normal subgroup of G contained in $\Omega_1(H)$ contains $\langle e_1 \rangle$. Put $\langle f_i \rangle = \langle e_i \rangle^\sigma$: then $\langle f_0 \rangle = \Omega_1(\langle a_1 \rangle) = Z(G^\sigma) \wedge \Omega_1(G^\sigma)$; and we can choose the symbols so that $f_1^{a_1} = f_0 f_1$, $\langle f_0 f_1 \rangle = \langle e_0 e_1 \rangle^\sigma$. Then $H^{\sigma a_1} \wedge \langle f_0, f_1 \rangle = \langle f_0 f_1 \rangle$, $\langle e_1 \rangle \not\leq H^{\sigma a_1 \sigma^{-1}}$, and if we put $(H^{\sigma a_1 \sigma^{-1}})_G = K^{\sigma a_1 \sigma^{-1}}$ we get $K^{\sigma a_1 \sigma^{-1}} \wedge \Omega_1(H) = 1$; hence K is cyclic, $\Omega_1(K^{\sigma a_1 \sigma^{-1}}) = \langle e_0 e_1 \rangle$, $\Omega_1(K) = \langle e_1 \rangle$. By induction H/K is abelian, and so $H' \leq \langle e_1 \rangle$, $H^{\sigma a_1 \sigma^{-1}} / \langle e_0 e_1 \rangle$ is a modular p -group.

If now $Q = H \wedge H^{\sigma a_1 \sigma^{-1}}$, we have $Q \triangleleft H^{\sigma a_1 \sigma^{-1}}$, $Q \wedge \langle e_1 \rangle = 1$ and Q is abelian. H is nilpotent of class ≤ 2 , so it is regular and $1 \neq \mathcal{O}_{r-1}(H) = \{x^{p^{r-1}} \mid x \in H\}$ is a subgroup of $\Omega_1(H)$ which is normal in G : but then $e_1 \in \mathcal{O}_{r-1}(H)$, i.e. $\langle e_1 \rangle = \Omega_1(\langle h \rangle)$ with $h \in H$, $|h| = p^r$; put $\langle k \rangle = \langle h \rangle^{\sigma a_1 \sigma^{-1}}$. We have $\langle h \rangle \wedge Q = \langle k \rangle \wedge Q = 1$ and, since $|H| \leq p^r |Q|$, we get $H = \langle h \rangle Q$, $H^{\sigma a_1 \sigma^{-1}} = \langle k \rangle Q$; furthermore $\langle k \rangle$, which contains $\langle e_0 e_1 \rangle$, is quasi-normal in $H^{\sigma a_1 \sigma^{-1}}$. For every $x \in Q$ $\langle x, k \rangle \wedge Q = \langle x \rangle \triangleleft \langle x, k \rangle$, i.e. k induces a power automorphism on the abelian group Q : $\langle k \rangle Q = H^{\sigma a_1 \sigma^{-1}}$ is modular, as well as H , H^σ , $H^{\sigma a_1}$.

Let Q have exponent p^s ($< p^r$); $\bar{O}_{s-1}(Q^\sigma)$ is a normal subgroup of G contained in $\Omega_1(H)$, hence $e_1 \in \bar{O}_{s-1}(Q^\sigma)$; we see that $Q^\sigma = \Omega_s(H)$; since $Q \triangleleft H^{\sigma a_1 \sigma^{-1}}$ and $G = \langle a \rangle H^{\sigma a_1 \sigma^{-1}}$ we have $Q^\sigma = Q^{\langle a \rangle}$ and, more, the interval $[H/Q]$ being a chain, $Q^\sigma = QQ^\sigma$. We shall now prove that $[k, a]$ centralizes Q^σ . First of all, $\langle k \rangle \wedge H = 1$ and $\Omega_r(G) = \Omega_r(\langle a \rangle)H$ imply $\langle k \rangle H = \Omega_r(G)$, i.e. $k = a^\beta h^\nu x$ where $|a^\beta| = p^r$, $x \in Q$; a simple induction based on [1] proves that from $|h^\nu| < p^r$ would follow the contradiction $\Omega_1(\langle a^\beta h^\nu \rangle) = \Omega_1(\langle kx^{-1} \rangle) = \langle e_0 \rangle \leq H^{\sigma a_1 \sigma^{-1}}$; moreover, we have $\langle h \rangle \Omega_{r-1}(H) \triangleleft G$: if $p^s < p^r$ this is H , while if $p^s = p^r$ it is the unique subgroup of H which is cyclic modulo $\Omega_{r-1}(H)$ and whose \bar{O}_{r-1} is $\langle e_1 \rangle$. From this it follows that $\langle h \rangle^\sigma \Omega_{r-1}(G^\sigma)$ is quasnormal of order p in $G^\sigma / \Omega_{r-1}(G^\sigma)$, hence $\langle k \rangle^\sigma \Omega_{r-1}(G^\sigma) \in \langle a^\beta \rangle^\sigma \cdot \langle h \rangle^\sigma \Omega_{r-1}(G^\sigma)$: and we may assume $x \in \Omega_{r-1}(H)$. If now the exponent of Q is $< p^r$, then $\Omega_{r-1}(H)$ is abelian and $[k, a] = [h^\nu, a]^x [x, a] \in \Omega_{r-1}(H)$, i.e. $[k, a]$ centralizes $Q^\sigma = \Omega_s(H) \leq \Omega_{r-1}(H)$; if the exponent of Q is p^r (and assuming that H is not abelian, since otherwise everything is trivial) $H/Z(H)$ has order p^2 , the order of $H/\Omega_{r-1}(H)$ is at least p^2 , $Z(H) \leq \Omega_{r-1}(H)$ i.e. $\Omega_{r-1}(H) = Z(H)$ and $[a, k]$ in this case also centralizes $Q^\sigma = H$. It follows that k and k^a induce the same automorphism on Q^σ ; k operates as a power on Q , k^a (hence k) gives the same power on Q^a , and finally k operates as a power on $QQ^a = Q^\sigma$ (if Q^σ is abelian this is obvious; if $Q^\sigma = H$, one checks that in a modular p -group with derived group of order p ($p \neq 2$) powers congruent to 1 (mod. p) are indeed automorphisms), so that the subgroup of the elements of G which induce a power automorphism in $\Omega_s(H)$ contains $\langle k \rangle \Omega_{r-1}(H) = \langle k \rangle \Omega_{r-1}(G)$; furthermore, we can determine s by the condition that $H/\Omega_s(H)$ is cyclic, but $H/\Omega_{s-1}(H)$ is not. We now look at $\langle h \rangle^{\sigma a_1^{-1} \sigma^{-1}} = \langle l \rangle$: we have $\langle l \rangle^\sigma = \langle h \rangle^{\sigma a_1^{-1}}$, $(\langle l \rangle \Omega_{r-1}(G))^\sigma \leq (\langle a^\beta \rangle \cdot \langle h \rangle \Omega_{r-1}(G))^\sigma$, $(\langle l \rangle \Omega_{r-1}(G))^\sigma$ is neither $(\langle h \rangle \Omega_{r-1}(G))^\sigma$ nor $(\langle k \rangle \Omega_{r-1}(G))^\sigma = (\langle h \rangle \Omega_{r-1}(G))^{\sigma a_1}$ because the centralizer of $\langle a_1 \rangle \Omega_{r-1}(G^\sigma) / \Omega_{r-1}(G^\sigma)$ in $\Omega_r(G^\sigma) / \Omega_{r-1}(G^\sigma)$ is $\langle a_1^\beta \rangle \Omega_{r-1}(G^\sigma) / \Omega_{r-1}(G^\sigma)$. The properties we proved above for k hold for l too, i.e. l is in the subgroup of the elements of G which induce a power on $\Omega_s(H)$; but then this subgroup contains $\langle a^\beta \rangle$ and $\langle h \rangle$: so $[h, Q] \leq \langle e_1 \rangle \wedge Q = 1$, and H is abelian, q.e.d.

COROLLARY. Let $\sigma: G \rightarrow G^\sigma$ be a projectivity, with G a finite group of odd order. If H is a normal subgroup of G and $T = (H_{G^\sigma}^\sigma)^{\sigma^{-1}}$, then H/T is abelian.

PROOF. Let G be a counterexample of minimum order; we shall show that G satisfies the hypotheses of the theorem. Obviously $T = 1$,

and the lemma and corollary imply that $G = H\langle a \rangle$ where $\langle a \rangle : \langle a \rangle \wedge H$ is a prime power. G is not a P -group ([4], p. 11), for proper normal subgroups of P -groups are abelian. H^σ is a Dedekind subgroup of G^σ and the interval $[G/H]$ is a chain: by [2] G^σ is a p -group. Since G is neither a P -group, nor cyclic, σ must be index-preserving, q.e.d.

2. – The hypotheses of the theorem do not imply that H^σ is abelian, as the following example shows.

Let E, F be elementary abelian p -groups with bases $e_0, e_1, \dots, e_p; f_0, f_1, \dots, f_p$ respectively (p is an odd prime), and consider the groups

$$K = \langle E, u | u^{p^2} = e_0, e_i^u = e_i e_{i-1} \ (i = 2, \dots, p), e_1^u = e_1 \rangle$$

and

$$K_1 = \langle F, x | x^{p^2} = f_0, f_i^x = f_i f_{i-1} \ (i = 1, \dots, p) \rangle.$$

K, K_1 have both order p^{p+3} ; the map $\alpha: K \rightarrow K_1$ such that

$$\left(\prod_{i=1}^p e_i^{r_i} \right) u^r \mapsto \left(\prod_{i=1}^p f_i^{r_i} \right) x^r,$$

when restricted to E , is an isomorphism $E \rightarrow F$; moreover, it induces a map $\bar{\alpha}: K/\langle e_0 \rangle \rightarrow K_1/\langle f_0 \rangle$ which is also an isomorphism. Since every subgroup of K , not contained in E , contains $\langle e_0 \rangle$, and similarly for K_1 , α induces a projectivity $K \rightarrow K_1$ [3]. Now extend K to $G = \langle K, v | [v, K] = 1, v^p = e_1^{-1} \rangle$ and K_1 to

$$G_1 = \langle K_1, y | x^y = x^{1+p}, f_i^y = f_i \ (i = 1, \dots, p-1), f_p^y = f_p f_1^{-1}, y^p = f_1^{-1} \rangle;$$

G, G_1 have order p^{p+4} ; G_1 is, apart from slight changes of notation, Thompson's example [5]. $\Omega_2(G) = \langle E, u^p, v \rangle$ is abelian, while $\Omega_2(G_1) = \langle F, x^p, y \rangle$ is modular non-abelian, for f_p operates on the abelian group $\langle x^p, y f_p^{-1}, f_2, \dots, f_{p-1} \rangle$ as the power $1 + p$. For every $a \in K_1$ $\langle (ay)^p \rangle = \langle a^p y^p \rangle$: if $a \in \Omega_2(G_1)$ $\langle a, y \rangle$ is modular, hence regular, with derived subgroup of order p at most, so $(ay)^p = a^p y^p$; if $|a| = p^3$ we have $\langle a^p, f_1 \rangle = \langle x^p, f_1 \rangle = [K_1, y]$, $\langle a^p \rangle \triangleleft \langle a, y \rangle$, $\langle a, y \rangle / \langle a^p \rangle$ is modular, so $\langle a, y \rangle$ is again modular, $\bar{O}_1(\langle a, y \rangle) = \langle f_0 \rangle = \langle a^{p^2} \rangle = \langle (ay)^{p^2} \rangle$, and $a^p y^p = (ay)^{p+m p^2}$ for some integer m . Hence, for every $k \in K$, $\langle k^\alpha y \rangle \wedge K_1 = (\langle kv \rangle \wedge K)^\alpha$. Every subgroup L of G , $L \not\leq K$, can be written as $L = (L \wedge K) \langle kv \rangle$ for some $k \in K$, and similarly for G_1 ; we

try to define a projectivity $\sigma: G \rightarrow G_1$ by the stipulation $L^\sigma = L^\alpha$ if $L \leq K$, $L^\sigma = (L \wedge K)^\alpha \langle k^\alpha y \rangle$ for $L = (L \wedge K) \langle kv \rangle \not\leq K$. The above discussion shows that this definition makes sense for cyclic subgroups; and for the intervals $[G/\langle u^p, e_1 \rangle]$, $[G_1/\langle x^p, f_1 \rangle]$ it is the map induced by the isomorphism

$$\beta: G/\langle u^p, e_1 \rangle \rightarrow G_1/\langle x^p, f_1 \rangle$$

such that $\bar{e}_i^\beta = \bar{f}_i$, $\bar{u}^\beta = \bar{x}$, $\bar{v}^\beta = \bar{y}$ (with the usual meaning of $\bar{}$). Let now $L = (L \wedge K) \langle kv \rangle$ be a non-cyclic subgroup of G , not containing $\langle u^p, e_1 \rangle$; it is easily seen that $L \leq \Omega_2(G)$, so that $(L \wedge K)^\alpha$ and $\langle k^\alpha y \rangle$ are contained in $\Omega_2(G_1)$, which is modular, and $(L \wedge K)^\alpha \langle k^\alpha y \rangle$ is indeed a subgroup. Again with $L \leq \Omega_2(G)$, let $k, k' \in K$ be such that

$$L = (L \wedge K) \langle kv \rangle = (L \wedge K) \langle k'v \rangle;$$

then $k' = tk$ with $t \in L \wedge K$, and one directly checks that $k'^\alpha = (t^\alpha)^{1+n} k^\alpha$ for some integer n , so that

$$(L \wedge K)^\alpha \langle k^\alpha y \rangle = (L \wedge K)^\alpha \langle k'^\alpha y \rangle.$$

This remark shows that σ is indeed a projectivity. If now $H = \langle v, e_2, \dots, e_p \rangle$, we have $H \triangleleft G$, $H_{G^\sigma}^\sigma = 1$, and H^σ is not abelian.

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