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On the Motion of a Non-Homogeneous Ideal Incompressible Fluid in an External Force Field.

HUGO BEIRÃO DA VEIGA - ALBERTO VALLI (*)

1. Introduction and main results.

In this paper we consider the motion of a non-homogeneous ideal incompressible fluid in a bounded connected open subset Ω of \mathbb{R}^3 .

We denote in the sequel by $v(t, x)$ the velocity field, by $\rho(t, x)$ the mass density and by $\pi(t, x)$ the pressure. The Euler equations of the motion are

$$(E) \quad \left\{ \begin{array}{ll} \rho \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right] = -\nabla \pi & \text{in } Q_T \equiv [0, T] \times \bar{\Omega}, \\ \operatorname{div} v = 0 & \text{in } Q_T, \\ \frac{\partial \rho}{\partial t} + v \cdot \nabla \rho = 0 & \text{in } Q_T, \\ v \cdot n = 0 & \text{on } [0, T] \times \Gamma, \\ v|_{t=0} = a & \text{in } \bar{\Omega}, \\ \rho|_{t=0} = \rho_0 & \text{in } \bar{\Omega}, \end{array} \right.$$

where $n = n(x)$ is the unit outward normal vector to the boundary Γ of Ω , $b = b(t, x)$ is the external force field and $a = a(x)$, $\rho_0 = \rho_0(x)$

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are the initial velocity field and the initial mass density respectively.

When the fluid is homogeneous, i.e. the density ϱ_0 (and consequently ϱ), is constant, equations (E) have been studied by several authors. As regards the two-dimensional case, we recall the papers of Wolibner [13], Leray [6], Hölder [3], Schaeffer [10], Yudovich [14], [15], Golovkin [2], Kato [5], Mc Grath [9] and Bardos [1]; for the case of a variable boundary see Valli [12]. For the n -dimensional case we recall the papers of Lichtenstein, Ebin and Marsden, Swann, Kato, Bourguignon and Brezis, Temam, Bardos and Frisch.

For non-homogeneous fluids, Marsden [8] has proved the existence of a local solution to problem (E), under the assumption that the external force field $b(t, x)$ is divergence free and tangential to the boundary, i.e. $\operatorname{div} b = 0$ in Q_T and $b \cdot n = 0$ on $[0, T] \times \Gamma$. The proof relies on techniques of Riemannian geometry on infinite dimensional manifolds. See also the reference [16].

In this paper we prove the existence of a local solution of problem (E) without any restriction on the external force field $b(t, x)$ but we need condition (A) on the initial mass density $\varrho_0(x)$ ⁽¹⁾.

Our techniques are based on the method of characteristics and on Schauder's fixed point theorem, and in this sense related to the methods of Kato [5] and Mc Grath [9].

We prove the following results ⁽²⁾.

THEOREM A. *Let Ω be of class $C^{3+\lambda}$, $0 < \lambda < 1$, and let $a \in C^{2+\lambda}(\bar{\Omega})$ with $\operatorname{div} a = 0$ in $\bar{\Omega}$ and $a \cdot n = 0$ on Γ , $\varrho_0 \in C^{2+\lambda}(\bar{\Omega})$ with $\varrho_0(x) > 0$ for each $x \in \bar{\Omega}$, and $b \in C^{0,1+\lambda}(Q_T) \cap C^{\lambda,0}(Q_T)$ with $\operatorname{rot} b \in C^{0,1+\lambda}(Q_T) \cap C^{\lambda,0}(Q_T)$.*

Moreover we assume that⁽¹⁾

$$(A) \quad \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} < \begin{cases} \frac{1}{K_1} & \text{if } \Omega \text{ is simply connected,} \\ \frac{1}{K_1(1 + K_3 K_4)} & \text{otherwise.} \end{cases}$$

⁽¹⁾ *Added in proofs.* In the authors' papers « On the Euler equations for non-homogeneous fluids » (I), (II) (to appear) condition (A) is dropped and the three dimensional case is proved.

⁽²⁾ The definition of K_1 is given in (3.4); those of K_3 and K_4 in (7.21), (7.11) and (7.24).

Then there exist

$$T_1 \in]0, T], \quad v \in C^{1,2+\lambda}(Q_{T_1}) \cap C^{1+\lambda,0}(Q_{T_1}),$$

$$\varrho \in C^{2+\lambda,2+\lambda}(Q_{T_1}), \quad \pi \in C^{0,2+\lambda}(Q_{T_1}) \cap C^{\lambda,1}(Q_{T_1}),$$

such that (v, ϱ, π) is a solution of (E) in Q_{T_1} .

THEOREM B. Assume that ϱ_0 and $\nabla \varrho_0$ belong to $L^\infty(\Omega)$, $\min \varrho_0 > 0$ and that b belongs to $L^1(0, T; L^\infty(\Omega))$. Then problem (E) has at most a solution (v, ϱ, π) in the class of vector functions $v \in L^\infty(Q_T)$ such that $\partial v / \partial t$, $\partial v / \partial x_1$ and $\partial v / \partial x_2$ are in $L^1(0, T; L^\infty(\Omega))$. The pressure is unique up to an arbitrary function of t which may be added to it. This result holds in dimension $n \geq 2$.

For other uniqueness theorems see also Serrin [11].

The paper consists of two parts. In Part I we prove Theorem A for a simply connected domain Ω , and Theorem B. In Part II we prove Theorem A in the general case, i.e. we assume that Γ consists of $m + 1$ simple closed curves $\Gamma_0, \Gamma_1, \dots, \Gamma_m$, where Γ_j ($j = 1, \dots, m$) are inside of Γ_0 and outside of one another.

PART I

2. Notations.

Let Ω be a bounded simply connected open subset of \mathbf{R}^2 .

We denote by $C^{k+\lambda}(\bar{\Omega})$, k non negative integer, $0 < \lambda < 1$, the space of k -times continuously differentiable functions in $\bar{\Omega}$ with λ -Hölder continuous derivatives of order k ; by $C^0(Q_T)$ the space of continuous functions in Q_T ; by $C^1(Q_T)$ the space of continuously differentiable functions in Q_T .

We set

$$D_i \varphi \equiv \frac{\partial \varphi}{\partial x_i}, \quad D^\alpha D_i^j \varphi \equiv \frac{\partial^{|\alpha|+j} \varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial t^j},$$

and

$$C^{k,h}(Q_T) \equiv \{ \varphi \in C^0(Q_T) \mid D^\alpha D_i^j \varphi \in C^0(Q_T) \text{ if } 0 \leq j \leq k, \\ |\alpha| \leq h \text{ and } j + |\alpha| \leq \max(k, h) \},$$

$$\begin{aligned}
C^{\lambda,0}(Q_T) &\equiv \{\varphi \in C^0(Q_T) \mid \varphi \text{ is } \lambda\text{-Hölder continuous in } t, \\
&\qquad\qquad\qquad \text{uniformly with respect to } x\}, \\
C^{0,\lambda}(Q_T) &\equiv \{\varphi \in C_0(Q_T) \mid \varphi \text{ is } \lambda\text{-Hölder continuous in } x, \\
&\qquad\qquad\qquad \text{uniformly with respect to } t\}, \\
C^{k+\lambda,h}(Q_T) &\equiv \{\varphi \in C^{k,h}(Q_T) \mid D^\alpha D_t^j \varphi \in C^{\lambda,0}(Q_T) \\
&\qquad\qquad\qquad \text{if } j + |\alpha| = \max(k, h) \text{ or if } j = h\}, \\
C^{k,h+\lambda}(Q_T) &\equiv \{\varphi \in C^{k,h}(Q_T) \mid D^\alpha D_t^j \varphi \in C^{0,\lambda}(Q_T) \\
&\qquad\qquad\qquad \text{if } j + |\alpha| = \max(k, h) \text{ or if } |\alpha| = h\}, \\
C^{k+\lambda,h+\lambda}(Q_T) &\equiv C^{k+\lambda,h}(Q_T) \cap C^{k,h+\lambda}(Q_T).
\end{aligned}$$

We denote by $\|\cdot\|_\infty$ the supremum norm, both in $\bar{\Omega}$ or in Q_T , and by $[\cdot]_\lambda$ the usual λ -Hölder seminorm in $\bar{\Omega}$. Furthermore we define

$$\begin{aligned}
[\varphi]_{\lambda,0} &\equiv \sup_{\substack{t,s \in [0,T] \\ t \neq s \\ x \in \Omega}} \frac{|\varphi(t,x) - \varphi(s,x)|}{|t-s|^\lambda}, \\
[\varphi]_{0,\lambda} &\equiv \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y \\ t \in [0,T]}} \frac{|\varphi(t,x) - \varphi(t,y)|}{|x-y|^\lambda}, \\
[\varphi]_{\text{lip},0} &\equiv \sup_{\substack{t,s \in [0,T] \\ t \neq s \\ x \in \bar{\Omega}}} \frac{|\varphi(t,x) - \varphi(s,x)|}{|t-s|}, \\
[\varphi]_{0,\text{lip}} &\equiv \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y \\ t \in [0,T]}} \frac{|\varphi(t,x) - \varphi(t,y)|}{|x-y|},
\end{aligned}$$

Finally, we set

$$\begin{aligned}
\|D\varphi\|_\infty &\equiv \sum_{|\alpha|=1} \|D^\alpha \varphi\|_\infty, \\
\|D^2\varphi\|_\infty &\equiv \sum_{|\alpha|=2} \|D^\alpha \varphi\|_\infty,
\end{aligned}$$

and analogously for the seminorms $[\cdot]_{\lambda,0}$ and $[\cdot]_{0,\lambda}$.

If $u = (u_1, u_2)$ is a vector field defined in Q_T , we write $u \in C^{\lambda,0}(Q_T)$ if $u_1, u_2 \in C^{\lambda,0}(Q_T)$, and we set $[u]_{\lambda,0} \equiv [u_1]_{\lambda,0} + [u_2]_{\lambda,0}$; the same convention is used for the other vector spaces and norms (in $\bar{\Omega}$ or in Q_T).

We put

$$\begin{aligned} \text{Rot } \varphi &\equiv \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right), \\ \text{rot } u &\equiv \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \end{aligned}$$

where φ is a scalar function and $u = (u_1, u_2)$ is a vector function.

3. Preliminaries.

Let $\varphi \in C^{1,1+\lambda}(Q_T)$ with $D_t \varphi \in C^{\lambda,0}(Q_T)$; we assume that

$$(3.1) \quad \begin{cases} \|\varphi\|_{0,1+\lambda} \equiv \|\varphi\|_{\infty} + \|D\varphi\|_{\infty} + [D\varphi]_{0,\lambda} \leq A, \\ \|D_t \varphi\|_{\infty} \leq B, \\ \|D_t \varphi\|_{0,\lambda} \equiv \|D_t \varphi\|_{\infty} + [D_t \varphi]_{0,\lambda} \leq C, \\ [D_t \varphi]_{\lambda,0} \leq D, \end{cases}$$

where A, B, C, D are positive constants that we will specify in the following (see (4.9)).

Let ψ be the solution of

$$(3.2) \quad \begin{cases} -\Delta \psi(t, x) = \varphi(t, x) & \text{in } \Omega, \\ \psi|_{\Gamma} = 0, \end{cases}$$

for each $t \in [0, T]$, i.e.

$$(3.2)' \quad \psi(t, x) = \int_{\Omega} G(x, y) \varphi(t, y) dy$$

where $G(x, y)$ is the Green function for the operator $-\Delta$ with zero boundary condition.

Put $\|\chi\|_{k+\lambda} \equiv \sum_{|\alpha| \leq k} \|D^{\alpha} \chi\|_{\infty} + \sum_{|\alpha|=k} [D^{\alpha} \chi]_{\lambda}$. It is well known that there exist constants $c = c(\lambda, \Omega)$ such that

$$(3.3) \quad \begin{aligned} \|\chi\|_{3+\lambda} &\leq c \|\Delta \chi\|_{1+\lambda}, \\ \|\chi\|_{2+\lambda} &\leq c \|\Delta \chi\|_{\lambda}, \\ [D\chi]_{\lambda} &\leq c \|\Delta \chi\|_{\infty}, \end{aligned}$$

for each $\chi \in C^{\infty}(\bar{\Omega})$ vanishing on Γ .

Moreover there exists $K_1 = K_1(\Omega)$ such that

$$(3.4) \quad \sup_{x \in \bar{\Omega}} \int_{\Omega} |\nabla_x G(x, y)| dy \leq \frac{1}{2} K_1.$$

It is sufficient to choose $K_1 = 4\pi K \text{diam } \Omega$, where K is such that

$$(3.4)' \quad |\nabla_x G(x, y)| \leq \frac{K}{|x - y|}, \quad \forall x, y \in \bar{\Omega}, \quad x \neq y,$$

(see for instance Lichtenstein [7], pag. 248).

We obtain

LEMMA 3.1. *Let $\varphi \in C^{1,1+\lambda}(Q_T)$ with $D_t \varphi \in C^{\lambda,0}(Q_T)$ and let ψ be defined in (3.2). Put*

$$(3.5) \quad v \equiv \text{Rot } \psi;$$

then $v \in C^{1,2+\lambda}(Q_T)$, $D_t v \in C^{\lambda,0}(Q_T)$ and

$$(3.6) \quad \begin{aligned} \|v\|_{0,2+\lambda} &\leq c \|\varphi\|_{0,1+\lambda}, \\ \|D_t v\|_{\infty} &\leq \frac{1}{2} K_1 \|D_t \varphi\|_{\infty}, \\ \|D_t v\|_{0,1+\lambda} &\leq c \|D_t \varphi\|_{0,\lambda}, \\ [D_t v]_{\lambda,0} &\leq \frac{1}{2} K_1 [D_t \varphi]_{\lambda,0}, \\ [D_t v]_{0,\lambda} &\leq c \|D_t \varphi\|_{\infty}. \end{aligned}$$

Moreover $\text{div } v = 0$ and $\text{rot } v = \varphi$ in Q_T , $v \cdot n = 0$ on $[0, T] \times \Gamma$.

PROOF. Since $\varphi \in C^{0,1+\lambda}(Q_T) \subset C^0([0, T]; C^{1+\lambda}(\bar{\Omega}))$ for each $\lambda' < \lambda$ (see for instance Kato [5], Lemma 1.2), it follows from Schauder's estimates that $v \in C^0([0, T]; C^{2+\lambda'}(\bar{\Omega}))$; hence $v, Dv, D^2v \in C^0(Q_T)$. Moreover estimate (3.6)₁ follows directly from (3.3)₁, i.e. $v \in C^{0,2+\lambda}(Q_T)$. Differentiating (3.2)' with respect to t , we have

$$(3.7) \quad D_t \psi(t, x) = \int_{\Omega} G(x, y) D_t \varphi(t, y) dy;$$

since $D_t \varphi \in C^{0,\lambda}(Q_T)$, arguing as above it follows that $D_t v \in C^{0,1+\lambda}(Q_T)$ and (3.6)₃ holds.

Applying the operator Rot to (3.7) we have

$$D_t v(t, x) = \int_{\Omega} \text{Rot}_x G(x, y) D_t \varphi(t, y) dy$$

and (3.4) yields (3.6)₂ and (3.6)₄.

Estimate (3.6)₅ follows directly from (3.5) and (3.3)₃. Finally remark that $\text{rot Rot} = -\Delta$ and that $v \cdot n$ is a tangential derivative of ψ at the boundary. \square

By using (3.1) one has

$$\begin{aligned}
 (3.8) \quad & \|v\|_{0,2+\lambda} < cA, \\
 & \|D_t v\|_\infty < \frac{1}{2} K_1 B, \\
 & \|D_t v\|_{0,1+\lambda} < cC, \\
 & [D_t v]_{\lambda,0} < \frac{1}{2} K_1 D, \\
 & [D_t v]_{0,\lambda} < cB.
 \end{aligned}$$

Now we construct the stream lines of the vector field $v(t, x)$. We denote by c, c_1, c_2, \dots , constants depending at most on λ and Ω .

We put $U(\sigma, t, x) \equiv y(\sigma)$, $\sigma, t \in [0, T]$, $x \in \bar{\Omega}$, where $y(\sigma)$ is the solution of the ordinary differential equation

$$(3.9) \quad \begin{cases} \frac{dy}{d\sigma} = v(\sigma, y(\sigma)) & \text{in } [0, T], \\ y(t) = x. \end{cases}$$

Such a solution is global since $v \cdot n = 0$ on $[0, T] \times \Gamma$; from $v \in C^{1,2}(Q_T)$ one has $U \in C^2([0, T] \times Q_T)$.

We denote by $\|DU\|_\infty \equiv \sup_{\sigma \in [0, T]} \|DU(\sigma, \cdot, \cdot)\|_\infty$ and analogously for each norm and seminorm involving U and its derivatives.

We have:

LEMMA 3.2. *The vector function $U(\sigma, t, x)$ satisfies the following estimates:*

$$\begin{aligned}
 (3.10) \quad & \|DU\|_\infty \leq 2 \exp [cTA], \\
 & \|D^2 U\|_\infty \leq cTA \exp [cTA], \\
 & [D^2 U]_{0,\lambda} \leq cTA(1 + TA) \exp [cTA], \\
 & [U]_{\text{lip},0} \leq cA \exp [cTA], \\
 & [DU]_{\lambda,0} \leq cT^{1-\lambda} A(1 + T^\lambda A^\lambda) \exp [cTA], \\
 & [D^2 U]_{\lambda,0} \leq cT^{1-\lambda} A(1 + T^\lambda A^\lambda)(1 + TA) \exp [cTA].
 \end{aligned}$$

PROOF. One obtains these estimates by direct computation of the resolutive formula

$$(3.11) \quad U(\sigma, t, x) = x + \int_t^\sigma v(\tau, U(\tau, t, x)) d\tau.$$

We give only the explicit proof of (3.10)_s. From (3.11) one gets

$$(3.12) \quad D_i U_j(\sigma, t, x) = \delta_{ij} + \int_t^\sigma \sum_h (D_h v_j)(\tau, U(\tau, t, x)) D_i U_h(\tau, t, x) d\tau$$

and

$$(3.13) \quad \begin{aligned} D_{ik}^2 U_j(\sigma, t, x) = & \\ & \int_t^\sigma \left[\sum_{r,h} (D_{rh}^2 v_j)(\tau, U(\tau, t, x)) D_k U_r(\tau, t, x) D_i U_h(\tau, t, x) + \right. \\ & \left. + \sum_h (D_h v_j)(\tau, U(\tau, t, x)) D_{ik}^2 U_h(\tau, t, x) \right] d\tau. \end{aligned}$$

Hence one obtains

$$\begin{aligned} \sum_{i,j,k} |D_{ik}^2 U_j(\sigma, t, x) - D_{ik}^2 U_j(\sigma, t, y)| \leq & T|x - y|^\lambda \{ [D^2 v]_{0,\lambda} [U]_{0,\text{lip}}^4 \|DU\|_\infty^2 + \\ & + 2 \|D^2 v\|_\infty \|DU\|_\infty [DU]_{0,\lambda} + \|D^2 U\|_\infty [Dv]_{0,\lambda} [U]_{0,\text{lip}}^4 \} + \\ & + \|Dv\|_\infty \left| \int_t^\sigma \sum_{i,k,h} |D_{ik}^2 U_h(\tau, t, x) - D_{ik}^2 U_h(\tau, t, y)| d\tau \right| \end{aligned}$$

and from Gronwall's lemma

$$\begin{aligned} \sum_{i,j,k} |D_{ik}^2 U_j(\sigma, t, x) - D_{ik}^2 U_j(\sigma, t, y)| \leq & T|x - y|^\lambda \cdot \\ & \cdot \{ [D^2 v]_{0,\lambda} [U]_{0,\text{lip}}^4 \|DU\|_\infty^2 + 2 \|D^2 v\|_\infty \|DU\|_\infty [DU]_{0,\lambda} + \\ & + \|D^2 U\|_\infty [Dv]_{0,\lambda} [U]_{0,\text{lip}}^4 \} \exp [T \|Dv\|_\infty]. \end{aligned}$$

From (3.10)₁, (3.10)₂ and (3.8)₁ one obtains (3.10)₃.

On proving (3.10)₄ and (3.10)₅, recall that

$$(3.14) \quad \frac{\partial U(\sigma, t, x)}{\partial t} = - \sum_h \frac{\partial U(\sigma, t, x)}{\partial x_h} v_h(t, x). \quad \square$$

We now study the equation

$$(3.15) \quad \begin{cases} \frac{\partial \varrho}{\partial t} + v \cdot \nabla \varrho = 0 & \text{in } Q_T, \\ \varrho|_{t=0} = \varrho_0 & \text{in } \bar{Q}. \end{cases}$$

LEMMA 3.3. *Let $\varrho_0 \in C^{2+\lambda}(\bar{Q})$ and $\varrho_0(x) > 0$ for each $x \in \bar{Q}$. Then the solution of (3.15) is given by*

$$(3.16) \quad \varrho(t, x) = \varrho_0(U(0, t, x)).$$

Moreover $\varrho \in C^{2+\lambda, 2+\lambda}(Q_T)$ and

$$(3.17) \quad \begin{aligned} \left\| \frac{D\varrho}{\varrho} \right\|_{\infty} &\leq 2 \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} \exp [cTA], \\ \left\| \frac{D^2\varrho}{\varrho} \right\|_{\infty} &\leq c \left(TA \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} + \left\| \frac{D^2\varrho_0}{\varrho_0} \right\|_{\infty} \right) \exp [cTA], \\ \left[\frac{D\varrho}{\varrho} \right]_{0,\lambda} &\leq c \left(TA \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} + \left[\frac{D\varrho_0}{\varrho_0} \right]_{\lambda} \right) \exp [cTA], \\ \left[\frac{D^2\varrho}{\varrho} \right]_{0,\lambda} &\leq c \left\{ TA(1 + TA) \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} + TA \left[\frac{D\varrho_0}{\varrho_0} \right]_{\lambda} + \right. \\ &\quad \left. + TA \left\| \frac{D^2\varrho_0}{\varrho_0} \right\|_{\infty} + \left[\frac{D^2\varrho_0}{\varrho_0} \right]_{\lambda} \right\} \exp [cTA], \\ \left[\frac{D\varrho}{\varrho} \right]_{\lambda,0} &\leq c \left\{ T^{1-\lambda} A(1 + T^{\lambda} A^{\lambda}) \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} + A^{\lambda} \left[\frac{D\varrho_0}{\varrho_0} \right]_{\lambda} \right\} \exp [cTA]. \end{aligned}$$

PROOF. One easily obtains (3.16) by using the method of characteristics. From (3.16) one has

$$\begin{aligned} \frac{D_i \varrho}{\varrho}(t, x) &= \sum_h \frac{D_h \varrho_0}{\varrho_0}(U(0, t, x)) D_i U_h(0, t, x), \\ \frac{D_{ik}^2 \varrho}{\varrho}(t, x) &= \sum_{r,h} \frac{D_{rh}^2 \varrho_0}{\varrho_0}(U(0, t, x)) D_k U_r(0, t, x) D_i U_h(0, t, x) + \\ &\quad + \sum_h \frac{D_h \varrho_0}{\varrho_0}(U(0, t, x)) D_{ik}^2 U_h(0, t, x). \end{aligned}$$

By using (3.10), we obtain easily estimates (3.17). □

4. The vorticity equation.

In this number we study the auxiliary equation

$$(4.1) \quad \begin{cases} \frac{\partial \zeta}{\partial t} + v \cdot \nabla \zeta = \gamma & \text{in } Q_T, \\ \zeta|_{t=0} = \alpha & \text{in } \bar{\Omega}, \end{cases}$$

where $\alpha(x) \equiv \text{rot } a(x)$, $\beta(t, x) \equiv \text{rot } b(t, x)$, and $\gamma(t, x)$ is defined in Q_T by

$$(4.2) \quad \gamma \equiv \beta + \frac{\text{Rot } \varrho}{\varrho} \left[\frac{\partial v}{\partial t} + (v \cdot \nabla)v - b \right],$$

where a and b are as in Theorem A.

One integrates (4.1) by the method of characteristics and one obtains

$$(4.3) \quad \zeta(t, x) = \alpha(U(0, t, x)) + \int_0^t \gamma(\tau, U(\tau, t, x)) d\tau.$$

We denote by $\bar{c}, \bar{c}_1, \bar{c}_2, \dots$, constants that depend at most on $\lambda, \Omega, \|D\varrho_0/\varrho_0\|_\lambda, \|D^2\varrho_0/\varrho_0\|_\lambda, \|b\|_{0,1+\lambda}, \|b\|_{\lambda,0}, \|\beta\|_{0,1+\lambda}$ and $\|\beta\|_{\lambda,0}$.

LEMMA 4.1. *Under the above conditions the following estimates hold:*

$$(4.4) \quad \begin{aligned} \|\gamma\|_\infty &\leq \bar{c}(A^2 + 1) \exp [cTA] + \left\| \frac{D\varrho_0}{\varrho_0} \right\|_\infty K_1 B \exp [cTA], \\ [\gamma]_{0,\lambda} &\leq \bar{c}(1 + TA)(A^2 + B + 1) \exp [cTA], \\ \|D\gamma\|_\infty &\leq \bar{c} \{ (1 + TA)(A^2 + B + 1) + C \} \exp [cTA], \\ [D\gamma]_{0,\lambda} &\leq \bar{c} \{ (1 + T^2A^2)(A^2 + B + 1) + (1 + TA)C \} \exp [cTA], \\ [\gamma]_{\lambda,0} &\leq \bar{c} \{ T^{1-\lambda}AC + 1 + A^\lambda(A^2 + B + 1)(1 + TA) \} \cdot \\ &\quad \cdot \exp [cTA] + \left\| \frac{D\varrho_0}{\varrho_0} \right\|_\infty K_1 D \exp [cTA]. \end{aligned}$$

PROOF. It follows by direct computations, using (4.2), (3.8) and (3.17). \square

Finally we have

LEMMA 4.2. *The solution $\zeta(t, x)$ of (4.1) satisfies:*

$$\begin{aligned}
 (4.5) \quad & \|\zeta\|_{0,1+\lambda} \leq 2\|\alpha\|_{1+\lambda} \exp [cTA] + \bar{c}T \{A\|D\alpha\|_{\infty} + \\
 & \quad + (1 + T^2 A^2)(A^2 + B + 1) + (1 + TA)C\} \exp [cTA], \\
 & \|D_t \zeta\|_{\infty} \leq c_1 A \|D\alpha\|_{\infty} \exp [cTA] + \bar{c}_1(A^2 + 1) \exp [cTA] + \\
 & \quad + \bar{c}TA[(1 + TA)(A^2 + B + 1) + C] \exp [cTA] + \\
 & \quad \quad \quad + \left\| \frac{D\rho_0}{\rho_0} \right\|_{\infty} K_1 B \exp [cTA], \\
 & [D_t \zeta]_{0,\lambda} \leq c_2 A (\|D\zeta\|_{\infty} + [D\zeta]_{0,\lambda}) + \\
 & \quad \quad \quad + \bar{c}_2(1 + TA)(A^2 + B + 1) \exp [cTA], \\
 & [D_t \zeta]_{\lambda,0} \leq c_4 A [D\zeta]_{\lambda,0} + cT^{1-\lambda} B \|D\zeta\|_{\infty} + \\
 & \quad + \bar{c}_3 A^{\lambda}(A^2 + B + 1)(1 + TA) \exp [cTA] + \\
 & \quad \quad \quad + \bar{c}_3(T^{1-\lambda}AC + 1) \exp [cTA] + \left\| \frac{D\rho_0}{\rho_0} \right\|_{\infty} K_1 D \exp [cTA],
 \end{aligned}$$

where $\|D\zeta\|_{\infty}$, $[D\zeta]_{0,\lambda}$ and $[D\zeta]_{\lambda,0}$ are bounded respectively by (4.6), (4.7) and (4.8).

PROOF. From (4.3), (3.10) and Lemma 4.1 it follows easily that

$$\begin{aligned}
 (4.6) \quad & \|\zeta\|_{\infty} \leq \|\alpha\|_{\infty} + \bar{c}T(A^2 + B + 1) \exp [cTA], \\
 & \|D\zeta\|_{\infty} \leq 2\|D\alpha\|_{\infty} \exp [cTA] + \\
 & \quad + \bar{c}T[(1 + TA)(A^2 + B + 1) + C] \exp [cTA],
 \end{aligned}$$

$$\begin{aligned}
 (4.7) \quad & [D\zeta]_{0,\lambda} \leq 2[D\alpha]_{\lambda} \exp [cTA] + \\
 & \quad + \bar{c}T[A\|D\alpha\|_{\infty} + (1 + T^2 A^2)(A^2 + B + 1) + (1 + TA)C] \exp [cTA],
 \end{aligned}$$

hence (4.5)₁ holds.

From (4.1), one has $D_t \zeta = -v \cdot \nabla \zeta + \gamma$, and by direct computation one obtains (4.5)₂, (4.5)₃ and (4.5)₄.

Finally, from (4.3) it follows that:

$$\begin{aligned}
 (4.8) \quad & [D\zeta]_{\lambda,0} \leq c_3 A^{\lambda} [D\alpha]_{\lambda} \exp [cTA] + \\
 & \quad + cT^{1-\lambda} A(1 + T^{\lambda} A^{\lambda}) \|D\alpha\|_{\infty} \exp [cTA] + \\
 & \quad + \bar{c}T^{1-\lambda} [(1 + TA)(A^2 + B + 1) + C] (1 + T^{1+\lambda} A^{1+\lambda}) \exp [cTA]. \quad \square
 \end{aligned}$$

We assume in the sequel that condition (A) of Theorem A holds, and we choose the constants A, B, C, D such that

$$(4.9) \quad \left\{ \begin{array}{l} A > 2 \|\alpha\|_{1+\lambda}, \\ B > \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} K_1 B + c_1 A \|D\alpha\|_{\infty} + \bar{c}_1(A^2 + 1), \\ C > c_1 A \|D\alpha\|_{\infty} + 2c_2 A \|D\alpha\|_{\lambda} + (\bar{c}_1 + \bar{c}_2)(A^2 + 1) + \\ \hspace{15em} + \left[\left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} K_1 + \bar{c}_2 \right] B, \\ D > \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} K_1 D + c_3 c_4 A^{1+\lambda} \|D\alpha\|_{\lambda} + \bar{c}_3(A^2 + B + 1) A^{\lambda} + \bar{c}_3. \end{array} \right.$$

From (4.5), (4.6), (4.7) and (4.8) it follows that there exists $T_1 \in]0, T]$ such that

$$(4.10) \quad \begin{aligned} \|\zeta\|_{0,1+\lambda} &\leq A, \\ \|D_t \zeta\|_{\infty} &\leq B, \\ \|D_t \zeta\|_{0,\lambda} &\leq C, \\ [D_t \zeta]_{\lambda,0} &\leq D, \end{aligned}$$

where the norms are taken on the cylinder $Q_{T_1} \equiv]0, T_1] \times \bar{\Omega}$. The set

$$(4.11) \quad S \equiv \left\{ \varphi \in C^1(Q_{T_1}) \mid \|\varphi\|_{0,1+\lambda} \leq A, \|D_t \varphi\|_{\infty} \leq B, \right. \\ \left. \|D_t \varphi\|_{0,\lambda} \leq C, [D_t \varphi]_{\lambda,0} \leq D \right\}$$

is a convex, bounded and closed subset of $C^1(Q_{T_1})$.

Moreover the map $F: \varphi \mapsto \zeta$ defined by (3.2), (3.5), (3.9), (3.15) and (4.3) satisfies

$$(4.12) \quad F(S) \subset S,$$

and, from (4.8),

$$(4.13) \quad [D\zeta]_{\lambda,0} \leq \text{const}, \quad \forall \varphi \in S.$$

By the Ascoli-Arzelà theorem and (4.11), (4.13) it follows that $F(S)$ is relatively compact in $C^1(Q_{T_1})$.

Finally, we shall see that F is continuous in the $C^1(Q_{T_1})$ topology, hence, by the Schauder fixed point theorem, one has

LEMMA 4.3. $F: S \rightarrow S$ has a fixed point.

PROOF. It is sufficient to prove that F is continuous from $C^1(Q_{T_1})$ in $C^0(Q_{T_1})$, since $F(S)$ is relatively compact in $C^1(Q_{T_1})$.

Let $\varphi_n \in S$, $\varphi_n \rightarrow \varphi$ in $C^1(Q_{T_1})$. From (3.2) and (3.5), one has

$$(4.14) \quad \begin{aligned} v^n &\rightarrow v && \text{in } C^0(Q_{T_1}), \\ Dv^n &\rightarrow Dv && \text{in } C^0(Q_{T_1}). \end{aligned}$$

Moreover, from (3.7) and (3.4')

$$(4.15) \quad \frac{\partial v^n}{\partial t} \rightarrow \frac{\partial v}{\partial t} \quad \text{in } C^0(Q_{T_1}).$$

On the other hand

$$\begin{aligned} |U^n(\sigma, t, x) - U(\sigma, t, x)| &\leq \left| \int_t^\sigma [|v^n(\tau, U^n(\tau, t, x)) - v(\tau, U^n(\tau, t, x))| + \right. \\ &\quad \left. + |v(\tau, U^n(\tau, t, x)) - v(\tau, U(\tau, t, x))|] d\tau \right| \leq \\ &\leq T_1 \|v_n - v\|_\infty + [v]_{0, \text{lip}} \left| \int_t^\sigma |U^n(\tau, t, x) - U(\tau, t, x)| d\tau \right|, \end{aligned}$$

and from Gronwall's lemma

$$|U^n(\sigma, t, x) - U(\sigma, t, x)| \leq T_1 \|v_n - v\|_\infty \exp [T_1 [v]_{0, \text{lip}}],$$

hence $U^n \rightarrow U$ uniformly in $[0, T_1] \times Q_{T_1}$.

Analogously, one evaluates $|D_i U_i^n(\sigma, t, x) - D_i U_i(\sigma, t, x)|$ by using (3.12), and this gives

$$\begin{aligned} \|DU^n - DU\|_\infty &\leq T_1 ([Dv^n]_{0, \text{lip}} \|DU^n\|_\infty \|U^n - U\|_\infty + \|DU\|_\infty \|Dv^n - Dv\|_\infty) \cdot \\ &\quad \cdot \exp [T_1 \|Dv^n\|_\infty]. \end{aligned}$$

Hence $DU^n \rightarrow DU$ uniformly in $[0, T_1] \times Q_{T_1}$. Consequently

$$(4.16) \quad \frac{\text{Rot } \varrho_n}{\varrho_n} \rightarrow \frac{\text{Rot } \varrho}{\varrho} \quad \text{in } C^0(Q_{T_1})$$

and

$$\gamma_n(\sigma, U^n(\sigma, t, x)) \rightarrow \gamma(\sigma, U(\sigma, t, x)) \quad \text{uniformly in } [0, T_1] \times Q_{T_1}.$$

From (4.3) the thesis follows. \square

This fixed point $\zeta = \varphi = F[\varphi]$, together with the corresponding v and ϱ , is a solution of the system

$$(4.17) \quad \begin{cases} \frac{\partial \zeta}{\partial t} + v \cdot \nabla \zeta = \beta + \frac{\text{Rot } \varrho}{\varrho} \cdot \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right] & \text{in } Q_{T_1}, \\ \zeta = \text{rot } v & \text{in } Q_{T_1}, \\ \text{div } v = 0 & \text{in } Q_{T_1}, \\ \frac{\partial \varrho}{\partial t} + v \cdot \nabla \varrho = 0 & \text{in } Q_{T_1}, \\ v \cdot n = 0 & \text{on } [0, T_1] \times \Gamma, \\ \zeta|_{t=0} = \alpha & \text{in } \bar{\Omega}, \\ \varrho|_{t=0} = \varrho_0 & \text{in } \bar{\Omega}. \end{cases}$$

5. Existence of a solution of system (E) when Ω is simply connected.

Since

$$\text{rot} [(v \cdot \nabla) v] = (\text{div } v) \text{rot } v + v \cdot \nabla (\text{rot } v)$$

one has from (4.17)₁, (4.17)₂ and (4.17)₄

$$\varrho \text{rot} \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right] = \text{Rot } \varrho \cdot \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right].$$

We recall the general identity

$$\text{rot} (\varrho w) = \varrho \text{rot } w - (\text{Rot } \varrho) \cdot w,$$

where ϱ is an arbitrary scalar and w an arbitrary vector, and applying it we obtain

$$(5.1) \quad \text{rot} \left\{ \varrho \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right] \right\} = 0 \quad \text{in } Q_{T_1}.$$

When Ω is simply connected, it is well known that there exists a scalar function $\pi \in C^{0,1}(Q_{T_1})$ such that $(E)_1$ holds in Q_{T_1} .

Moreover $\pi \in C^{\lambda,1}(Q_{T_1}) \cap C^{0,2+\lambda}(Q_{T_1})$: in fact $\pi(t, x)$ is determined as the integral of $\nabla\pi \cdot ds$ from a fixed point x_0 to x , along a path independent of t . Since $\nabla\pi \in C^{\lambda,0}(Q_{T_1})$, it follows that $\pi \in C^{\lambda,0}(Q_{T_1})$. The other statement follows directly from $(E)_1$. Furthermore

$$\begin{cases} \operatorname{rot} (v|_{t=0} - a) = 0 & \text{in } \bar{\Omega}, \\ \operatorname{div} (v|_{t=0} - a) = 0 & \text{in } \bar{\Omega}, \\ (v|_{t=0} - a) \cdot n = 0 & \text{on } \Gamma, \end{cases}$$

and consequently $(E)_5$ holds.

Hence we have found a solution (v, π, ϱ) to problem (E) in Q_{T_1} . This solution verifies the regularity conditions stated in Theorem A, as follows from Lemmas 3.1 and 3.3.

6. Uniqueness of the solution of system (E) .

Let (v, π, ϱ) and $(\tilde{v}, \tilde{\pi}, \tilde{\varrho})$ be two solutions of (E) in $[0, T] \times \bar{\Omega}$, under the conditions of Theorem B. We set $u \equiv \tilde{v} - v$, $\sigma \equiv \tilde{\pi} - \pi$, $\eta \equiv \tilde{\varrho} - \varrho$. On subtracting the two equations $(E)_1$, we obtain

$$(6.1) \quad \tilde{\varrho} \left[\frac{\partial u_i}{\partial t^s} + (\tilde{v} \cdot \nabla) u + (u \cdot \nabla) v \right] = -\nabla\sigma - \eta \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right].$$

On the other hand from $(E)_3$ one gets

$$\left(\tilde{\varrho} \frac{\partial u}{\partial t}, u \right) = \frac{1}{2} \frac{d}{dt} (\tilde{\varrho} u, u) + \frac{1}{2} ((\tilde{v} \cdot \nabla \tilde{\varrho}) u, u),$$

where $(,)$ denotes the scalar product in $L^2(\Omega)$ or in $[L^2(\Omega)]^2$. Taking the scalar product of (6.1) with u it follows

$$(6.2) \quad \frac{1}{2} \frac{d}{dt} (\tilde{\varrho} u, u) = -(\tilde{\varrho} (u \cdot \nabla) v, u) - \left(\eta \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v - b \right], u \right),$$

since

$$(\tilde{\varrho}(\tilde{v} \cdot \nabla)u, u) + \frac{1}{2}((\tilde{v} \cdot \nabla \tilde{\varrho})u, u) = 0 ;$$

recall that $\operatorname{div} \tilde{v} = 0$ and $\tilde{v} \cdot n = 0$.

Moreover, on subtracting the two equations (E)₃, we obtain

$$(6.3) \quad \frac{\partial \eta}{\partial t} + v \cdot \nabla \eta = -u \cdot \nabla \tilde{\varrho}$$

and taking the scalar product of (6.3) with η it follows

$$(6.4) \quad \frac{1}{2} \frac{d}{dt} (\eta, \eta) = - (u \cdot \nabla \tilde{\varrho}, \eta) ,$$

since $(v \cdot \nabla \eta, \eta) = 0$.

From (6.2) and (6.4) one obtains

$$(6.5) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\tilde{\varrho}|u|^2 + \eta^2) dx = - \int_{\Omega} \tilde{\varrho} [(u \cdot \nabla)v] \cdot u dx - \\ - \int_{\Omega} \eta \left[\frac{\partial v}{\partial t} + (v \cdot \nabla)v - b + \nabla \tilde{\varrho} \right] \cdot u dx .$$

Set

$$f(t) \equiv \frac{1}{2} \int_{\Omega} (\tilde{\varrho}|u|^2 + \eta^2) dx .$$

Obviously $f(0) = 0$; moreover from (3.16) and (3.11)

$$\|\nabla \tilde{\varrho}\|_{\infty} \leq 2 \|\nabla \varrho_0\|_{\infty} \exp [\|Dv\|_{L^1(0,T;L^{\infty}(\Omega))}]$$

and consequently from (6.5)

$$f'(t) \leq c(t)f(t)$$

where $c(t) \in L^1(0, T)$. By Gronwall's lemma $f(t)$ vanishes identically in $[0, T]$, i.e. $\tilde{v} = v$ and $\tilde{\varrho} = \varrho$ in Q_T .

Finally, from (E)₁ it follows that $\nabla \pi = \nabla \tilde{\pi}$ in Q_T , i.e. $\pi = \tilde{\pi}$ up to an arbitrary function of t ,

PART II

7. Existence of a solution of system (E) when Ω is not simply connected.

Let Ω be a bounded connected open subset of \mathbb{R}^2 . We assume that Γ consists of $m + 1$ simple closed curves $\Gamma_0, \Gamma_1, \dots, \Gamma_m$, where Γ_j ($j = 1, \dots, m$) are inside of Γ_0 and outside of one another.

We denote by v the vector field defined in (3.5) and by $u^{(k)}$, $k = 1, \dots, m$, the vector fields introduced at the end of §1 in [4]. We have $u^{(k)} \in C^{2+\lambda}(\bar{\Omega})$, $\text{rot } u^{(k)} = 0$, $\text{div } u^{(k)} = 0$ in $\bar{\Omega}$ and $u^{(k)} \cdot n = 0$ on Γ . We put

$$(7.1) \quad \bar{v}(t, x) \equiv v(t, x) + \sum_{k=1}^m \theta_k(t) u^{(k)}(x) \equiv v(t, x) + v'(t, x),$$

and consequently we have $\text{div } \bar{v} = 0$ and $\text{rot } \bar{v} = \varphi$ in Q_T , $\bar{v} \cdot n = 0$ on $[0, T] \times \Gamma$.

We define $\bar{q}(t, x)$ to be the solution of

$$(7.2) \quad \begin{cases} \frac{\partial \bar{q}}{\partial t} + \bar{v} \cdot \nabla \bar{q} = 0 & \text{in } Q_T, \\ \bar{q}|_{t=0} = \varrho_0 & \text{in } \bar{\Omega}. \end{cases}$$

Now we prove that there exist $\theta_k(t) \in C^{1+\lambda}([0, T])$ such that

$$(7.3) \quad \left(\bar{q} \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u^{(k)} \right) = 0 \quad \forall t \in [0, T],$$

$$(7.4) \quad (\bar{v}|_{t=0} - a, u^{(k)}) = 0$$

for each $k = 1, \dots, m$. We are going to use the Schauder fixed point theorem.

We consider the map $\bar{\theta}_k \mapsto \bar{v}$ from $C^0([0, T])$ in $C^{0,2+\lambda}(Q_T)$ defined by (7.1), the map $\bar{v} \mapsto \bar{q}$ from $C^{0,2+\lambda}(Q_T)$ in $C^{\lambda,0}(Q_T)$ defined by (7.2) and finally the map $(\bar{v}, \bar{q}) \mapsto \theta_k$ defined by (7.3), (7.4), i.e.

$$(7.3)' \quad \sum_{s=1}^m \mu_{ks}(t) \frac{d\theta_s(t)}{dt} + \sum_{s,h=1}^m \mu_{ksh}(t) \theta_s(t) \theta_h(t) + \sum_{s=1}^m [\nu_{ks}(t) + \eta_{ks}(t)] \theta_s(t) + \mu_k(t) + \nu_k(t) + \eta_k(t) = 0 \quad \text{in } [0, T],$$

$$(7.4)' \quad \theta_k(0) = (a, u^{(k)}),$$

for each $k = 1, \dots, m$. We have defined

$$(7.5) \quad \begin{aligned} \mu_{ks}(t) &\equiv (\bar{\varrho} u^{(s)}, u^{(k)}), & \mu_{ksh}(t) &\equiv (\bar{\varrho}(u^{(s)} \cdot \nabla) u^{(h)}, u^{(k)}), \\ \nu_{ks}(t) &\equiv (\bar{\varrho}(v \cdot \nabla) u^{(s)}, u^{(k)}), & \eta_{ks}(t) &\equiv (\bar{\varrho}(u^{(s)} \cdot \nabla) v, u^{(k)}), \\ \mu_k(t) &\equiv \left(\bar{\varrho} \frac{\partial v}{\partial t}, u^{(k)} \right), & \nu_k(t) &\equiv (\bar{\varrho}(v \cdot \nabla) v, u^{(k)}), \\ & & \eta_k(t) &\equiv -(\bar{\varrho} b, u^{(k)}). \end{aligned}$$

Since $u^{(k)} \in C^{2+\lambda}(\bar{\Omega})$, $v \in C^{1,2+\lambda}(Q_T) \cap C^{1+\lambda,0}(Q_T)$ and $\bar{\varrho} \in C^{\lambda,0}(Q_T)$, all these coefficients belong to $C^\lambda([0, T])$.

The notation \tilde{c} , \tilde{c}_1 , \tilde{c}_2 , ... will be used for constants depending at most on λ , Ω , a , b , ϱ_0 , m , $u^{(k)}$.

Assume that estimates (3.1) hold and moreover

$$(7.6) \quad \sup_{t \in [0, T]} \left[\sum_{k=1}^m \tilde{\theta}_k(t)^2 \right]^{\frac{1}{2}} \equiv \|\tilde{\theta}\|_\infty \leq E,$$

where $\tilde{\theta} \equiv (\tilde{\theta}_1, \dots, \tilde{\theta}_m)$ and E is a constant that will be fixed in the following.

One has

$$(7.7) \quad \begin{aligned} \|\bar{v}\|_\infty &\leq \|v\|_\infty + \sum_k \|\tilde{\theta}_k\|_\infty \|u^{(k)}\|_\infty \leq \tilde{c}(A + E), \\ \|D\bar{v}\|_\infty &\leq \|Dv\|_\infty + \sum_k \|\tilde{\theta}_k\|_\infty \|Du^{(k)}\|_\infty \leq \tilde{c}(A + E). \end{aligned}$$

Define $\bar{U}(\sigma, t, x)$ to be the solution of

$$\begin{cases} \frac{d\bar{U}}{d\sigma}(\sigma, t, x) = \bar{v}(\sigma, \bar{U}(\sigma, t, x)), \\ \bar{U}(t, t, x) = x; \end{cases}$$

one has, as in (3.10)₁:

$$(7.8) \quad [\bar{U}]_{\text{lip},0} = \left\| \frac{\partial \bar{U}}{\partial t} \right\|_\infty = \|D\bar{U}\|_\infty \|\bar{v}\|_\infty \leq 2 \|\bar{v}\|_\infty \exp [T \|D\bar{v}\|_\infty].$$

It follows from (7.7) and (7.8) that

$$(7.9) \quad [\bar{U}]_{\text{lip},0} \leq \tilde{c}(A + E) \exp [\tilde{c}T(A + E)].$$

From (7.2) one has $\bar{\varrho}(t, x) = \varrho_0(\bar{U}(0, t, x))$, hence

$$(7.10) \quad \begin{aligned} \|\bar{\varrho}\|_\infty &\leq \|\varrho_0\|_\infty, \\ [\bar{\varrho}]_{\lambda,0} &\leq [\varrho_0]_{\text{lip}}[\bar{U}]_{\lambda,0} \leq T^{1-\lambda}[\varrho_0]_{\text{lip}}[\bar{U}]_{\text{lip},0}. \end{aligned}$$

Define

$$(7.11) \quad K_2 \equiv \sup_k \|u^{(k)}\|_{L^1(\Omega)}.$$

We have from (7.5)

$$(7.12) \quad \begin{aligned} \|\mu_{ks}\|_\infty &\leq K_2^2 \|\varrho_0\|_\infty, & [\mu_{ks}]_\lambda &\leq \tilde{c} T^{1-\lambda} [\varrho_0]_{\text{lip}} [\bar{U}]_{\text{lip},0}, \\ \|\mu_{ksh}\|_\infty &\leq \tilde{c} \|\varrho_0\|_\infty, & [\mu_{ksh}]_\lambda &\leq \tilde{c} T^{1-\lambda} [\varrho_0]_{\text{lip}} [\bar{U}]_{\text{lip},0}, \\ \|\nu_{ks}\|_\infty + \|\eta_{ks}\|_\infty &\leq \tilde{c} A \|\varrho_0\|_\infty, \\ & & [\nu_{ks}]_\lambda + [\eta_{ks}]_\lambda &\leq \tilde{c} A (\|\varrho_0\|_\infty + T^{1-\lambda} [\varrho_0]_{\text{lip}} [\bar{U}]_{\text{lip},0}), \\ \|\mu_k\|_\infty &\leq K_1 K_2 B |\Omega|^{\frac{1}{2}} \|\varrho_0\|_\infty, \\ & & [\mu_k]_\lambda &\leq \tilde{c} (D \|\varrho_0\|_\infty + T^{1-\lambda} [\varrho_0]_{\text{lip}} [\bar{U}]_{\text{lip},0} B), \\ \|\nu_k\|_\infty &\leq \tilde{c} A^2 \|\varrho_0\|_\infty, & [\nu_k]_\lambda &\leq \tilde{c} A^2 (\|\varrho_0\|_\infty + T^{1-\lambda} [\varrho_0]_{\text{lip}} [\bar{U}]_{\text{lip},0}), \\ \|\eta_k\|_\infty &\leq \tilde{c} \|\varrho_0\|_\infty, & [\eta_k]_\lambda &\leq \tilde{c} (\|\varrho_0\|_\infty + T^{1-\lambda} [\varrho_0]_{\text{lip}} [\bar{U}]_{\text{lip},0}), \end{aligned}$$

where $|\Omega| \equiv \text{meas } \Omega$.

Let $M(t)$ be the $(m \times m)$ -symmetric matrix $\{\mu_{ks}(t)\}$. One sees easily that $|\xi|^2 \min_{\bar{\Omega}} \varrho_0 < M(t) \xi \cdot \xi < \max_{\bar{\Omega}} \varrho_0 |\xi|^2$ for each $\xi \in \mathbf{R}^m$, and

$$(7.13) \quad 0 < \left(\min_{\bar{\Omega}} \varrho_0\right)^m < \det M(t) < \left(\max_{\bar{\Omega}} \varrho_0\right)^m \quad \forall t \in [0, T].$$

The element $\bar{\mu}_{ks}(t)$ of $[M(t)]^{-1}$ has the form

$$(7.14) \quad \bar{\mu}_{ks}(t) = \frac{(-1)^{k+s} M_{ks}(t)}{\det M(t)},$$

where $M_{ks}(t)$ is the minor of the matrix $M(t)$ corresponding to the (k, s) -element of $M(t)$.

Hence

$$(7.15) \quad \begin{aligned} \|\bar{\mu}_{ks}\|_\infty &\leq (m-1)! K_2^{2(m-1)} \frac{\|\varrho_0\|_\infty^{m-1}}{(\min_{\bar{\Omega}} \varrho_0)^m}, \\ [\bar{\mu}_{ks}]_\lambda &\leq \sum_{i,j,h,\tau} \left(\|\mu_{ij}\|_\infty^{m-2} \frac{1}{(\min_{\bar{\Omega}} \varrho_0)^m} + \|\mu_{ij}\|_\infty^{2(m-1)} \frac{1}{(\min_{\bar{\Omega}} \varrho_0)^{2m}} \right) [\mu_{h\tau}]_\lambda, \end{aligned}$$

and by using (7.12)

$$(7.16) \quad [\bar{\mu}_{ks}]_{\lambda} \leq \tilde{c} T^{1-\lambda} [\varrho_0]_{\text{lip}} [\bar{U}]_{\text{lip},0} \left[\frac{\|\varrho_0\|_{\infty}^{m-2}}{(\min_{\bar{\Omega}} \varrho_0)^m} + \frac{\|\varrho_0\|_{\infty}^{2(m-1)}}{(\min_{\bar{\Omega}} \varrho_0)^{2m}} \right].$$

Applying $[M(t)]^{-1}$ to (7.3)', one obtains

$$(7.17) \quad \frac{d\theta_k(t)}{dt} = \sum_{s,h=1}^m \bar{\mu}_{ksh}(t) \theta_s(t) \theta_h(t) + \sum_{s=1}^m [\bar{\nu}_{ks}(t) + \bar{\eta}_{ks}(t)] \theta_s(t) + \\ + \bar{\mu}_k(t) + \bar{\nu}_k(t) + \bar{\eta}_k(t) \quad \text{in } [0, T],$$

where

$$(7.18) \quad \bar{\mu}_k(t) \equiv - \sum_{i=1}^m \bar{\mu}_{ki}(t) \mu_i(t)$$

and analogously for the other coefficients.

Obviously the system (7.17), (7.4)' has an unique local solution $\theta_k(t)$, $k = 1, \dots, m$.

Moreover, taking the scalar product of (7.17) with $\theta(t)$, one has

$$(7.19) \quad \left\{ \begin{array}{l} \frac{1}{2} \frac{d}{dt} |\theta(t)|^2 \leq \tilde{c} \frac{\|\varrho_0\|_{\infty}^m}{(\min_{\bar{\Omega}} \varrho_0)^m} \cdot \\ \cdot \{ |\theta(t)|^3 + A |\theta(t)|^2 + (A^2 + B + 1) |\theta(t)| \}, \\ |\theta(0)|^2 \leq \tilde{c}_5^2 \|a\|_{\infty}^2. \end{array} \right.$$

Hence, if we choose $E > \tilde{c}_5 \|a\|_{\infty}$ in (7.6), we see that there exists $T^* \in]0, T]$ such that

$$|\theta(t)| \leq E \quad \text{in } [0, T^*].$$

If we put

$$S_1 \equiv \{ \bar{\theta}(t) \in C^0([0, T^*]) \mid \|\bar{\theta}\|_{\infty} \leq E \}$$

and we denote by F_1 the map $\bar{\theta} \mapsto \theta$ defined by (7.1), (7.2), (7.17) and (7.4)', we have $F_1(S_1) \subset S_1$.

Moreover from (7.17) and the Ascoli-Arzelà theorem, it follows that $F_1(S_1)$ is relatively compact in $C^0([0, T^*])$.

Finally, we see easily that $F_1: S_1 \rightarrow S_1$ is continuous, consequently F_1 has a fixed point in S_1 .

Hence equation (7.3), (7.4) has a local solution $\theta(t) \in C^{1+\lambda}$. We want to prove that $\theta(t)$ is a global solution.

From (7.3) we have

$$0 = \left(\bar{\varrho} \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], v' \right) = \left(\bar{\varrho} \frac{\partial v'}{\partial t}, v' \right) + \left(\bar{\varrho} \frac{\partial v}{\partial t}, v' \right) + \\ + (\bar{\varrho} (\bar{v} \cdot \nabla) v', v') + (\bar{\varrho} (\bar{v} \cdot \nabla) v, v') - (\bar{\varrho} b, v').$$

Moreover

$$(\bar{\varrho} (\bar{v} \cdot \nabla) v', v') = -\frac{1}{2} ((\bar{v} \cdot \nabla \bar{\varrho}) v', v'),$$

and from (7.2)

$$\frac{1}{2} \frac{d}{dt} (\bar{\varrho} v', v') = \left(\bar{\varrho} \frac{\partial v'}{\partial t}, v' \right) - \frac{1}{2} ((\bar{v} \cdot \nabla \bar{\varrho}) v', v').$$

Hence

$$0 = \frac{1}{2} \frac{d}{dt} (\bar{\varrho} v', v') + \left(\bar{\varrho} \frac{\partial v}{\partial t}, v' \right) + (\bar{\varrho} (v' \cdot \nabla) v, v') + (\bar{\varrho} (v \cdot \nabla) v, v') - (\bar{\varrho} b, v'),$$

i.e.

$$\frac{1}{2} \frac{d}{dt} \sum_{k,s} \mu_{ks}(t) \theta_k(t) \theta_s(t) = \\ = - \sum_k \mu_k(t) \theta_k(t) - \sum_{k,s} \eta_{ks}(t) \theta_k(t) \theta_s(t) - \sum_k \nu_k(t) \theta_k(t) - \sum_k \eta_k(t) \theta_k(t).$$

Consequently

$$\frac{1}{2} \frac{d}{dt} [M(t) \theta(t) \cdot \theta(t)] \leq \tilde{c}_6 \| \varrho_0 \|_\infty [A |\theta(t)|^2 + (A^2 + B + 1) |\theta(t)|] \leq \\ \leq \tilde{c}_6 \| \varrho_0 \|_\infty \left[\frac{A}{\min \varrho_0} M(t) \theta(t) \cdot \theta(t) + \frac{(A^2 + B + 1)}{\sqrt{\min \varrho_0}} \sqrt{M(t) \theta(t) \cdot \theta(t)} \right],$$

$$M(0) \theta(0) \cdot \theta(0) \leq \tilde{c}_7 \| \varrho_0 \|_\infty \| a \|_\infty^2.$$

Set

$$\alpha_1 \equiv 2\tilde{c}_6 \frac{\| \varrho_0 \|_\infty}{\sqrt{\min \varrho_0}} (A^2 + B + 1), \quad \alpha_2 \equiv 2\tilde{c}_6 \frac{\| \varrho_0 \|_\infty}{\bar{\varrho}} A;$$

the solution $y(t)$ of

$$\begin{cases} y'(t) = \alpha_1 \sqrt{y(t)} + \alpha_2 y(t), \\ y(0) = \tilde{c}_7 \|\varrho_0\|_\infty \|a\|_\infty^2, \end{cases}$$

satisfies

$$\alpha_1 + \alpha_2 \sqrt{y(t)} = [\alpha_1 + \alpha_2 \sqrt{y(0)}] \exp [(\alpha_2/2)t],$$

Hence by comparison theorems

$$|\theta(t)| \leq \frac{A^2 + B + 1}{A} \left(\exp \left[\frac{\alpha_2}{2} t \right] - 1 \right) + \sqrt{\tilde{c}_7 \frac{\|\varrho_0\|_\infty}{\min \varrho_0}} \|a\|_\infty \exp \left[\frac{\alpha_2}{2} t \right] \quad \forall t \in [0, T],$$

i.e. $\theta(t)$ is a global solution in $[0, T]$ and

$$(7.20) \quad \|\theta\|_\infty \leq \frac{A^2 + B + 1}{A} (\exp [\tilde{c}AT] - 1) + \tilde{c}_8 \exp [\tilde{c}AT] < \tilde{c}(A^2 + B + 1)T \exp [\tilde{c}AT] + \tilde{c}_8 \exp [\tilde{c}AT].$$

Define

$$(7.21) \quad K_3 \equiv m! \frac{\|\varrho_0\|_\infty^m}{(\min \varrho_0)^m} K_2^{2m-1} |\Omega|^{\frac{1}{2}}.$$

From (7.12), (7.15) and (7.18) one has

$$\|\tilde{\mu}_k\|_\infty \leq K_3 K_1 B.$$

Consequently, from (7.17) and (7.15), (7.16)

$$(7.22) \quad \left\| \frac{d\theta}{dt} \right\|_\infty \leq \tilde{c}_9 (\|\theta\|_\infty^2 + A^2 + 1) + K_3 K_1 B,$$

$$(7.23) \quad \left[\frac{d\theta}{dt} \right]_\lambda \leq \tilde{c}_{10} (A^2 + A \|\theta\|_\infty + 1) + K_3 K_1 D + \tilde{c} T^{1-\lambda} ([\bar{U}]_{\text{lip},0} + \|\theta\|_\infty + A) (\|\theta\|_\infty^2 + A^2 + B + 1).$$

Finally, define

$$(7.24) \quad K_4 \equiv \sqrt{2} \left[\sum_k \|u^{(k)}\|_\infty^2 \right]^{\frac{1}{2}};$$

from (7.1) and (3.8) one has,

$$\begin{aligned}
 \|\bar{v}\|_{0,2+\lambda} &\leq \tilde{c}(A + \|\theta\|_\infty), \\
 \|D_t \bar{v}\|_\infty &\leq K_1 B + K_4 \left\| \frac{d\theta}{dt} \right\|_\infty, \\
 \|D_t \bar{v}\|_{0,1+\lambda} &\leq \tilde{c} \left(C + \left\| \frac{d\theta}{dt} \right\|_\infty \right), \\
 [D_t \bar{v}]_{\lambda,0} &\leq K_1 D + K_4 \left[\frac{d\theta}{dt} \right]_\lambda, \\
 [D_t \bar{v}]_{0,\lambda} &\leq \tilde{c} \left(B + \left\| \frac{d\theta}{dt} \right\|_\infty \right),
 \end{aligned}
 \tag{7.25}$$

which replace estimates (3.8).

Set

$$\bar{\gamma} \equiv \beta + \frac{\text{Rot } \bar{\varrho}}{\bar{\varrho}} \cdot \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right] \quad \text{in } Q_T;$$

by replacing v, U, ϱ, γ with $\bar{v}, \bar{U}, \bar{\varrho}, \bar{\gamma}$ in the proofs of Lemmas 3.2, 3.3, 4.1 and by using (7.25) one obtains:

LEMMA 7.1. *Let $\bar{\xi}(t, x)$ be the solution of*

$$\begin{cases} \frac{\partial \bar{\xi}}{\partial t} + \bar{v} \cdot \nabla \bar{\xi} = \bar{\gamma} & \text{in } Q_T, \\ \bar{\xi}|_{t=0} = \alpha & \text{in } \bar{\Omega}. \end{cases}$$

Then Lemma 4.2 is true if we substitute in (4.5), (4.6), (4.7) and (4.8) ξ with $\bar{\xi}$, A with $A + \|\theta\|_\infty$, $K_1 B$ with $K_1 B + K_4 \|d\theta/dt\|_\infty$, B with $B + \|d\theta/dt\|_\infty$, C with $C + \|d\theta/dt\|_\infty$, $K_1 D$ with $K_1 D + K_4 [d\theta/dt]_\lambda$. Constants c and \bar{c} , c_i and \bar{c}_i must be replaced respectively by \tilde{c} , \tilde{c}_i . We will denote these new estimates by (4.5)', (4.6)', (4.7)' and (4.8)'.

Hence the existence of a solution of system (4.17) will be a consequence of the existence of a fixed point for the map $\bar{F}: \varphi \mapsto \bar{\xi}$.

First of all, we prove that there exists $T_1 \in]0, T]$ such that $\bar{F}(S) \subset S$, provided that A, B, C, D are chosen in a suitable way in (4.11).

By using (7.20), (7.22) and (7.23), we obtain from Lemma 7.1

that in Q_T one has

$$(7.36) \quad \begin{aligned} \|\bar{\xi}\|_{0,1+\lambda} &\leq f_1(T, A, B, C), \\ \|D_t \bar{\xi}\|_{\infty} &\leq f_2(T, A, B, C), \\ \|D_t \bar{\xi}\|_{0,\lambda} &\leq f_3(T, A, B, C), \\ [D_t \bar{\xi}]_{\lambda,0} &\leq f_4(T, A, B, C, D), \end{aligned}$$

where the functions f_i are continuous, non-negative, and non-decreasing with respect to each variable. Hence, if we fix A, B, C, D such that

$$(7.27) \quad \begin{aligned} f_1(0, A, B, C) &< A, \\ f_2(0, A, B, C) &< B, \\ f_3(0, A, B, C) &< C, \\ f_4(0, A, B, C, D) &< D, \end{aligned}$$

there exists $T_1 \in]0, T]$ for which

$$(7.28) \quad \begin{aligned} \|\bar{\xi}\|_{0,1+\lambda} &\leq f_1(T_1, A, B, C) \leq A, \\ \|D_t \bar{\xi}\|_{\infty} &\leq f_2(T_1, A, B, C) \leq B, \\ \|D_t \bar{\xi}\|_{0,\lambda} &\leq f_3(T_1, A, B, C) \leq C, \\ [D_t \bar{\xi}]_{\lambda,0} &\leq f_4(T_1, A, B, C, D) \leq D, \end{aligned}$$

in Q_{T_1} .

It is easy to verify that (7.27) has a solution, provided that condition (A) is satisfied. For example, one can choose successively

$$(7.29) \quad \begin{aligned} A &> 2 \|\alpha\|_{1+\lambda}, \\ B &> \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} K_1(1 + K_3 K_4) B + \tilde{c}_1(A + \tilde{c}_8) \|D\alpha\|_{\infty} + \\ &\quad + \tilde{c}_1[(A + c_8)^2 + 1] + \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} K_4 \tilde{c}_9(\tilde{c}_8^2 + A^2 + 1), \\ C &> B + 2\tilde{c}_2(A + \tilde{c}_8) \|D\alpha\|_{\lambda} + \\ &\quad + \tilde{c}_2\{(A + \tilde{c}_8)^2 + B + \tilde{c}_9(\tilde{c}_8^2 + A^2 + 1) + K_3 K_1 B + 1\}, \\ D &> \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} K_1(1 + K_3 K_4) D + \tilde{c}_3 \tilde{c}_4(A + \tilde{c}_8)^{1+\lambda} [D\alpha]_{\lambda} + \\ &\quad + \tilde{c}_3[(A + \tilde{c}_8)^2 + B + \tilde{c}_9(\tilde{c}_8^2 + A^2 + 1) + K_3 K_1 B + 1] \cdot \\ &\quad \cdot (A + \tilde{c}_8)^{\lambda} + \tilde{c}_3 + \left\| \frac{D\varrho_0}{\varrho_0} \right\|_{\infty} K_4 \tilde{c}_{10}(A^2 + \tilde{c}_8 A + 1). \end{aligned}$$

Lemma 4.3 is proved as before, provided that $\bar{F}: S \mapsto S$ is continuous from $C^1(Q_{T_1})$ in $C^0(Q_{T_1})$.

Hence, we must prove that if $\varphi_n \rightarrow \varphi$ in $C^1(Q_{T_1})$, $\varphi_n \in S$, then \bar{v}_n and \bar{v} satisfy (4.14) and (4.15). Since v^n and v satisfy these last conditions, it is sufficient to prove that

$$\begin{aligned} \theta^n &\rightarrow \theta && \text{uniformly in } [0, T_1], \\ \frac{d\theta^n}{dt} &\rightarrow \frac{d\theta}{dt} && \text{uniformly in } [0, T_1]; \end{aligned}$$

we begin by recalling that \bar{v}^n and $\bar{\varrho}_n$ satisfy

$$(7.30) \quad \begin{cases} \frac{\partial \bar{\varrho}_n}{\partial t} + \bar{v}^n \cdot \nabla \bar{\varrho}_n = 0 & \text{in } Q_{T_1}, \\ \bar{\varrho}_n|_{t=0} = \varrho_0 & \text{in } \bar{D}, \end{cases}$$

$$(7.31) \quad \left(\bar{\varrho}_n \left[\frac{\partial \bar{v}^n}{\partial t} + (\bar{v}^n \cdot \nabla) \bar{v}^n - b \right], u^{(k)} \right) = 0 \quad \forall t \in [0, T_1],$$

$$(7.32) \quad (\bar{v}^n|_{t=0} - a, u^{(k)}) = 0,$$

for each $k = 1, \dots, m$.

Set now $\eta \equiv \bar{\varrho}_n - \bar{\varrho}$, $u' \equiv v'^n - v'$, $u \equiv v^n - v$, $\bar{u} \equiv \bar{v}^n - \bar{v} = u + u'$; one obtains from (7.3)

$$(7.33) \quad \left(\bar{\varrho}_n \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u^{(k)} \right) - \left(\eta \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u^{(k)} \right) = 0.$$

On substrating (7.33) from (7.31) one has

$$\left(\bar{\varrho}_n \left[\frac{\partial \bar{u}}{\partial t} + (\bar{v}^n \cdot \nabla) \bar{u} + (\bar{v} \cdot \nabla) \bar{v} \right], u^{(k)} \right) = - \left(\eta \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u^{(k)} \right),$$

and multiplying by $\theta_k^n - \theta_k$

$$(7.34) \quad \begin{aligned} &\left(\bar{\varrho}_n \left[\frac{\partial u'}{\partial t} + (\bar{v}^n \cdot \nabla) u' + (u' \cdot \nabla) \bar{v} \right], u' \right) = \\ &= - \left(\bar{\varrho}_n \left[\frac{\partial u}{\partial t} + (\bar{v}^n \cdot \nabla) u + (u \cdot \nabla) \bar{v} \right], u' \right) - \left(\eta \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u' \right). \end{aligned}$$

From (7.30)₁

$$\left(\bar{\varrho}_n \frac{\partial u'}{\partial t}, u' \right) = \frac{1}{2} \frac{d}{dt} (\bar{\varrho}_n u', u') + \frac{1}{2} ((\bar{v}^n \cdot \nabla \bar{\varrho}_n) u', u')$$

and moreover

$$(\bar{\varrho}_n (\bar{v}^n \cdot \nabla) u', u') = \frac{1}{2} \sum_{i,j} \int_{\Omega} \bar{\varrho}_n \bar{v}_i^n \frac{\partial u'_j}{\partial x_i} dx = -\frac{1}{2} ((\bar{v}^n \cdot \nabla \bar{\varrho}_n) u', u').$$

Hence (7.34) becomes

$$(7.35) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} (\bar{\varrho}_n u', u') + (\bar{\varrho}_n (u' \cdot \nabla) \bar{v}, u') = \\ & = - \left(\bar{\varrho}_n \left[\frac{\partial u}{\partial t} + (\bar{v}^n \cdot \nabla) u + (u \cdot \nabla) \bar{v} \right], u' \right) - \left(\eta \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u' \right). \end{aligned}$$

On other hand, from (7.30) and (7.2) one obtains

$$\frac{\partial \eta}{\partial t} + \bar{v} \cdot \nabla \eta = -u' \cdot \nabla \bar{\varrho}_n - u \cdot \nabla \bar{\varrho}_n,$$

and taking the scalar product with η

$$(7.36) \quad \frac{1}{2} \frac{d}{dt} (\eta, \eta) = -(\eta u', \nabla \bar{\varrho}_n) - (\eta u, \nabla \bar{\varrho}_n).$$

Set $f(t) \equiv \frac{1}{2} (\bar{\varrho}_n u', u') + \frac{1}{2} (\eta, \eta)$: from (7.35) and (7.36) one has

$$\begin{aligned} \frac{d}{dt} f(t) & \leq c (\bar{\varrho}_n u', u') + \left\| \frac{\partial v^n}{\partial t} - \frac{\partial v}{\partial t} \right\|_{\infty} \int_{\Omega} \bar{\varrho}_n |u'| dx + \\ & + c \|Dv^n - Dv\|_{\infty} \int_{\Omega} \bar{\varrho}_n |u'| dx + c \|v^n - v\|_{\infty} \int_{\Omega} \bar{\varrho}_n |u'| dx + \\ & + c \int_{\Omega} |\eta| |u'| dx + c \|v^n - v\|_{\infty} \int_{\Omega} |\eta| dx, \end{aligned}$$

since $\|\bar{v}^n\|_\infty$ and $\|\nabla\bar{\varrho}_n\|_\infty$ are bounded, and $0 < \min_{\bar{\Omega}} \varrho_0 \leq \bar{\varrho}_n(t, x) \leq \|\varrho_0\|_\infty$.
Hence

$$\begin{cases} f'(t) \leq cf(t) + c_n \sqrt{f(t)}, \\ f(0) = 0 \end{cases}$$

where $c_n \rightarrow 0$, and consequently by comparison theorems

$$(7.37) \quad f(t) \leq \left(\frac{c_n}{c}\right)^2 \left(\exp\left[\frac{ct}{2}\right] - 1\right)^2 \quad \text{in } [0, T_1].$$

Estimate (7.37) gives

$$(7.38) \quad \begin{aligned} \sup_{t \in [0, T_1]} \|v'^n - v'\|_{L^s(\Omega)} &\xrightarrow{n} 0, \\ \sup_{t \in [0, T_1]} \|\bar{\varrho}_n - \bar{\varrho}\|_{L^s(\Omega)} &\xrightarrow{n} 0, \end{aligned}$$

i.e.

$$(7.39) \quad \sup_{t \in [0, T_1]} |\theta^n(t) - \theta(t)| = \sup_{t \in [0, T_1]} \|v'^n - v'\|_{L^s(\Omega)} \xrightarrow{n} 0.$$

Moreover from (7.5) one has

$$\|\mu_{ks}^n - \mu_{ks}\|_\infty \leq c \sup_{t \in [0, T_1]} \|\bar{\varrho}_n - \bar{\varrho}\|_{L^s(\Omega)} \xrightarrow{n} 0,$$

and analogously for the other coefficients.

Consequently from (7.14) it follows that

$$\|\bar{\mu}_{ks}^n - \bar{\mu}_{ks}\|_\infty \xrightarrow{n} 0,$$

and the same is true for each coefficient in (7.17); hence we conclude that

$$\left\| \frac{d\theta^n}{dt} - \frac{d\theta}{dt} \right\|_\infty \xrightarrow{n} 0.$$

As in § 4, we have proved that

$$\text{rot} \left\{ \bar{\varrho} \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right] \right\} = 0 \quad \text{in } Q_{T_1},$$

and moreover

$$\left(\bar{\varrho} \left[\frac{\partial \bar{v}}{\partial t} + (\bar{v} \cdot \nabla) \bar{v} - b \right], u^{(k)} \right) = 0 \quad \forall t \in [0, T_1],$$

$$(\bar{v}|_{t=0} - a, u^{(k)}) = 0,$$

for each $k = 1, \dots, m$.

As in Kato [5], Lemma 1.6 (see also Hopf [4]), it follows that there exists a scalar function $\bar{\pi} \in C^{0,1}(Q_{T_1})$ such that (E)₁ holds in Q_{T_1} . The further regularity properties of $\bar{\pi}$ are proved as in § 4, since \bar{v} has the same regularity of v .

Finally, by using (7.4) one obtains that $\bar{v}|_{t=0} = a$ in \bar{D} , i.e. we have found a solution $(\bar{v}, \bar{\pi}, \bar{\varrho})$ of system (E) in Q_{T_1} .

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