Rendiconti
del
Seminario Matematico
della
Università di Padova

Livio Clemente Piccinini

G-convergence for ordinary differential equations with Peano phaenomenon

Rendiconti del Seminario Matematico della Università di Padova, tome 58 (1977), p. 65-86

<http://www.numdam.org/item?id=RSMUP_1977__58__65_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 58 (1977)

G-Convergence for ordinary differential equations with Peano phaenomenon.

LIVIO CLEMENTE PICCININI (*) (1)

SUMMARY : In this paper we give a general theory of G-convergence for first order ordinary differential equations under very general assumptions, including the existence of Peano phaenomenon. An abstract compactness theorem is given from which theorems on G-convergence follow for wide classes of equations.

0. Introduction.

Usually in order to study the G-convergence of differential operators strong hypothesis are made about the uniqueness of solutions.

When the object of study are non-linear differential equations these assumptions are too restrictive, since they cannot take in account Peano phaenomenon. Here we solve this problem attaching to every equations not single solutions, but sets, large enough, of solutions ; the G-convergence will consist of the convergence of these sets, that we shall call Peano processes.

In sections 1 and 3 Peano processes are treated axiomatically, while in sections 2 and 4 differential equations are considered. In this paper we restrict ourselves to the study of first order equations, but it is possible to extend this theory, with minor complications, to any finite dimension. We do not give examples and do not afford homogeneization problems, since they can be found in [1].

(*) Address of the author : Facoltà di Statistica Università di Padova

(1) Lavoro eseguito nell'ambito del G.N.A.F.A.

[1] L. C. PICCININI, *Homogeneization properties for ordinary differential equations* Rend. Circ. Mat. Pal. (1978).

Further extensions of the theory can be given in order to include also equations with transmission problems; this will be treated in a following paper.

Throughout the paper we shall use standard notations, but for this one: for any two sets A and $B \subset \mathbb{R}$ we set

$$(0.1) \quad \|A \,\Delta\, B\| = \inf \{\delta : (\forall\, x \in A, \exists\, y \in B : |x-y| < \delta), (\forall x \in B, \exists\, y \in A : |x-y| < \delta)\}$$

and, if the sets depend on a parameter t, say $A \equiv A(t)$, $B \equiv B(t)$, then

$$(0.2) \qquad \|A \,\Delta\, B\| = \sup_t (\|A(t) \,\Delta\, B(t)\|).$$

1. - Abstract formulation of the problem.

In this section we define axiomatically a Peano process; the definition of solution of a Cauchy problem follows, and starting from this an equivalence relation is introduced. The section ends with the definition of convergence for a sequence of equivalence classes.

Def. 1.1. *A Peano process is a family of subsets of* $\mathbb{R} \times [a, b]$ *depending on three indexes* $x \in R$, $t \in [a, b]$ $p \in]0,1]$ *satisfying the monotonicity, consistency, regularity, equation properties listed below.*

Such sets will be denoted $S_{xt,p}$. For reader's convenience we suggest to read at the same time also section 2, where a Peano process is actually built starting from a differential equation.

We denote by $S_{xt,p}(t_0)$ the section

$$(1.1) \qquad S_{xt,p}(t_0) = S_{xt,p} \cap \{\tau = t_0\}.$$

Since we are dealing with one x-variable it will be simpler to use also the following functions

$$(1.2) \qquad S^+_{xt,p}(t_0) = \sup_\xi S_{xt,p}(t_0)$$

$$(1.3) \qquad S^-_{xt,p}(t_0) = \inf_\xi \; S_{xt,p}(t_0).$$

Peano processes satisfy the following monotonicity properties:

M1.

(1.4) $$S_{xt,p}(t) = \{x\}$$

M2.

(1.5) $$S_{xt,p}(\tau) \neq \varnothing \text{ for any } \tau \in [a, b]$$

$$S_{xt,p}(\tau) \text{ is connected for any } \tau \in [a, b]$$

M3. If $p_1 > p_2$, then

(1.6) $$S_{xt,p_1} \supset S_{xt,p_2}$$

M4. Let $t_0 > t$ (resp. $t_0 < t$), then for any $\tau > t_0$ (resp. $\tau < t_0$), for any $q > p > 0$ it holds

(1.7) $$S_{xt,p}(\tau) \subset \{\bigcup_{\xi} S_{\xi t_0,q}(\tau)\ ;\ \xi \in S_{xt,p}(t_0)\}$$

(1.8) $$S_{xt,q}(\tau) \supset \{\bigcup_{\xi} S_{\xi t_0,p}(\tau)\ ;\ \xi \in S_{xt,q}(t_0)\}.$$

Peano processes satisfy the following regularity property

R1 (Equiabsolute continuity) For each set $K \subset\subset \mathbb{R}$ and for any $\varepsilon > 0$, there exists $\delta_K(\varepsilon) > 0$ such that for any finite number of disjoint intervals $[t_i', t_i'']$ with

(1.9) $$\sum_{i=1}^{N} |t_i'' - t_i'| < \delta_K(\varepsilon)$$

it holds for any $x \in K$, $t \in [a, b]$, $p \in]0, 1]$

(1.10) $$\sum_{i=1}^{N} \|S_{xt,p}(t_i'')\ \Delta\ S_{xt,p}(t_i')\| < \varepsilon\,.\text{(1)}$$

---

(1) Since we are dealing only with one x-variable this statement is the same as equiabsolute continuity of S^+ and S^-.

Peano processes satisfy the following consistence properties :

C1. For each $K \subset\subset \mathbb{R}$ there exists a continuous function satisfying

$$\varphi_K(0, d, \delta) = d \tag{1.11}$$

$$\lim_{(\tau,d,\delta)\to 0} \left[\frac{\varphi_K(\tau, d, \delta) - d}{\tau} - \delta \right] = 0 \tag{1.12}$$

such that if $x_1, x_2 \in K, t \in [a, b], p \in]0, 1], p \leq q < p + \delta$ and

$$|x_1 - x_2| < d \tag{1.13}$$

then

$$\|S_{x_1 t,q}(\tau) \Delta S_{x_2 t,q}(\tau)\| \leq \varphi_K(|\tau - t|, d, \delta). \tag{1.14}$$

C2. For each $K \subset\subset \mathbb{R}$ there exists a function of class C^1, $\zeta_K(\tau, \delta)$ satisfying the conditions $\zeta_K(0, \delta) = \dfrac{\partial \zeta_K}{\partial \tau}(0, \delta) = 0$
such that for any p it holds

$$\|S_{xt,p+\delta}(\tau) \Delta S_{xt,p}(\tau)\| \leq \zeta_K(|t - \tau|, \delta) + \delta |t - \tau| \tag{1.15}$$

and there exists a function $\vartheta_K(\tau, \delta)$ continuous satisfying the conditions $\vartheta_K(0, \delta) = 0, \vartheta_K(\tau, \delta) < \tau \cdot \delta)$ for $\tau > 0, \delta > 0$, such that

$$\operatorname{dist}\left(S_{xt,p}(\tau), R/S_{xt,p+\delta}(\tau)\right) \geq \delta |t - \tau| - \vartheta_K(|t - \tau|, \delta). \tag{1.16}$$

C3. For each $K \subset\subset \mathbb{R}$ there exists a continuous function $\gamma_K(d, \tau)$, strictly positive for $d > 0, \tau > 0$ such that if $x_1, x_2 \in K$ and $|x_1 - x_2| < \gamma_K(d, \tau)$, then for $|t - t_0| > \tau, p_2 > p_1 + d$ it holds

$$S_{x_2 t_0, p_2}(t) \supset S_{x_1 t_0, p_1}(t). \tag{1.17}$$

A Peano process associated with an equation satisfies the following equation property :

E1. For each $K \subset\subset \mathbb{R}$ there exists a function $\varepsilon_K(\tau)$ of class C^1 such that $\varepsilon_K(0) = \varepsilon'_K(0) = 0$. For any $x \in K$, for any t, p it holds

(1.18) $$S^{+}_{xt,p}(\tau) - S^{-}_{xt,p}(\tau) \leq \varepsilon_K(|t-\tau|) + p\,|t-\tau|$$

We come now to treat the solutions of a Peano process.

Definition. 1. 2. *We call solution of the Cauchy problem of data* $x(t_0) = x_0$ *for a Peano process the following intersection*

(1.19) $$R_{x_0t_0} = \bigcap_{p>0} S_{x_0t_0,p}\,.$$

It holds the following locality theorem :

THEOREM 1.1. *Let* $t_1 > t_0$ *(resp.* $t_1 < t_0$*), then for any* $\tau > t_1$ *(resp.* $\tau < t_1$*) it holds.*

(1.20) $$R_{x_0t_0}(\tau) = \{\bigcup_{\xi} R_{\xi t_1}(\tau)\,;\ \xi \in R_{x_0t_0}(t_1)\}\,.$$

Proof. We prove first that for any interval I, $E = \{\bigcup_{\xi} R_{\xi t_1}(\tau)\,,\ \xi \in \mathrm{I}\}$ is still an interval. Let us suppose E has at least two connected components, $E_1, E_2, \ldots$. In correspondence there are disjoint sets $I_1, I_2, \ldots$ such that $\bigcup_n I_n = I$ and.

$\bigcup_{\xi \in I} R_{\xi t_1}(\tau) = E_n$. Since I is connected there are at least two sets, say I_1, I_2 such that $I_1 \cap \partial I_2 \neq \varnothing$. Let $\xi_0 \in I_1 \cap \partial I_2$; using C3, for any $p > 0$, for a fixed $\delta > 0$, it exists $\bar{\xi} \in I_2$ such that

$$|\xi_0 - \bar{\xi}| < \gamma_K\left(\delta, \frac{|\tau - t_1|}{2}\right), \text{ hence for } t - t_1 > \frac{\tau - t_1}{2} \text{ it is}$$

$$S_{\xi_0t_1,p+\delta}(t) \supset S_{\bar{\xi}t_1,p}(t)\,.$$

For the monotonicity with respect to p

$$S_{\xi_0t_1,p+\delta}(t) \supset R_{\bar{\xi}t_1}(t)\,.$$

Hence for any given p, δ it holds for $t = \tau$

$$S_{\xi_0t_1,p+\delta}(\tau) \cap \Big(\bigcup_{\xi \in I_2} R_{\xi t_1}(\tau)\Big) \neq \varnothing$$

and consequently

$$\partial R_{\xi_0 t_1}(\tau) \cap \left(\bigcup_{\xi \in I_2} R_{\xi t_1}(\tau)\right) \neq \varnothing .$$

Thus we are lead to a contradiction, since E_1 and E_2 are not disjoint connected components. We remark now that $R_{xt}(\tau)$ is a closed set. This comes obviously from (1.16). It will then be enough to prove that

$$R^+_{xt}(\tau) = R^+_{\bar{\xi} t_1}(\tau) , \bar{\xi} = R^+_{xt}(t_1)$$

and the corresponding relation for R^-. Since $R^+_{xt}(t_1) = \lim_{p \to 0} S^+_{xt,p}(t_1)$ monotonically, for any $\varepsilon > 0, p < p_0$ it holds $R^+_{xt}(t_1) < S^+_{xt,p}(t_1) < R^+_{xt}(t_1) + \varepsilon$. For any fixed $\delta > 0$, we set $\varepsilon = \gamma_K\left(\frac{\delta}{2}, \frac{|\tau - t_0|}{2}\right)$, hence by C3, for $|\bar{\tau} - t_1| > \frac{|\tau - t_1|}{2}$, and in particular for τ, letting $\xi' = S^+_{xt,p}(t_1)$, using M4, it holds

$$S^+_{\bar{\xi} t_1, p+\delta}(\tau) \geq S^+_{\xi' t_1, p+\frac{\delta}{2}}(\tau) \geq S^+_{xt,p}(\tau) \geq R^+_{xt}(\tau)$$

Since it holds for all p and δ it follows

$$R^+_{\bar{\xi} t_1}(\tau) \geq R^+_{xt}(\tau) .$$

To prove the opposite inclusion we remark that for any p

$$S^+_{xt,p}(\tau) \geq S^+_{\xi' t_1, p/2}(\tau) \geq S^+_{\bar{\xi} t_1, p/2}(\tau) \geq R^+_{\bar{\xi}, t_1}(\tau)$$

and for the independence from p

$$R^+_{xt}(\tau) \geq R^+_{\bar{\xi} t_1}(\tau) .$$

The relation for R^- is proved in the same way. *Q.E.D.*

DEFINITION 1.3 (*Equivalence*). *Two Peano processes are said to be equivalent if for $x \in \mathbb{R}, t \in [a, b], p \in]0, 1]$, it holds for any $\varepsilon > 0$*

$$S^{(1)}_{xt,p-\varepsilon} \subseteq S^{(2)}_{xt,p} \subseteq S^{(1)}_{xt,p+\varepsilon} \quad (1.21)$$

M3 ensures that (1.21) is reflexive. From the arbitrariness of ε and p symmetry and transitivity follow.

THEOREM 1.2 *Two equivalent Peano processes have the same solutions.*

Proof. This property comes trivially from the definitions.

We give now the definition of convergence for a sequence of equivalence classes :

DEFINITION 1.4. *A sequence* $\mathcal{C}^{(n)}$ G-*converges to the equivalence class* $\mathcal{C}$ *if for any choice of a sequence of Peano processes* $\mathcal{S}^{(n)} \in \mathcal{C}^{(n)}$ *and* $\mathcal{S} \in \mathcal{C}$ *, for any* $\varepsilon > 0$ *, for any* $\mu > 0$ *, for any* $x \in \mathbb{R}$, $t \in [a, b]$, $p \in]0, 1]$ *there exists an index* n_0 *such that for* $n > n_0$ *it holds*

$$S^{+}_{kt,p-\mu} - \varepsilon \leq S^{(n)+}_{xt,p} \leq S^{+}_{xt,p+\mu} + \varepsilon \tag{1.22}$$

$$S^{-}_{xt,p-\mu} + \varepsilon \geq S^{(n)-}_{xt,p} \geq S^{-}_{xt,p+\mu} - \varepsilon \tag{1.23}$$

LEMMA 1.3. *Let* $\mathcal{S}^{(n)}$, $\mathcal{S}$ *satisfy uniformly* $R1$. *Then* (1.22) *and* (1.23) *are equivalent to the condition*

$$S_{xt,p-\mu} \subseteq \min\lim_{n} S^{(n)}_{xt,p} \subseteq \max\lim_{n} S^{(n)}_{xt,p} \subseteq S_{xt,p+\mu} \tag{1.24}$$

and this condition is equivalent to the following : *if a subsequence* $S^{(kn)}_{xt,p}$ *converges to a set* $T_{xt,p}$ *that is*

$$\lim_{n\to\infty} \|S^{(kn)}_{xt,p} \,\Delta\, T_{xt,p}\| = 0 \tag{1.25}$$

then

$$S_{xt,p-\mu} \subseteq T_{xt,p} \subseteq S_{xt,p+\mu}\,. \tag{1.26}$$

In order to prove lemma 1.3 we need the following :

LEMMA 1.4. *Suppose a sequence of sets* S^n *satisfy* $R1$ *uniformly. Then it is possible to find a subsequence converging (in the sense of* (1.25)) *to a set* T *, which still satisfies* $R1$ *with the same function* $\delta_K(\varepsilon)$.

Proof. It is enough to prove that there exist a subsequence such that $S^{(k_n)\pm}$ converges uniformly. This is possible using Ascoli-

Arzellà theorem. The limiting functions $T^{\pm}$ are still absolutely continuous with the same modulus since they are uniform limit of equiabsolutely continuous functions. *Q.E.D.*

Proof of lemma 1.3.

We prove first that (1.24) implies (1.26). Let us suppose that (1.25) holds ; then for any n

$$T_{xt,p} \subset \overline{\{\bigcup_{i=n}^{\infty} S_{xt,p}^{(k_i)}\}} \subset \overline{\{\max \lim_n S_{xt,p}^{(n)}\}} \subset \overline{S_{xt,p+\mu}} \subset S_{xt,p+2\mu}$$

by (1.16). (1.25) implies also that for any n

$$T_{xt,p} \supset \{\bigcap_{i=n}^{\infty} S_{xt,p}^{(k_i)}\}^{0} \cup \{(x\,,\,t)\} \supset \{\min \lim_n S_{xt,p}^{(n)}\}^{0} \cup \{(x,t)\} \supset$$

$$\supset \overset{0}{S}_{xt,p-\mu} \cup \{(x\,,\,t)\} \supset S_{xt,p-2\mu}\,.$$

Conversely let $x \in \max \lim_n S_{xt,p}^{(n)}$; then there exist a subsequence $S_{xt,p}^{(k_n)} \ni x$. From this subsequence, by lemma 1.4 we can chose a new subsequence, converging to a limit $T_{xt,p}$, that can be chosen in such a way that it contains x. Hence $\max \lim_n S_{xt,p}^{(n)} \subset S_{xt,p+\mu}$.

Let now $x \in S_{xt,p-\mu}$, $x \notin \min \lim_n S_{xt,p}^{(n)}$; then there is a subsequence $S_{xt,p}^{(k_n)} \not\ni x$. Passing to a converging subsequence according to lemma 1.4, and choosing the limit $T_{xt,p}$ such that $x \notin T_{xt,p}$ we are lead to an absurd, since $T_{xt,p} \supset S_{xt,p-\mu} \ni x$.

We prove now that (1.22), (1.23) are equivalent to (1.26). (1.22) and (1.23) imply (1.26) by (1.16). The converse follows supposing (1.22) or (1.23) are false and using lemma 1.4, what leads to an absurd.

THEOREM 1.5. *Definition* 1.4 *does not depend on the choice of the processes* $\mathcal{S}^{(n)}$ *representing the sequence* $\mathcal{C}^{(n)}$ *and* $\mathcal{S}$ *for* $\mathcal{C}$.

Proof. Let $\mathcal{S}'^{(n)}$, $\mathcal{S}'$ be other representants of the same equivalence classes ; for any $\eta > 0$, it holds by (1.22) and (1.21).

$$S'^{+(n)}_{xt,p} \subset S^{+(n)}_{xt,p+\eta} \subset S^{+}_{xt,p+2\eta} + \varepsilon \subset S'^{+}_{xt,p+3\eta} + \varepsilon$$

All other relations are proved in the same way. *Q.E.D.*

2. – Connection between Peano processes and equations.

We present here a class of equations to which Peano processes can be associated. We make some assumptions in order to be sure that solutions are defined on the whole interval $[a, b]$. Actually these assumptions can be weakened by a truncation method of which we give a hint at the end of the section.

We consider the equation

$$x' = f(t, x)$$

where f is a function defined in $[a, b] \times \mathbb{R}$ satisfying the following requirements:

i) For each H there exists $\delta_H^*(\mu)$ continuous with $\delta_H^*(0) = 0$ such that if E is a measurable subset of $[a, b]$ of Lebesgue measure $|E|$, then for $|x| < H$

$$\int_E |f(t, x)| \, dt \leq \delta_H^*(|E|) \tag{2.1}$$

ii) There exists a continuous strictly increasing, function $\psi_H^*(d)$, with $\psi_H^*(0) = 0$, $\psi_H^{*\prime}(d)$ continuous in $]0, +\infty[$ such that whenever $|x_1|, |x_2| < H$ then for any $t \in [a, b]$

$$|f(t, x_1) - f(t, x_2)| \leq \psi_H^*(|x_1 - x_2|) . \tag{2.2}$$

We shall make the following global assumption, in order to be sure that all the solutions we require to build a Peano process are defined in the whole $[a, b]$:

iii) There exists M such that for $H > M$

$$\delta_H^*(|b - a|) \leq C^2 H \tag{2.3}$$

$$\psi_H^*(H) \leq \frac{C'^2 \cdot H}{b - a} - 1 \tag{2.4}$$

where $C^2 + C'^2 \leq 1/2$.

We prove that under this assumptions the solutions of the equations

$$x' = f(t, x) \pm 1$$

esist on the whole $[a, b]$. Let $|x_0| < H|2$; we get then

$$(2.5) \qquad x(t) = x_0 + \int_a^t [f(\tau, x(\tau)) \pm 1]\, d\tau .$$

Suppose now that at a certain t $|x(t) - x_0| = H/2$, it follows

$$H/2 = |x(t) - x_0| \leq \int_a^t |f(\tau, x_0)|\, d\tau + \int_a^t \psi_H^*(|x(\tau) - x_0|)\, d\tau + |t - a| \leq$$

$$\leq C^2 H + \frac{C'^2 H}{(b-a)} |t - a| \leq H/2 ,$$

the last equality holding only if $t = b$; we have thus proved that the first value of t for which the solution can be larger than H is b.

We shall require a lemma concerning some simple continuity properties of a special equation :

LEMMA 2.1.

Let $g(x) = \psi_H^*(x)$ *satisfying the requirements of* ii) *and* (2.5). *For any* $\delta \geq 0$, $\eta \geq 0$ *we call* $\bar{x}(\delta, \eta, t)$ *the greatest solution of the equation.*

$$(2.6) \qquad \begin{cases} x' = g(x) + \delta \\ x(0) = \eta \end{cases} \qquad 0 \leq t \leq b - a .$$

Then $\bar{x}(\delta, \eta, t)$ *is a continuous function in the set of its variables.*

Proof. If Peano phaenomenon does not appear, that is if there exist a unique solution for any given value of δ, η, continuity is a classical result. Now we remark that under our hypothesis Peano phaenomenon can appear only for $\delta = 0$, $\eta = 0$, and is equivalent to the condition

$$\int_0^a f^{-1}(x)\, dx < +\infty \tag{2.7}$$

Let now be $|x_0| < H/2$, $H/2 > M$, so that the solutions of (2.6) are defined in the whole inteval $[0, b\text{-}a]$ and are less than H. In this case, for $\eta_1 < \eta_2$ we get

$$\int_{\eta_1}^{\bar{x}} (f(x) + \delta)^{-1}\, dx - \int_{\eta_2}^{\bar{x}} (f(x) + \delta)^{-1}\, dx =$$

$$= \int_{\eta_1}^{\eta_2} (f(x) + \delta)^{-1}\, dx < \int_{\eta_1}^{\eta_2} f(x)^{-1}\, dx .$$

Since (2.7) holds, then for any $\varepsilon > 0$ there exists d such that whenever $|\eta_2 - \eta_1| < d$, then

$$\int_{\eta_1}^{\eta_2} f(x)^{-1}\, dx < \varepsilon .$$

Hence for any $\delta \geq 0$, provided $|\eta_2 - \eta_1| < d$,

$$|\bar{x}(\delta, \eta_2, t) - \bar{x}(\delta, \eta_1, t)| \leq \varepsilon \max_{|x|<H} |f(x)| .$$

The dependence on t is uniformly continuous with respect to the other variables by (2.6), so that in order to prove global continuity we only need to prove continuity with respect to δ.

The mapping $\delta \rightsquigarrow \bar{x}(\delta, \eta, t)$ is monotone increasing. If $\eta > 0$, $\{\bar{x}(\delta, \eta, \mathrm{t}) ; 0 \leq \delta \leq 1\}$ is connected by the usual theorems on continuous dependence on the parameters. In order to prove the continuity we thus need only to prove that

$$\{\bar{x}(\delta, 0, t) ; 0 \leq \delta \leq 1\}$$

is connected. This is immediate since it is enough to consider the backward problem from time t, using the existence and uniqueness of the backward solution. *Q.E.D.*

We come now to the actual construction of the Peano process.

We shall define $S_{x_0 t_0, p_0}$ to be the reachable set in $[a, b] \times \mathbf{R}$ for the problem

$$\begin{cases} x' = f(t, x) + g(x) \\ x(t_0) = x_0 \end{cases}$$

under the condition $\|g(t)\|_\infty < p_0$.

Obviously we get that $S^+_{x_0 t_0 p_0}(t)$ for $t > t_0$ is the greatest solution of the problem

$$(2.8)_1 \qquad \begin{cases} S^{+\prime} = f(t, S^+) + p_0 \\ S^+(t_0) = x_0 \end{cases}$$

and for $t < t_0$ is the greatest solution of the problem

$$(2.8)_2 \qquad \begin{cases} S^{+\prime} = f(t, S^+) - p_0 \\ S^+(t_0) = x_0 \end{cases}$$

Analogous relations hold for S^-.

Now we must prove that this is actually a Peano process, that is, it satisfies all the required properties.

M1 is trivial; M2 comes from the elementary theory, and the set is not empty because of C2, that will be proved later. M3 depends on the monotonicity of solutions with respect to the parameter p. M4 holds in a stronger form since we have chosen maximal (resp. minimal) solutions of (2.8).

We prove now R1. Let $|x_0| < H$; we have already proved that by iii) that $|S^\pm| < 2H$. Suppose N disjoint intervals $[t_i', t_i'']$ are given, such that $\sum_{i=1}^{N} |t_i'' - t_i'| < \delta$. Then we have

$$(2.9) \quad \sum_{i=1}^{N} |S^\pm(t_i'') - S^\pm(t_i')| = \sum_{i=1}^{N} \left| \int_{t_i'}^{t_i''} f(\tau, S^\pm(\tau))\, d\tau \right| \le$$

$$\le \sum_{i=1}^{N} \int_{t_i'}^{t_i''} f(\tau, x_0)\, d\tau + \delta\, \psi^*_{2H}(2H) \le \delta^*_H(\delta) + \delta\, [\psi^*_{2H}(2H)].$$

The function

$$\varepsilon(\delta) = \delta_H^*(\delta) + \delta\, \psi_{2H}^*(2H) \tag{2.10}$$

is monotone increasing, continuous, $\varepsilon(0) = 0$. Its inverse is the function required in R1 whenever $K \subset [-H, H]$.

We prove now C1. Let $|x_1|, |x_2| < H$, $|x_1 - x_2| < d$, $p_1 \leq p_2 < p + \delta$ We get

$$S_2^+ - S_1^+ = x_2 - x_1 + \int_{t_0}^{t} [f(\tau, S_2^+) - f(\tau, S_1^+)] + (p_2 - p_1)]\, d\tau,$$

$$|S_2^+ - S_1^+| \leq |x_2 - x_1| + \int_{t_0}^{t} [\psi_{2H}^*(S_2^+ - S_1^+) + (p_2 - p_1)]\, d\tau.$$

Let $\bar{x}(\delta, d, t)$ the greatest solution of the problem

$$\begin{cases} x' = \psi_{2H}^*(\bar{x}) + \delta \\ \bar{x}(\delta, d, 0) = 0. \end{cases} \tag{2.11}$$

By LEMMA 2.1 we get that $\bar{x}$ is continuous. We must prove (1.12). We get

$$\frac{\bar{x}(\delta, d, t) - d}{\tau} - \delta = \frac{\bar{x}(\delta, d, t) - \bar{x}(\delta, d, 0)}{\tau} - \delta =$$

$$= \bar{x}'\big(\delta, d, \tau^*(\delta, d)\big) - \delta = \psi_{2H}^*\big(\bar{x}(\delta, d, \tau^*(\delta, d))\big)$$

where $0 \leq \tau^*(\delta, d) \leq \tau$. By the continuity of $\bar{x}$ and of ψ_{2H}^* we get then (1.12). The function $\varphi_K(\delta, d, \tau) = \bar{x}(\delta, d, \tau)$ for any $K \subset [-H, H]$ is the one required in C1 (Remark: for S^- we go on exactly in the same way).

We prove now the first part of C2. For $|x_0| < H$ we have

$$|S_{p+\delta}^+ - S_p^+| \leq \int_{t_0}^{t} \big[\psi_{2H}^*(S_{p+\delta}^+ - S_p^+) + \delta\big]\, d\tau. \tag{2.12}$$

Let then $\bar{x}(\tau, \delta)$ be the greatest solution of the problem

$$\begin{cases} \bar{x}' = \psi_{2H}^{*}(\bar{x}) + \delta \\ \bar{x}(0, \delta) = 0 \end{cases}$$

$\bar{x}$ is continuous by lemma 2.1. The function $\zeta_K(\tau, \delta) = \bar{x}(\tau, \delta) - \delta|\tau|$ is the function required in (1.15) for $K \subset [-H, H]$.

In order to prove (1.16) we are lead from (2.12) to the inequality $|S_{p+\delta}^{+} - S_{p}^{+}| \geq \int_{t_0}^{t} [\delta - \psi_{2H}^{*}(S_{p+\delta}^{+} - S_{p}^{+})] \, d\tau$, hence, calling $\bar{x}(\tau, \delta)$ the solution of the problem

$$\begin{cases} \bar{x}' = \delta - \psi_{2H}^{*}(\bar{x}) \\ \bar{x}(0, \delta) = 0 \end{cases}$$

this solution is strictly positive for $\tau > 0$. Continuity in this case is obvious since there is always a unique solution. The required function is $\vartheta_K(\tau, \delta) = \delta\tau - \bar{x}(\tau, \delta) < \delta\tau$ for $\tau > 0, \delta > 0$.

We prove C3. Let $|x_1|, |x_2| < H$, let $x_1 < x_2$ (otherwise there is nothing to prove). We get, passing through the integral equation

$$S_{x_1 t_0, p_1}^{+}(t) - S_{x_2 t_0, p_2}^{+}(t) = S_1^{+} - S_2^{+} =$$

$$= x_1 - x_2 + \int_{t_0}^{t} [f(\tau, S_1^{+}) - f(\tau, S_2^{+}) + p_1 - p_2] \, d\tau$$

hence

$$S_1^{+} - S_2^{+} \geq x_1 - x_2 + \int_{\tau_0}^{t} [(p_1 - p_2) - \psi_{2H}^{*}(|S_1 - S_2|)] \, d\tau$$

Let $\bar{x}$ be the solution of the equation

$$\begin{cases} \bar{x}' = p_1 - p_2 - \psi_{2H}^{*}(|\bar{x}|) \\ \bar{x}(0) = x_1 - x_2 < 0 . \end{cases}$$

We set $d = p_1 - p_2 > 0$, and we fix $\tau_0 > 0$. Let $x_1 - x_2 = \bar{x}(0)$ be such that

$$\psi^*_{2H}(\bar{x}(0)) \leq \frac{1}{2}\left(d - \frac{\bar{x}(0)}{\tau_0}\right) \tag{2.13}$$

Then
$$\bar{x}' > \frac{|x_2 - x_1|}{\tau_0} \text{ for } \tau < \tau_0$$

so that $\bar{x} > 0$ for $\tau = \tau_0$.

The function $\gamma_K(d, \dot{\tau}_0)$ is given by the bound on $\bar{x}(0)$ that appears in (2.13). It satisfies the required conditions for $|t - t_0| > \tau_0$ in virtue of $M4$. For S^- the proof is similar.

We prove now $E1$. We have

$$S^+ - S^- = \int_{t_0}^{t} \left\{[f(\tau, S^+) - f(\tau, S^-)] + 2p\right\} d\tau \leq$$

$$\leq \int_{t_0}^{t} [\psi^*_{2H}(S^+ - S^-) + 2p]\, d\tau.$$

Let $\bar{x}$ be the greatest solution of

$$\begin{cases} x' = \psi^*_{2H}(\bar{x}) + 2p \\ \bar{x}(0, p) = 0. \end{cases}$$

We get $\bar{x}'(0, p) = 2p$. Let us consider the function $y(t) = \bar{x}(t, 1) - 2t$.

For this function we have $y(0) = y'(0) = 0$; we remark also that it is $y = \sup\limits_{p \leqslant 1} \bar{x}(t, p) - 2pt$, since ψ^*_{2H} is increasing. Hence $\varepsilon_K(t) = y(t)$ is the function required in E1.

Now that we have shown how to build a Peano process starting from an equation, we deal with the reverse problem, namely to reconstruct the coefficients of an equation from a Peano process and check that its solutions are actually the reachable sets for the solutions of that differential equation.

In order to reconstruct the coefficients we put

$$f(t_0, x_0) = \frac{d}{dt}\,[S^+_{x_1 t_1, p_1}](t_0) - p_1 = \frac{d}{dt}\,[S^-_{x_2 t_2, p_2}](t_0) + p_2$$

under the condition that $S^{\pm}_{x_i t_i, p_i}(t_0) = x_0$. Since the function $S^{\pm}$ are absolutely continuous the coefficients are thus constructed almost everywhere. The definition does not depend on the particular element of the Peano process passing through the point t_0, x_0, in virtue of M4, does not depend on the particular p in virtue of C2 and does not depend on the choice of S^+ or S^- because of E1. Furthermore from C1 it follows

$$(2.13)\qquad \begin{aligned} &|f(t_0, x_2) - f(t_0, x_1)| = \\ &= \lim_{t\to t_0} \frac{|S_2^+(t) - S_2^+(t_0) - S_1^+(t) + S_1^+(t_0)|}{|t - t_0|} = \\ &= \lim_{t\to t_0} \frac{|S_2^+(t) - S_1^+(t) - [S_2^+(t_0) - S_1^+(t_0)]|}{|t - t_0|} \le \\ &\le \lim_{\tau\to 0} \frac{\varphi_K(\tau, d, 0) - d}{\tau} = \tilde{\varphi}_K(d) \end{aligned}$$

this last function $\tilde{\varphi}_K(d)$ is continuous by (1.12) and $\tilde{\varphi}_K(0) = 0$.

That is, letting $\psi^*_H(d) = \tilde{\varphi}_K(d)$ for $K =]-H, H[$, (2.2) is satisfied.

Now let $|x_0| < H$; we get

$$\left|\int_{t'}^{t''} f(\tau, x_0)\,d\tau\right| = \left|\int_{t'}^{t''} f(\tau, S^+(\tau))\,d\tau + \int_{t'}^{t''} [f(\tau, x_0) - f(\tau, S^+(\tau))]\,d\tau\right| \le$$

$$\le |S^+(t'') - S^+(t')| + |t'' - t'| \cdot \psi^*_{2H}(2H),$$

from this relation and from R1 we get the uniform summability stated in (2.1).

By M4, E1, C2 it holds that the solutions of the Peano process are just solutions (as reachable sets) for the equation we have derived from the Peano process. We must still prove that the elements of the Peano process are exactly the maximum reachable set from any given initial data. Suppose that a solution $\bar{x}$ of the equation is at some value greater than $S^+_{xt,p}$. Since these last function is derivable almost everywhere with derivative $f(x, t) + p$, for any $p > 0$ $S^+_{xt,p}(\tau)$ for $\tau > t$ is greater than $\bar{x}$. Hence $R_{xt}(\tau) \supset \bar{x}(\tau)$, thus contraddicting the assumption. The same argument holds for S^- and for $\tau < t$.

3. - Abstract compactness theorem.

THEOREM 3.1 *Let* $\mathcal{C}^{(k)}$ *be a sequence of equivalense classes of Peano processes. Let* $\mathcal{C}^{(k)}$ *be equibounded, that is* R1, C1, C2, C3, E1 *hold with the same functions for any* k. *Then it is possible to find a convergent subsequènce. The limiting equivalence class* $\mathcal{C}$ *satisfes* R1, C1, C2, C3, E1 *with the same functions.*

Proof. We fix arbitrarily a sequence of Peano processe $\mathfrak{S}^{(k)} \in \mathcal{C}^{(k)}$. We fix three countable sets : $\{t_i\}$ dense in $[a, b]$, $\{x_i\}$ dense in $\mathbb{R}$, p_1 dense in $]0, 1]$. We order the indexes in a sequence $\{\mathfrak{I}_s\} = (t_i, x_j, p_1)$. We shall call $\mathfrak{S}_s$ the element associated to the *s-th* index.

Consider first the sequence $S_1^{(k)}$. In virtue of lemma 1.4 it is possible, to find a converging subsequence, let S_1 be its limit. By the diagonal procedure we can thus find a subsequence converging for any index s to a limiting element S_s. For sake of simplicity we still call $\mathcal{C}^k$ the subsequence. If x, t, p does not belong to the system $\mathfrak{I}_s$ we choose arbitrarily a converging subsequence and we let $S_{xt,p}$ be its limit. So we have defined a family of sets $\mathfrak{S}$. We have to prove that this is actually a Peano process, and we must prove that $\mathfrak{S}^k$ G-converges to $\mathfrak{S}$. We prove first the last statement. We use condition (1.26). Let $T_{xt,p}$ be the limit of a subsequence $S^{(s_h)}_{xt,p}$ and $T_{xt,p+\varepsilon}$ be the limit of another subsequence $S^{(r_h)}_{xt,p+\varepsilon}$. It is enough to prove that

$$T_{xt,p} \subset T_{xt,p+\varepsilon} .$$

We shall actually prove that

$$T^+_{xt,p} \leq T^+_{xt,p+\varepsilon} \tag{3.1}$$

$$T^-_{xt,p} \geq T^-_{xt,p+\varepsilon} . \tag{3.2}$$

Let x be such that $S_{xt,p+\varepsilon}(\tau) \subset K$. For any fixed τ_0, let $\bar{t}$, $\bar{x}$, $\bar{p}$ be a point of the convergence system such that $p + \varepsilon/2 < \bar{p} < p$, $|\bar{x} - \bar{x}| < 1/2\, \gamma_K\left(\frac{1}{4}\varepsilon, \tau_0\right)$, $t > \bar{t} > t - \delta_K\left(\frac{1}{2}\gamma_K\left(\frac{1}{4}\varepsilon, \tau_0\right)\right)$. We get by R1, C3, M4 that for $p + \varepsilon/2 < p^* < \bar{p}$, with p^* belonging to the system, it holds for $\tau > t + \tau_0$

$$S^{+(sk)}_{\bar{x}\bar{t},p^*}(\tau) \leq S^{+(sk)}_{xt,p+\varepsilon}$$

$$S^{+(rk)}_{\bar{x}\bar{t},p^*}(\tau) \geq S^{+(rk)}_{xt,p}\,.$$

Since $S^{+(sk)}_{xt,p^*}$, $S^{+(rk)}_{\bar{x}\bar{t},p^*}$ converge to the same limit $S^{+}_{\bar{x}\bar{t},p^*}$ it follows for $\tau > t + \tau_0$

$$T^{+}_{xt,p}(\tau) < S^{+}_{\bar{x}\bar{t},p^*}(\tau) < T^{+}_{xt,p+\varepsilon}(\tau)\,.$$

For the arbitrariness of τ we have thus proved (3.1) for $\tau < t$. The other parts of the proof are the same.

Now we prove that the limiting family satisfies all the properties required by a Peano process. Properties M1, M2, R1, E1 are concerned with single sets and are preserved by the uniform convergence. For the other properties we prove first that they still hold when we restrict ourselves to the family of sets S_s; this is quite obvious, for properties M3, C1, C2 and C3 are preserved by the uniform convergence (that is the case when dealing with S_s's).

M4 is not required for the present and will be proved later.

We shall use the following

LEMMA 3. 2. *Let S^k be a sequence of Peano processes converging on a dense system of indexes; then for any fixed $x_0\, t_0\, p_0$, for any $\tau > 0$, $d > 0$, there exist $x_s t_s p_s$ in the convergence system with $p_0 < p_s < < p_0 + d$, such that for $|t - t_0| > \tau$ it holds*

$$S^{+(k)}_{s}(t) > S^{+}_{x_0 t_0 p_0}(t) \tag{3.3}$$

$$S^{-(k)}_{s}(t) < S^{-}_{x_0 t_0 p_0}(t) \tag{3.4}$$

and there exist $x_s\, t_s\, p_s$ such that $p_0 - d < p_s < p_0$ and for $|t - t_0| > \tau$ it holds

(3.3)′ $$S_s^{+k}(t) < S^+_{x_0t_0p_0}(t)$$

(3.4)′ $$S_s^{-k}(t) > S^-_{x_0t_0p_0}(t)\,.$$

Proof. We prove only (3.3), the other relations being similar. Let $x \in K$, let $a = \gamma_K\left(\frac{d}{2}, \frac{\tau}{2}\right)$; let $l = \delta_K\left(\frac{1}{2}\gamma_K\left(\frac{d}{2}, \frac{\tau}{2}\right)\right)$. We choose t_s so that $|t_s - t_0| < \min\left(l, \frac{\tau}{2}\right)$ and x_s so that $|x_s - x_0| < \frac{1}{2}\gamma_K\left(\frac{d}{2}, \frac{\tau}{2}\right)$.

Let $p_0 + \frac{3}{4}d < p_s < p_0 + d$. Then at $t^* = t_s$ if $t_s > t_0$, $t^* = t_0$ if $t_s < t_0$

$$|S_s^{+(k)}(t^*) - S^{+(k)}_{x_0t_0p_0}(t^*)| < \gamma_K\left(\frac{d}{2}, \frac{\tau}{2}\right)$$

and, calling y_s, y_0 the two values of the S^+ at t^*, we get by C3 for $\varepsilon < \frac{1}{8}d$

$$S^{+(k)}_{y_s,t^*,p_s-\varepsilon}(t) > S^{+k}_{y_0t^*,p_s+\varepsilon}(t) \text{ for } |t - t^*| > \frac{\tau}{2}$$

By M4 then it follows

$$S_s^{+(k)}(t) > S^{+(k)}_{x_0t_0p_0} \text{ for } |t - t_0| > \tau \qquad Q.E.D.$$

Now we prove that M3, C1, C2, C3 hold in general.
M3. We must prove that if $p_1 = p_2 + d$, $d > 0$ it follows

(3.5) $$S^+_{x_0t_0,p_1}(t) \supseteq S^+_{x_0t_0p_2}(t)\,.$$

We use lemma 3.2 choosing $x_s\, t_s\, p_s$ in such a way that it holds together for $|t - t_0| > \tau$

$$S^{+(k)}_{x_0 t_0, p_2}(t) \leq S^{+(k)}_s(t) \leq S^{+(k)}_{x_0 t_0, p_1}(t)$$

$$S^{-(k)}_{x_0 t_0, p_2} \geq S^{-(k)}_s(t) \geq S^{-(k)}_{x_0 t_0, p_1}(t)\,.$$

Hence taking the limit we get

$$\mathrm{S}_{x_0 t_0, p_2}(t) \subseteq \max_k \lim S^{k}_{x_0 t_0, p_2}(t) \subseteq S_s(t) \subseteq \min_k \lim S^{(k)}_{x_0 t_0, p_1}(t) \subseteq \\ \subseteq S_{x_0 t_0, p_1}(t)$$

For the arbitrariness of τ it follows (3.5)

C1. It is enough to prove that when $p_2 \leq p_1$, $|x_1 - x_2| < d$ $|p_2 - p_1| < \delta$

$$\|S_{x_1 t_0 p_1}(\tau)\, \Delta\, S_{x_2 t_0 p_2}(\tau)\| \leq \varphi_k\big(|t_0 - \tau|\,,\, d\,,\, \delta\big).$$

Using lemma 3.2 for $\tau > 0$ $d = \varepsilon > 0$, choosing x_{s_1}, x_{s_2} so that $|x_{s_1} - x_{s_2}| < d$ $t_{s_1} = t_{s_2} = t_s$, $|t_s - t_0| < \tau$, $p_1 < p_{s_1} < p_1 + \varepsilon$, $p_2 > p_{s_2} > p_2 - \varepsilon$ we get for $|t - t_0| > \tau$

$$S^{(k)}_{x_2 t_0 p_2}(t) \supset S^{(k)}_{s_2}(t) \qquad S^{(k)}_{s_1}(t) \supset S^{(k)}_{x_1 t_0 p_1}(t) \tag{3.6}$$

Furthermore

$$\|S^{(k)}_{s_2}(t)\, \Delta\, S^{(k)}_{s_1}(t)\| \leq \varphi_K\big(|t - t_s|, d, \delta + 2\,\varepsilon\big) \tag{3.7}$$

and since both sequences converge (3.7) holds also for the limits, so that

$$\|S_{s_2}(t)\, \Delta\, S_{s_1}(t)\| \leq \varphi_K\big(|t - t_s|, d, \delta + 2\,\varepsilon\big). \tag{3.8}$$

By the inclusion relations (3.6) it follows then for $|t - t_0| > \tau$

$$\|S_{x_2 t_0 p_2}(t)\, \Delta\, S_{x_1 t_0 p_1}(t)\| \leq \|\max_k \lim S^{(k)}_{x_2 t_0 p_2}(t)\, \Delta \min_k \lim S^{(k)}_{x_1 t_0 p_1}(t)\| \leq \\ \leq \|S_{s_2}(t)\, \Delta\, S_{s_1}(t)\| \leq \varphi_K\big(|t - t_s|, d, \delta + 2\,\varepsilon\big)\,.$$

In virtue of the continuity of φ_K and the independence from τ , ε it follows C1 with the same function.

In order to prove C2 we still use lemma 3.2 with $\tau > 0$ $d = \varepsilon > 0$ and we go on in the same way using the continuity of $\zeta_K(\tau, d)$ and of $\vartheta_K(\tau, d)$. In order to prove C3 we use an argument similar to that used to prove C1.

We are now lead to prove M4. First we remark that by C3 and M2 for any $\tau > t_0 > t$ (resp. $\tau < t_0 < t$) the set

$$R^{(k)}_{\pm\varepsilon}(\tau) = \{\bigcup_{\xi} S^{(k)}_{\xi t_0, p\pm\varepsilon}(\tau) \; ; \; \xi \in S^{(k)}_{xt,p}(t_0)\}$$

is connected, that is, is an interval. Then it is enough to prove that for $\tau > t_0$ (resp. $\tau < t_0$) it is

$$R^{+}_{\varepsilon}(\tau) > S^{+}_{xt,p}(\tau) \quad , \quad R^{+}_{-\varepsilon}(\tau) < S^{+}_{xt,p}(\tau) \tag{3.9}$$

$$R^{-}_{\varepsilon}(\tau) < S^{-}_{xt,p}(\tau) \quad R^{-}_{-\varepsilon}(\tau) > S^{-}_{xt,p}(\tau) \, . \tag{3.10}$$

Let $S^{(r_k)}_{xt,p}$ be a subsequence converging to $S_{xt,p}$. For any fixed $\tau_1 > 0$ let $\delta = \dfrac{1}{2}\gamma_K\left(\dfrac{\varepsilon}{3}, \tau_1\right)$. Then there esists $(r_k)_0$ such that for $r_k > (r_k)_0$ it holds $|S^{+(r_k)}_{xt,p}(t_0) - S^{+}_{xt,p}(t)| < \delta$. Hence by C3, for $\tau > t_0 + \tau_1$, calling $\bar{x} = S^{+}_{xt,p}(t_0) - \delta$;

$$S^{+(r_k)}_{\bar{x},t_0,p+\frac{2}{3}\varepsilon}(\tau) > S^{+(r_k)}_{\bar{x}+2\delta,t_0,p+\frac{\varepsilon}{3}}(\tau) > S^{+(r_k)}_{xt,p}(\tau) \, . \tag{3.11}$$

Taking the limit we get

$$\max_k \lim S^{+(k)}_{x,t,p+\frac{2}{3}\varepsilon}(\tau) > S^{+}_{xt,p}(\tau)$$

hence by the definition of G-convergence

$$S^{+}_{x,t,p+\varepsilon}(\tau) > S^{+}_{xt,p}(\tau)$$

and still by monotonicity

$$\mathrm{R}^{+}_{\varepsilon}(\tau) = \sup_{\bar{x} < S^{+}_{xt,p}(t_0)} \left(S^{+}_{\bar{x},t_0,p+\varepsilon}(\tau)\right) > S^{+}_{xt,p}(\tau) \, .$$

Since this last relation does not depend on τ_1 we have proved the first of (3.9) the other required inequalities are proved in the same way.

4. – Compactness theorem for differential equations.

In section 2 we have proved that a Peano process (or better its equivalence class) can represent and can be represented by a differential equation. So it is meaningful to give the following definitions:

DEFINITION 4.1 *We call Peano equation a differential equation that represents a Peano process.*

Remark that in section 2 we have introduced a vary large class of Peano equantions.

DEFINITION 4.2. (*G-convergence*). *A sequence of Peano differential equations G-converges to a Peano equation if and only if the associated equivalence classes of Peano processes converge to the associate equivalence class for the limiting equation.*

THEOREM 4.1. *Let* $f_k(t, x)$ *be a sequence of functions satisfying uniformly* (2.1), (2.2), (2.3), (2.4). *Then it is possible to find a subsequence such that the equations*

$$x_k' = f_k(t, x_k)$$

G-converge to an equation

$$x' = f(t, x)$$

and this last equation still a Peano equation, furthermore ψ_H^* *is the same as for the approwximating functions.*

There is nothing to prove since this theorem is a consequence of the results of sections 2 and 3. In particular the last statement follows from (2.11) for $\delta = 0$ and from (2.13). This last fact is important because it allows to consider special stable subclasses of Peano equations, among which very important is the class of equilipschitzian equations. If (2.3) and 2.4) do not hold our definitions can be easily generalized, taking for each H a function $f^H(t, x)$ equal to $f(t, x)$ for $|x| < H$, and satisfying (2.3) and (2.4) for $K > 2H$. Then G-convergence is given on each of the approximating function.

Manoscritto pervenuto in Redazione il 1° giugno 1977.