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A generalization of Ulm subgroups

KENNETH MESSA *

Abstract: The investigation of U_n -submodules yields a generalization of Ulm subgroups. Through this generalization, we study the α -th Ulm submodule of M , M^α , and the functor $M \mapsto M^\alpha$, in relation to its preserving sums, products and certain quotients. We also discover conditions for which every module is the first Ulm submodule of some module. Finally, the well-known result that the first Ulm subgroup of an algebraically compact abelian group is divisible is extended to semi-hereditary rings.

1. - Introduction.

This paper deals with certain submodules, called U_n -submodules, of arbitrary left R -modules. When $n = 1$, we get a generalization of the concept of Ulm subgroup of an Abelian group. We are chiefly concerned with this generalization.

It is well-known in Abelian Group Theory that the first Ulm subgroup of an algebraically compact group is divisible. See Fuchs [1]. One aim of this paper is to extend this theorem to semi-hereditary rings. We will also show that the map $M \mapsto \alpha$ -th Ulm submodule of M preserves sums, products and certain quotients. A theorem relating purity to the first Ulm submodule is proved. In addition, necessary and sufficient conditions for an arbitrary

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module to be equal to the first Ulm submodule of some module are given.

Throughout this paper R will be an associative ring with identity. All modules considered will be unitary left R -modules. The symbol $\dim R$ will denote the left homological dimension of R , and hereditary and noetherian will mean left hereditary and left noetherian. The letters k, n, m are always taken from the set $\{1, 2, 3, \dots\} \cup \{\infty\}$.

We will let $L_n = \{P \twoheadrightarrow Q \mid \text{where } Q/P \text{ is finitely presented of dimension } \leq n \text{ and } Q \text{ is finitely generated free}\}$.

The author wishes to thank his thesis advisor Laszlo Fuchs for his inspiration and guidance. Parts of this paper are included in the author's doctoral dissertation.

2. - U_n -submodules.

A submodule A of M will be called a U_n -submodule if given a monomorphism $P \twoheadrightarrow Q \in L_n$, every map $f: P \rightarrow A$ can be extended to a map $g: Q \rightarrow M$.

We can interpret U_n -submodules in terms of systems of equations. A system of equations

$$\sum_{j=1}^s r_{ij} x_j = a_i \quad (i \in I)$$

will be called an n -system in A if the a_i 's are in A and if $P \twoheadrightarrow Q \in L_n$ where Q is free with generators x_j and P is generated by the $\sum r_{ij} x_j$ for $i \in I$. Thus a submodule A of M is a U_n -submodule if and only if any finite compatible n -system of equations in A is solvable in M .

Some easy relations between the various U_n -submodules are given below:

A. A submodule of a U_n -submodule of M is again a U_n -submodule of M .

B. Every U_n -submodule is a U_k -submodule for all $k \leq n$.

C. The union of an ascending chain of U_n -submodules is again one. Hence, every U_n -submodule is contained in a maximal one.

D. Every injective submodule is a U_n -submodule for all n .

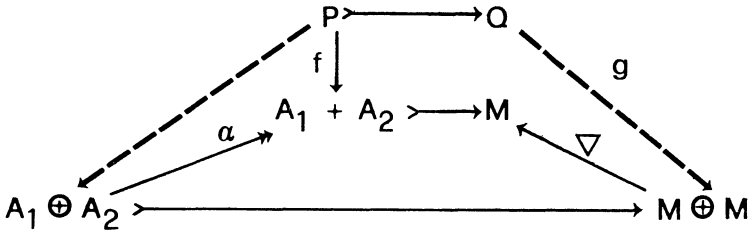
E. If $\dim R = n$ then every U_n -submodule is also a U_k -submodule for all $k \geq n$.

The particular case when R is a P.I.D. reduces the definition of a U_1 -submodule to solutions of $rx = a$ for all $r \in R$ which are not zero divisors. That is, A is a U_1 -submodule of M if $rx = a$ is solvable in M for each a in A and each non-zero divisor r in R . In particular when $R =$ the ring of integers, A is a U_1 -subgroup of an abelian group M if and only if A is contained in the first Ulm subgroup of M . See Fuchs [1].

We can generalize this somewhat in the theorem following lemma 1 :

LEMMA 1 : An arbitrary sum of U_1 -submodules is again one.

Proof : Let $A_i, i \in I$, be U_1 -submodules of M and let $P \twoheadrightarrow Q \in L_1$ with $f : P \twoheadrightarrow A_i$. As P is finitely generated, we may assume that I is finite. It suffices to verify lemma 1 for $|I| = 2$. Consider the following diagram where $\alpha : A_1 \oplus A_2 \rightarrow A_1 + A_2$ is the epimorphism given by $\alpha(a_1, a_2) = a_1 + a_2$ and ∇ is the codiagonal map :



As P is projective, f factors through α . As A_1 and A_2 are U_1 -modules, there exists a map $g : Q \rightarrow M \oplus M$ making the outside rectangle commute. Thus $\nabla \circ g$ makes the inside rectangle commute. Hence, $A_1 + A_2$ is a U_1 -submodule of M .

The proof of theorem 1 is immediate.

THEOREM 1 : Every module contains a unique maximal U_1 -submodule. This maximal U_1 -submodule of M will be called the *first Ulm submodule* of M and will be denoted M^1 .

In abelian groups, the first Ulm subgroup of a group A is the intersection of all subgroups nA where n ranges over all positive

integers. Clearly then, if R is the ring of integers, the first Ulm submodule is exactly the first Ulm subgroup. In fact, for a P.I.D. both concepts coincide.

Just as for abelian groups, we define for each ordinal α , $M^{\alpha+1} = (M^\alpha)^1$ and for each limit ordinal β , set $M^\beta = \bigcap_{\alpha < \beta} M^\alpha$. By this process we get a descending sequence of Ulm submodules of M . We call the sequence

$$M \supset M^1 \supset M^2 \supset \dots \supset M^\alpha \supset \dots$$

the *Ulm sequence* for M . A cardinality argument gives us that $M^\gamma = M^{\gamma+1}$ for some ordinal γ . We will denote this Ulm submodule by $M^\#$.

It is important to note that as an injective module is a U_1 -submodule in any module containing it, any injective submodule of M is contained in $M^\#$.

For a fixed ordinal α , the function $M \mapsto M^\alpha$ is functorial; i. e. if $f: M \rightarrow N$ then $f(M^\alpha) \subset N^\alpha$. Thus M^α is a fully invariant submodule of M . We then have:

THEOREM 2: If $C \subset M^\alpha$ then $(M/C)^\alpha = M^\alpha/C$.

Proof by transfinite induction: the previous discussion shows that M^1/C is a U_1 -submodule of M/C and hence is contained in $(M/C)^1$. For the reverse inclusion, let A/C be a U_1 -submodule of M/C . We must show that $A \subset M^1$. Consider the following diagram for arbitrary $P \twoheadrightarrow Q \in L_1$ and $f: P \rightarrow A$:

$$\begin{array}{ccc}
 P & \xrightarrow{i} & Q \\
 f \downarrow & & \swarrow s \\
 A & \xrightarrow{i} & M \\
 h \downarrow & & \searrow k \\
 A/C & \xrightarrow{\quad} & M/C \\
 & & \downarrow t
 \end{array}$$

where h and k are the canonical maps. By assumption, $h \circ f$ can be extended to $t: Q \rightarrow M/C$. As Q is free, there exists a map $s: Q \rightarrow M$

such that $k \circ s = t$. Now $s \circ i - j \circ f: P \rightarrow \ker k = C$. As $C \subset M^1$, we get an extension $g: Q \rightarrow M$ of $s \circ i - j \circ f$. Then $s + g$ extends f .

It is clear that if β is a limit ordinal or if $\beta = \alpha + 1$ and $M^\alpha/C = (M/C)^\alpha$ for all $\alpha < \beta$, then we have $(M/C)^\beta = M^\beta/C$.

The function $M \rightarrow M^\alpha$, for fixed α , also preserves direct sums and direct products. The proof of lemma 2 is straightforward.

LEMMA 2.
$$(\oplus A_i)^\alpha = \oplus A_i^\alpha,$$

$$(\prod A_i)^\alpha = \prod A_i^\alpha,$$

It also follows immediately that the canonical map $M \rightarrow M/M^\alpha$ preserves all Ulm submodules up to the α -th; i.e. the image of the β -th Ulm submodule is equal to the β -th Ulm submodule of M/M^α , for $\beta \leq \alpha$.

The next theorem relates, to a certain extent, the first Ulm submodule to purity. Recall, A is pure in M if every finite system of equations which is solvable in M is solvable in A . See Warfield [6]. The result generalizes that in abelian groups.

THEOREM 3: A pure in M implies $A^1 = M^1 \cap A$.

Proof: It is clear that $A^1 \subset M^1 \cap A$. For the reverse, we must show that $M^1 \cap A$ is a U_1 -submodule of A . Given any compatible n -system of equations in $M^1 \cap A \subset M^1$, this system has a solution in M . As A is pure in M , the system is solvable in A . Hence $M^1 \cap A$ is a U_1 -submodule of A .

COROLLARY: Let R be hereditary and noetherian. If M^1 is pure in M then M^1 is injective.

Proof: The proof is immediate upon observing that $M^\#$ is injective with respect to all elements of L_1 .

In closing this section, we would like to give necessary and sufficient conditions for an R -module to be the first Ulm submodule of some module.

THEOREM 4: Every R -module A is the first Ulm submodule of some module M if and only if the first Ulm submodule of all quotients $Q/P \in L_1$, is of the form X/P where $X = P \oplus Y$ for some Y .

Proof : For necessity, let $P \twoheadrightarrow Q \in L_1$. Consider the diagram below where $P = M^1$:

$$\begin{array}{ccccc}
 P & \twoheadrightarrow & Q & \longrightarrow & Q/P \\
 \parallel & & \downarrow f & & \downarrow g \\
 P & \twoheadrightarrow & M & \longrightarrow & M/P
 \end{array}$$

The map f is from the universal property of the Ulm submodules and g is the natural extension of f . Let $(Q/P)^1 = X/P \subset Q/P$. As $(M/P)^1 = M^1/P = 0$, $f(X) \subset P$. Hence $P \twoheadrightarrow X$ splits.

For sufficiency, consider the set I of all diagrams

$$\begin{array}{ccccc}
 P & \twoheadrightarrow & X & \twoheadrightarrow & Q \\
 & \searrow f & \downarrow f \oplus 0 & & \\
 & & A & &
 \end{array}$$

where $X = P \oplus Y$ and $P \twoheadrightarrow Q$ ranges over the set L_1 . Now take M to be the pushout of the two maps $\bigoplus_I X \twoheadrightarrow \bigoplus_I Q$ and $\bigoplus_I f: \bigoplus_I X \rightarrow A$. Clearly, A is a U_1 -submodule of M . $A = M^1$ follows immediately as $(Q/X)^1 = 0$.

3 - n -algebraically Compact Modules.

Recall : a module M is called algebraically compact if any compatible system of equations in M which is finitely solvable is also solvable in M . See Warfield [6]. This concept can be generalized as follows : call a module M *n -algebraically compact* if any compatible n -system of equations in M which is finitely solvable in M is solvable in M . Any n -algebraically compact module is also k -algebraically compact for $n \geq k$. If $\dim R = k$ then any k -algebraically compact module is n -algebraically compact for $n \geq k$.

It is known that if A is an algebraically compact abelian group then A^1 is the maximum injective subgroup of A . The next theorem generalizes this.

THEOREM 5: Let R be semi-hereditary. If M is 1-algebraically compact then M^1 is injective; hence, M^1 is the maximum injective submodule.

Proof: Let \hat{M} be an injective hull of M . Let E be the injective hull of M^1 in \hat{M} . As M^1 contains all injective submodules of M , it suffices to show that an injective submodule of M contains M^1 . Let E be generated by $M^1 \cup \{e_i : i \in I\}$. Consider all relations in E between e_i and the elements of M^1 (i. e. a presentation of $E \text{ mod } M^1$), $\sum_{\text{finite}} r_{ij} e_j = m_i$ where $r_{ij} \in R$, $m_i \in M^1$ and $i \in I$. As R is semi-hereditary, any finite subsystem is a l -system. Thus, any finite subsystem in M^1 is solvable in M . As M is l -algebraically compact, the entire system is solvable in M i. e. there exists a map $\rho : E \rightarrow M$ which leaves the elements of M^1 fixed. As E is an essential extension of M^1 , we have $\ker \rho = 0$. Therefore $M^1 \subset \rho E \subset M$ where ρE is injective. Hence $M^1 = \rho E$ is injective.

Remark: for all modules M , $M^\#$ is injective if and only if R is hereditary and noetherian.

Proof: the necessity is obvious. For the sufficiency, let I be a left ideal of R . As R is hereditary and noetherian, $I \rightarrow R \in L_1$. Thus, as $M^\#$ is injective with respect to all elements of L_1 , Baer's criterion implies $M^\#$ is injective.

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