H. W. K. ANGAD-GAUR

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The Homological Dimension of a Torsion-Free Abelian Group of Finite Rank as a Module Over Its Ring of Endomorphisms

by

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1. Introduction.

Every abelian group can be considered as a module over its endomorphism ring and it is natural to inquire what its projective dimension is.

Douglas-Farahat [3] proved that the projective dimension is ≤ 1 if the group is torsion or divisible. They described classes of torsion-free groups of finite rank with projective dimension 0 or ∞. Richman-Walker [7] found mixed groups of projective dimension 2.

The problem whether or not every positive integer can occur as a projective dimension of some group has been solved in the affirmative by Bobylev [1]. Using Corner's [2] construction he proved that for every positive integer or ∞, there exists a reduced, torsion-free group of countable rank with the prescribed dimension.

The question if the same holds for torsion-free groups of finite rank remained open. Here we wish to settle this by proving a result analogous to Bobylev's. Our proof is simpler than Bobylev's.

(*) Indirizzo dell'A.: Tulane University - Department of Mathematics - New Orleans Louisiana 70118.

2. – In this section we construct for every integer $n \geq 1$, a reduced torsion-free ring of finite rank with 1 whose global dimension is n. In doing so, we use an idea due to Jans ([5] p. 63, exercise 5).

Let A be a left Q_p-module with basis $e_1, \ldots, e_{n+1}, m_1, \ldots, m_n$ where Q_p denotes the localization of Z at p, *i.e.* the set of those rational numbers which, in their lowest form, have denominators relatively prime to a fixed prime p. Define the multiplication in A via

$$e_i\, e_j = \delta_{ij}\, e_i\,,\; e_i\, m_j = \delta_{ij}\, m_j\,,\; m_i\, e_j = \delta_{i,j-1}\, m_i\,,\; m_i\, m_j = 0$$

for $j = 1, \ldots, n\,;\, i = 1, \ldots, n+1$ where δ is the Kronecker delta. This is enough to extend the multiplication to all of A. Clearly, A becomes in this way a Q_p-algebra with identity $e_1 + \ldots + e_{n+1} = 1$. The additive group is reduced torsion-free of finite rank: it is the direct sum of $2n+1$ copies of Q_p.

The projective dimension of a left R-module M will be denoted by $\dim_R M$. We shall need the following two well-known results.

LEMMA A. *If*

$$0 \to B \to D \to C \to 0$$

is an exact sequence of left R-modules, then

$$\dim_R D \leq \max\,(\dim_R B, \dim_R C).$$

Equality holds except possibly when $\dim_R C = \dim_R B + 1$.

Proof. See Kaplansky [6], p. 169.

LEMMA B. *If M is a direct sum of modules B_i, then*

$$\dim_R M = \sup \dim_R B_i.$$

Proof. See Kaplansky [6], p. 169, example 4.

We prove a few lemmas before we can find the left global dimension of A.

LEMMA 1. i) $\dim_A Ae_i = 0$ *for all* i,

ii) $\dim_A Am_i = i - 1$ *for all* i.

Proof. i) follows from the fact that the e_i' s form a complete set of orthogonal idempotents and hence all Ae_i are projective A-modules.

ii) For $i = 1$: $Ae_1 \cong Am_1$, under an isomorphism which maps e_1 onto m_1. Application of i) gives the desired result.

$i > 1$: Consider the following A-exact sequence

$$0 \to Am_{i-1} \xrightarrow{h} Ae_i \xrightarrow{g} Am_i \to 0 \tag{1}$$

where g is defined by $g(e_i) = e_i m_i = m_i$ and h is the inclusion map.

For $i = 2$, we get $\dim_A Am_2 \leq 1$. If Am_2 was projective, then Am_2 would be isomorphic to a summand Af of A with f an idempotent. To prove that this is not possible, suppose the contrary. Then under an isomorphism, some rm_2 ($r \in Q_p$) is mapped onto f. We can write

$$f = \sum_{i=1}^{n+1} q_i e_i + \sum_{j=1}^{n} r_j m_j \quad \text{with } q_i, r_j \in Q_p.$$

Since for the annihilators we have

$$Ann\, rm_2 = Ann\, m_2 = Ann\, f$$

and

$$Ann\ m_2 = Ae_1 \oplus Am_1 \oplus Ae_3 \oplus \dots \oplus Ae_{n+1}.$$

we get by a simple calculation that f has the form $f = r_2 m_2$ ($r_2 \in Q_p$). But Am_2 contains no idempotents, so $f \in Am_2$ leads to a contradiction. Thus $dim_A Am_2 = 1$.

Continuing inductively for i $= 3, 4, \dots$, application of Lemma A to (1) gives $dim_A Am_i = i - 1$.

LEMMA 2. *Suppose k is a non-negative integer and $\mu \in Q_p$; then*

i) $A\ p^k e_i \cong Ae_i$,

ii) $A\ (p^k e_i + \mu m_i) \cong Ae_i$,

iii) $A\ p^k m_i \cong Am_i$.

Proof. i) and iii) are obvious since $e_i \mapsto p^k e_i$ and $m_i \mapsto p^k m_i$ induce isomorphisms. To prove ii), note that the map $f: Ae_i \to A\,(p^k e_i + \mu\, m_i)$ defined by $f(e_i) = p^k e_i + \mu\, m_i$ is an isomorphism.

Let L denote an arbitrary left ideal of A. Then by passing to A/N where N is the ideal of A generated by the m_i's, L becomes a direct sum : $(L + N)/N = \oplus\, Ap^{k_i}(e_i + N)$ for some i's. By taking coset representatives, $x_i = p^{k_i} e_i + \mu\, m_i$ $(\mu \in Q_p)$ one can now prove :

$$L = B + C \tag{2}$$

where $B = \oplus\, A\, x_i$ and $C = \oplus\, A\, p^{l_i} m_i$ with l_i a non-negative integer for some i's. Let $D = B \cap C$ then $D \cong \oplus\, A\, m_i$ with some i's.

Lemma 3. *For every left ideal* L *of* A, $dim_A\, L \leq n$.

Proof. If we decompose L as in (2) then we can consider the exact sequence

$$0 \to D \xrightarrow{h} B \oplus C \xrightarrow{g} L \to 0$$

where $\oplus$ denotes the outer direct sum, g is the natural epimorphism and $h(d) = (d, -d)$. By Lemmas 1 and B, $dim_A\, D \leq n - 1$. By Lemmas 1, 2 and B, $dim_A\,(B \oplus C) \leq n - 1$. Now by application of Lemma A to the exact sequence above we get $dim_A\, L \leq n$.

We now exhibit a left ideal of dimension n.

Lemma 4. *If* $L_i = Ape_{i+1} + Am_i$ *then* $dim_A\, L_i = i$ $(i = 0, \dots, n)$.

Proof. We prove this by induction on i. If $i = 0$, apply Lemma 2. If $i = 1$, we have the following exact sequence :

$$0 \to Ae_1 \xrightarrow{f} Ae_1 \oplus Ae_2 \xrightarrow{g} L_1 \to 0$$

where $f(e_1) = (pe_1, -m_1 e_2)$ and $g(ae_1, be_2) = ae_1 m_1 + bpe_2$ $(a, b \in A)$. From Lemmas 1 and A we know that $dim_A\, L_1 \leq 1$. If the above sequence splits then there exists a homomorphism $h: Ae_1 \oplus Ae_2 \to Ae_1$ such that $h \circ f = 1$ on Ae_1. Let $h(e_1, 0) = \lambda e_1$ $(\lambda \in Q_p)$. We must have $h(0, e_2) = 0$ since $h(0, e_2) = h(0, e_2^2) = e_2\, h(0, e_2) = 0$. Then $h(f(e_1)) = \lambda pe_1 = e_1$ and so p divides 1 in Q_p, a contradiction. Hence the sequence does not split and $dim_A\, L_1 = 1$.

For $i > 1$, the inductive step can be applied by observing that the sequence

$$0 \to L_{i-1} \xrightarrow{r} Ae_i \oplus Ae_{i+1} \xrightarrow{s} L_i \to 0$$

with $r(pe_i) = (pe_i, -m_i)$, $r(m_{i-1}) = (m_{i-1}, 0)$, $s(e_i, 0) = e_i m_i = m_i$ and $s(0, e_{i+1}) = pe_{i+1}$ is A-exact.

We can now prove:

THEOREM 1. *The left global dimension of A is equal to $n+1$.*

Proof. Because A is not semisimple, left global dimension of $A = \sup \{dim_A L | L$ is a left ideal of $A\} + 1$ (see [5], p. 56). The left ideal L_n has, by Lemma 4, projective dimension n. This together with Lemma 3 gives $\sup \{dim_A L | L$ is a left ideal of $A\} = n$, and hence left global dimension of A is $n+1$.

3. – Equip A with the p-adic topology, i.e. A has a linear topology with a neighborhood system consisting of the subgroups $p^k A$ $(k = 1, \ldots)$. Since A is p-reduced and torsion-free, this topology is Hausdorff. Form its completion $\hat{A}$ in the p-adic topology by considering Cauchy nets or inverse limits (see Fuchs [4]). $\hat{A}$ is a Q_p^*-ring with basis $e_1, \ldots, e_{n+1}, m_1, \ldots, m_n$ where Q_p^* denotes the ring of the p-adic integers. Since A is a free Q_p-module of finite rank, another way of obtaining $\hat{A}$ is by tensoring A by Q_p^*, i.e. $\hat{A} = A \otimes_{Q_p} Q_p^*$. Since the topology on A is Hausdorff, A can be considered to be a pure subring of $\hat{A}$. $\hat{A}$ becomes a left A-module, too.

4. – In this section we combine Corner's construction (see Corner [2]) with Bobylev's idea (see Bobylev [1]) in order to find a torsion-free group of finite rank whose endomorphism ring is isomorphic to the ring A described in 2.

First we want to state a lemma which we shall need.

LEMMA C. *If $\varrho_1 a_1 + \ldots + \varrho_n a_n = 0$ where $a_1, \ldots, a_n \in A$ and $\varrho_1, \ldots, \varrho_n$ are p-adic integers linearly independent over Q_p, then $a_1 = \ldots = a_n = 0$.*

Proof. See Corner [2] Lemma 2.1.

Choose in A a Q_p-basis $a_1, \dots, a_{2n+1}$ such that $a_1 = 1$. Choose in Q_p^* algebraically independent elements $\varrho_1, \dots, \varrho_{2n+1}, \beta$ over Q_p. Let

$$\varepsilon = \varrho_1 a_1 + \dots + \varrho_{2n+1} a_{2n+1},$$

and define G to be the pure subgroup

$$G = < A, A\varepsilon, m_n \beta >_*$$

in $\hat{A}$. It is clear that G is torsion-free of finite rank. If End G denotes the endomorphism ring of G, then we claim:

THEOREM 2. End $G \cong A$.

Proof. G is a left A-module. For if $g \in G$, then for some integer $q \neq 0$,

$$qg = a + b\varepsilon + c\beta m_n \quad \text{with } a, b, c \quad \text{in } A.$$

Therefore for any d in A,

$$d(qg) = q(dg) = da + db\varepsilon + dc\beta m_n \in G,$$

and hence by the purity of G, $dg \in G$.

Since $1 \in G$, we have that A is isomorphic to a subring of End G. It remains to prove that every endomorphism of G is multiplication by some element of A. Let $\eta \in$ End G. Then it is known that η can be extended in a unique way to a Q_p^*-endomorphism $\hat{\eta}$ of $\hat{A}$. Consider

$$\eta(\varepsilon) = \hat{\eta}(\varepsilon) = \varrho_1 \eta(a_1) + \dots + \varrho_{2n+1} \eta(a_{2n+1})$$

Since $\eta(\varepsilon)$, $\eta(a_i)$ are elements of G, for some integer $q \neq 0$ we have

$$q\eta(\varepsilon) = a + b\varepsilon + c\beta m_n \quad \text{with } a, b, c \quad \text{in } A$$

and $$q\eta(a_i) = a_i + b_i \varepsilon + c_i \beta m_n \quad \text{with } a_i, b_i, c_i \quad \text{in } A.$$

Substitution gives

$$a + b\Big(\sum_{i=1}^{2n+1} \varrho_i\, a_i\Big) + c\,\beta m_n = \sum_{k=1}^{2n+1} \varrho_k \Big(a_k + b_k \Big(\sum_{j=1}^{2n+1} \varrho_j\, a_j\Big) + c_k\, \beta m_n\Big).$$

By our choice of the ϱ_i's and β, all products of these elements are linearly independent over Q_p, from Lemma C we conclude

$$a = cm_n = c_k m_n = 0\ ,\ ba_k = a_k\ ,\ b_k\, a_j + b_j\, a_k = 0 \qquad \text{for all } j\ ,\ k.$$

If we let $j = k = 1$ in the last equation then $b_1 = 0$. Letting $j = 1$ in the last equation gives now $b_k = 0$, for all k. Then $q\,\eta(a_i) = ba_i$. For $i = 1$ this gives $q\,\eta(1) = b \in A$ and by purity of A, $\eta(1) \in A$.

Consequently, $q\,\eta(a_i) = q\,\eta(1)\, a_i$ and by torsion-freeness

$$\eta\,(a_i) = \eta\,(1)\, a_i \qquad \text{for all } i.$$

But then $\hat{\eta}$ is multiplication by $\eta\,(1)$ on $\hat{A}$ and hence η is multiplication by $\eta\,(1) \in A$ on G.

5. – In this section we will prove that for $n \geq 2$ the group G constructed in 4 has projective dimension n over its endomorphism ring A.

Consider the following short exact sequence of left A-modules

$$0 \to G \xrightarrow{k} \hat{A} \xrightarrow{\pi} \hat{A}/G \to 0 \qquad (3)$$

where π is the projection and k is the inclusion map. Let $\bar{\beta} = \pi\,(\beta e_{n+1})$. Then $\bar{\beta} \neq 0$ in $\hat{A}/G$, because if $\bar{\beta} = 0$ then $\beta e_{n+1} \in G$ and hence for some integer $q \neq 0$ we have $q\,(\beta e_{n+1}) = a_1 + a_2\,\varepsilon + a_3\,\beta m_n$ with a_1, a_2, a_3 in A. By Lemma C of 4, we get $qe_{n+1} = a_3 m_n$. Hence $q = 0$ which is a contradiction.

LEMMA 5. (i) $dim_A\ Ae_k \otimes_{Q_p} Q = 1$ for all k,

(ii) $dim_A\ Am_l \otimes_{Q_p} Q = l$ for all l.

Proof. Since $dim_{Q_p}\ Q = 1$, we have an exact sequence of Q_p-modules

$$0 \to F_1 \xrightarrow{r} F_0 \xrightarrow{s} Q \to 0 \qquad (4)$$

where $F_1 \subseteq F_0$ *are* free Q_p-modules and r is the inclusion map.

(i) Tensoring the above sequence from the left with the right flat Q_p-module Ae_k, we obtain the exact sequence of left A-modules

$$0 \to Ae_k \otimes_{Q_p} F_1 \to Ae_k \otimes_{Q_p} F_0 \to Ae_k \otimes_{Q_p} Q \to 0 .$$

Since F_0, F_1 are free Q_p-modules, $dim_A\, Ae_k \otimes_{Q_p} F_1 = dim_A\, Ae_k \otimes_{Q_p} F_0 = 0$ and by Lemma A we obtain that $dim_A\, Ae_k \otimes_{Q_p} Q \leq 1$. Since the additive group of $Ae_k \otimes_{Q_p} Q$ is divisible, it cannot be a projective A-module. Hence $dim_A\, Ae_k \otimes_{Q_p} Q = 1$.

(ii) We apply induction on l. If $l = 1$, the result follows from the fact that $Am_1 \cong Ae_1$ and (i). For the case $l = 2$ consider the exact sequences

$$0 \to M \xrightarrow{t} Ae_2 \otimes_{Q_c} F_0 \xrightarrow{g \otimes s} Am_2 \otimes_{Q_p} Q \to 0 \tag{5}$$

$$0 \to Am_1 \otimes_{Q_p} F_1 \xrightarrow{v} Ae_2 \otimes_{Q_p} F_1 \oplus Am_1 \otimes_{Q_p} F_0 \xrightarrow{n} M \to 0 \tag{6}$$

where $M = Ae_2 \otimes_{Q_p} F_1 + Am_1 \otimes_{Q_p} F_0 \subseteq Ae_2 \otimes_{Q_p} F_0$, $t(e_2 \otimes s) = e_2 \otimes r(s)$, $t(m_1 \otimes \bar{s}) = m_1 \otimes \bar{s}$, n denotes the natural epimorphism, $v(m_1 \otimes s) = \big(m_1 \otimes (-s), m_1 \otimes r(s)\big)$ and g is the map in (1) ($s \in F_1$ and $\bar{s} \in F_0$). From (6) we obtain by the above and Lemma's B and A that $dim_A\, M \leq 1$. Suppose by way of contradiction that $dim_A\, M = 0$. Then sequence (6) splits and hence there exists a surjection

$$h : Ae_2 \otimes_{Q_p} F_1 \oplus Am_1 \otimes_{Q_p} F_0 \to Am_1 \otimes_{Q_p} F_1$$

such that $h \circ v = 1$ on $Am_1 \otimes_{Q_p} F_1$. We must have $h(Ae_2 \otimes_{Q_p} F_1) = 0$, because $h(e_2 \otimes s) = h(e_2^2 \otimes s) = e_2 h(e_2 \otimes s) \in e_2(Am_1 \otimes_{Q_p} F_1) = 0$. Then there is a split exact sequence

$$0 \to Am_1 \otimes_{Q_p} F_1 \to Am_1 \otimes_{Q_p} F_0 \to Am_1 \otimes_{Q_p} Q \to 0 .$$

This is not possible since $Am_1 \otimes_{Q_p} F_0$ is reduced and $Am_1 \otimes_{Q_p} Q$ is divisible. Hence $dim_A M = 1$. This together with Lemma A gives from (5) that $dim_A\, Am_2 \otimes_{Q_p} Q = 2$.

For the case $l > 2$, tensoring (1) from the right by Q gives us the exact sequence

$$0 \to Am_{i-1} \otimes_{Q_p} Q \to Ae_i \otimes_{Q_p} Q \to Am_i \otimes_{Q_p} Q \to 0 .$$

By using the case $l = 2$, (i) and Lemma A we complete the proof of (ii) by applying induction to the above exact sequence on i.

Let L be the left A-submodule in $\hat{A}/G$ generated by $\{\overline{\beta}\}$. Since $\hat{A}/G$ is divisible, we can consider the divisible hull, N, of L. By torsion freeness of $\hat{A}/G$ we obtain $N = L \otimes_{Q_p} Q$.

Note that

$$m_n\overline{\beta} = m_n\pi(\beta e_{n+1}) = \pi(\beta \cdot m_n) = 0 .$$

Hence we get the following short exact sequence of left A-modules

$$0 \to Am_n \xrightarrow{f} Ae_{n+1} \xrightarrow{g} L \to 0$$

where $g(e_{n+1}) = e_{n+1} \cdot \overline{\beta} = \overline{\beta}$ and f is the inclusion map. Tensoring this sequence from the right by Q gives us the exact sequence

$$0 \to Am_n \otimes_{Q_p} Q \to Ae_{n+1} \otimes_{Q_p} Q \to N \to 0$$

From Lemma's 5 and A we obtain $\dim_A N = n + 1$ $(n > 1)$.

Consider the following short exact sequence of left A-modules

$$0 \to N \xrightarrow{h} \hat{A}/G \xrightarrow{\pi} (\hat{A}/G)/N \to 0$$

where π is the projection map and h the inclusion map. If $\dim_A \hat{A}/G = m < n + 1$ then by Lemma A, $\dim_A (\hat{A}/G)/N = n + 2$ which is a contradiction to Theorem 1. If $\dim_A \hat{A}/G = m > n + 1$, then we have a contradiction to Theorem 1.

Hence

$$\dim_A \hat{A}/G = n + 1. \tag{7}$$

Since $\dim_{Q_p} Q_p^* = 1$, we have a short exact sequence of Q_p-modules

$$0 \to R_1 \to R_0 \to Q_p^* \to 0$$

where R_0 and R_1 are free Q_p-modules. Tensoring from the left by the $A - Q_p$ -bimodule$_A$ A_{Q_p} gives the following short exact sequence of A-modules:

$$0 \to A \otimes_{Q_p} R_1 \to A \otimes_{Q_p} R_0 \to A \otimes_{Q_p} Q_p^* \to 0.$$

The left exactness follows from the fact that A_{Q_p} is torsion-free and hence flat as a right Q_p-module. Since tensor product commutes with direct sums, the first two terms of the sequence are free A-modules. We know that $A \otimes_{Q_p} Q_p^* \cong \hat{A}$. From Lemma A we infer

$$\dim_A \hat{A} \leq 1. \tag{8}$$

From Lemma A, (7) and (8), the exact sequence (3) implies that

$$\dim_A G = n \quad (n \geq 2). \tag{9}$$

We can now formulate our main result:

THEOREM 3. *To every integer $m \geq 0$ there exists a torsion free abelian group of finite rank such that its projective dimension over its endomorphism ring is equal to m.*

Proof. If $m = 0$, use $G = Q$. If $m = 1$, let G be any indecomposable group of rank > 1 such that End $G \cong Z$ (see [4]).

For $m \geq 2$, use the group G constructed in 4 with $n = m$. Then $n \geq 2$ and by Theorem 2 and (9) we get

$$\dim_{End\ G} G = m.$$

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