RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

GIULIANO ARTICO
UMBERTO MARCONI

On the compactness of minimal spectrum

Rendiconti del Seminario Matematico della Università di Padova, tome 56 (1976), p. 79-84
<http://www.numdam.org/item?id=RSMUP_1976__56__79_0>

Rend. Sem. Mat. Univ. Padova, Vol. 56 (1977)

On the Compactness of Minimal Spectrum.

Giuliano Artico - Umberto Marconi (*)

0. Introduction.

Let A be a commutative ring with 1; denote by Spec(A) the set of all prime ideals of A equipped with the hull-kernel topology, by Min(A) the subspace consisting of minimal prime ideals. Henriksen and Jerison [HJ] found some sufficient conditions for the compactness of Min(A); subsequently, Quentel [Q] discovered an equivalent condition. Here we give another characterization of the compactness of Min(A), which seems to give more light to the topological situation; this characterization, among other things, allows us to show that the class of (weakly) Baer rings coincides with the class of rings such that: 1) their minimal spectrum is compact; and 2) every prime ideal contains a unique minimal prime ideal.

We shall always deal with rings without non-zero nilpotents; but of course all purely topological results are independent of this hypothesis.

1. All rings are commutative and with 1. Spec(A) denotes the set of prime ideals of A, equipped with the Zariski topology; i.e. Spec(A) has as a base of open sets the sets $D(a) = \text{Spec}(A) - V(a) = \{P \in \text{Spec}(A) \colon a \notin P\}$. Thus, the subspace Min(A) of minimal prime ideals has $\{D^0(a) = D(a) \cap \text{Min}(A) \colon a \in A\}$ as a base of open sets. For the sake of simplicity, we assume that A is semiprime (that is, A has no non-zero nilpotents); however, it will be clear that all results

(*) Indirizzo dell'A.: Istituto di Matematica Applicata, Università di Padova, Padova, Italy.

Lavoro eseguito nell'ambito dei Gruppi di Ricerca Matematica del C.N.R.

obtained here hold in the general case, with some obvious modification (e.g., the nilradical of A in place of the zero ideal).

The sets $D^0(a)$ are clopen in $\mathrm{Min}(A)$; for, denoting by $\mathrm{Ann}(a)$ the annihilator of a, we have $D^0(a) = \mathrm{Min}(A) - V^0(a) = V^0(\mathrm{Ann}(a))$ (if I is an ideal of A, $V(I)$ is its hull in $\mathrm{Spec}(A)$, and $V^0(I) = V(I) \cap \cap \mathrm{Min}(A)$), (see [$HJ$]). Thus $\mathrm{Min}(A)$ is a space with a clopen basis, and, being T_0, it is also a Hausdorff space.

LEMMA. *Let A be a semiprime ring, P a prime ideal of A. The following are equivalent:*

i) *P is a minimal prime.*

ii) *For every $a \in P$,* $\mathrm{Ann}(a) \not\subseteq P$.

iii) *For every finitely generated ideal I contained in P,* $\mathrm{Ann}(I) \not\subseteq P$. *Thus, if A is semiprime and I is finitely generated,* $\mathrm{Ann}(I) = 0$ *iff* $V^0(I) = \emptyset$.

PROOF. The equivalence of i) and ii) is proved in [HJ, 1.1]. iii) implies ii): trivial, ii) implies iii): let $a_1, \dots, a_n \in P$ generate I; for each $i = 1, \dots, n$ choose $b_i \in \mathrm{Ann}(a_i) - P$; then $b = b_1 \dots b_n \in \in \mathrm{Ann}(I) - P$.

Plainly, iii) shows that no minimal prime ideal can contain a finitely generated ideal whose annihilator is zero; conversely, if I is finitely generated and $\mathrm{Ann}(I)$ contains a non-zero element b, then, since A is semiprime, there exist some minimal prime ideal which does not contain b; every such prime necessarily belongs to $V^0(I)$.

THEOREM. *Let A be a semiprime ring. The following are equivalent:*

1) *The family of sets $\{V^0(a) \colon a \in A\}$ is a subbase for the topology of* $\mathrm{Min}(A)$.

2) $\mathrm{Min}(A)$ *is a compact space.*

3) *For every element $a \in A$, there exists a finite number of elements $a_1, \dots, a_n \in A$ such that $aa_i = 0$ for each $i = 1, \dots, n$ and* $\mathrm{Ann}(a_1, \dots, a_n, a) = 0$.

PROOF. 1) implies 2). By Alexander's subbase theorem it is enough to show that, if B is a subset of A such that $\bigcap_{a \in B} D^0(a) = \emptyset$, then there exists a finite number of elements in B, say $a_1, \dots, a_n$ such that $\bigcap_{i=1}^{n} D^0(a_i) = \emptyset$. Let us observe that $\bigcap_{a \in B} D^0(a)$ coincides with the set of minimal prime ideals disjoint from B. If S is the multiplicative set generated by B, a prime ideal doesn't meet S if and only if it doesn't meet B. Now, zero belongs to S for, otherwise, there would exist a prime ideal, and then a minimal prime one, disjoint from B. But if zero belongs to S, there exist $a_1, \ldots, a_n \in B$ such that their product is zero, and so $\bigcap_{i=1}^{n} D^0(a_i) = D^0(a_1 \ldots a_n) = D^0(0) = \emptyset$.

2) implies 3). If $\mathrm{Min}(A)$ is a compact space, then $V^0(a)$ is an open compact set, and therfore it is a finite union of basic open sets, that is $V^0(a) = D^0(a_1) \cup \ldots \cup D^0(a_n)$. Since $D^0(a_i)$ is contained in $V^0(a)$ for each $i = 1, \ldots, n$, every minimal prime ideal contains aa_i, and so $aa_i = 0$ for each i. Moreover, the above relation implies that $V^0(a_1, \ldots, a_n, a) = \emptyset$; by the Lemma, $\mathrm{Ann}(a_1, \ldots, a_n, a) = 0$.

3) implies 1). Choose a basic open set $D^0(a)$. Let $a_1, \ldots, a_n$ be the elements given by 3). By the Lemma, the ideal $I = (a_1, \ldots, a_n, a)$ is contained in no minimal prime; this implies that $D^0(a) \supseteq V^0(a_1) \cap \ldots \cap V^0(a_n)$; but since $aa_i = 0$ for every $i = 1, \ldots, n$, equality actually holds.

REMARK 1. Condition 3) is due to Quentel [*Q*, Proposition 4].

REMARK 2. Condition 1) allows us to state Theorem 3.4 of [H.J] in the following way:

« The following conditions on a ring A without non zero nilpotents are equivalent:

a) $\mathrm{Min}(A)$ is compact and, for every $x, y \in A$, there exists $z \in A$ such that $\mathrm{Ann}(x) \cap \mathrm{Ann}(y) = \mathrm{Ann}(z)$.

b) The family of sets $\{V^0(x) \colon x \in A\}$ is a base for the open sets of Min(A).

c) For each $x \in A$ there exists $x' \in A$ such that $\mathrm{Ann}(\mathrm{Ann}(x')) = \mathrm{Ann}(x)$. » ; thus, the assumption of compactness of $\mathrm{Min}(A)$ in condition *b*) is redundant.

Notice also that condition 3) of the Theorem may be written as follows:

3 *bis*) *For each* $a \in A$ *there exist* $a_1, \ldots, a_n$ *such that*

$$\mathrm{Ann}(a) = \mathrm{Ann}(\mathrm{Ann}(a_1, \ldots, a_n)),$$

which thus appears as a weakening of condition *c*) in the above theorem.

2. In paper [K], Kist proves the equivalence of the following conditions:

a) There exists a continuous function of $\mathrm{Spec}(A)$ onto $\mathrm{Min}(A)$ which is the identity on $\mathrm{Min}(A)$.

b) A is a Baer ring, that is the annihilator ideal of each element in A is generated by an idempotent.

This implies that, in a Baer ring, every prime ideal contains a unique minimal prime ideal and that $\mathrm{Min}(A)$ is compact. We shall prove that these two last conditions characterize the Baer rings. First, we need two Lemmas:

LEMMA α. *Let P be a prime ideal of A and let O_P be the intersection of the prime ideals contained in P. Then O_P coincides with the ideal of the elements of A whose annihilator is not contained in P.*

(For a proof, one may look at [DMO, p. 460]).

LEMMA β. *Let A be a semiprime ring. The following are equivalent:*

i) *Every prime ideal contains a unique minimal prime ideal.*

ii) *If a, b are elements of A such that $ab = 0$, then* $\mathrm{Ann}(a) + \mathrm{Ann}(b) = A$.

iii) For every $a, b \in A$, $\mathrm{Ann}(a) + \mathrm{Ann}(b) = \mathrm{Ann}(ab)$.

PROOF. i) implies ii). If i) holds, then for every maximal ideal M, O_M is the unique minimal prime contained in M. If $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ is contained in M, then, since O_M is prime, either a or b belong to O_M: this is absurd for the characterization of O_M given in Lemma α.

ii) implies iii). Of course $\mathrm{Ann}(a) + \mathrm{Ann}(b)$ is contained in $\mathrm{Ann}(ab)$. If x belongs to $\mathrm{Ann}(ab)$, then $(xa)b = x(ab) = 0$ and so there exist $y \in \mathrm{Ann}(xa)$ and $z \in \mathrm{Ann}(b)$ such that $1 = y + z$, hence $x = xy + xz$, with $xy \in \mathrm{Ann}(a)$ and $xz \in \mathrm{Ann}(b)$.

iii) implies i). If P is a prime ideal, let us see that O_P is prime, too. In fact, if ab belongs to O_P, there exists an element $x \in \mathrm{Ann}(ab)$ that doesn't belong to P. According to iii), $x = y + z$, with $y \in \mathrm{Ann}(a)$ and $z \in \mathrm{Ann}(b)$; hence either $y \notin P$, or $z \notin P$; by Lemma α, this is equivalent to $a \in O_P$ or $b \in O_P$.

Now we can state the following theorem.

THEOREM. *Let A be a semiprime ring. The following are equivalent:*

1) *A is a Baer ring.*

2) *Every prime ideal contains a unique minimal prime ideal and* $\mathrm{Min}(A)$ *is a compact space.*

3) $\mathrm{Min}(A)$ *is a retract of* $\mathrm{Spec}(A)$, *that is there exists a continuous function φ of* $\mathrm{Spec}(A)$ *onto* $\mathrm{Min}(A)$ *which is the identity on* $\mathrm{Min}(A)$.

PROOF. 1) implies 2). Trivially if A is a Baer ring, condition 3) of Theorem 1 is satisfied and then $\mathrm{Min}(A)$ is a compact space. Let us see that every prime ideal contains a unique minimal prime ideal, proving that condition ii) of Lemma β holds. Let a, b be elements such that $ab = 0$ and let e, f be the idempotents which generate $\mathrm{Ann}(a)$ and $\mathrm{Ann}(b)$, respectively. Since $b \in \mathrm{Ann}(a) = (e)$, there exists $c \in A$ such that $b = ce$, hence $be = ce^2 = ce = b$ and so $(1-e)b = 0$. Then $(1-e) \in \mathrm{Ann}(b) = (f)$, so that $\mathrm{Ann}(a) + \mathrm{Ann}(b) = (e) + (f) = A$.

2) implies 3). Let φ be the map from $\mathrm{Spec}(A)$ to $\mathrm{Min}(A)$ defined by $\varphi(P) = O_P$. Since $\mathrm{Min}(A)$ is compact, to prove that φ is a continuous function it is enough to show that $\varphi^{\leftarrow}[D^0(a)]$ is a closed set (Theorem 1). This is trivial because, from the characterization of O_P given by the Lemma α, we have $\varphi^{\leftarrow}[D^0(a)] = V(\mathrm{Ann}(a))$.

3) implies 1). First we prove that, if Q is a minimal prime ideal contained in a prime ideal P, then Q is the image of P by the retraction φ. In fact $P \in \mathrm{clos}_{\mathrm{Spec}(A)}\{Q\}$, that is contained in $\varphi^{\leftarrow}[\varphi(Q)]$, so that $\varphi(P) = \varphi(Q) = Q$. Hence the retraction maps a prime ideal into the unique minimal prime ideal contained in it; therefore $V(\mathrm{Ann}(a)) = \varphi^{\leftarrow}[D^0(a)]$ is a clopen set; then $\mathrm{Ann}(a)$ is a direct summand in A, because in a semiprime ring an ideal I is a direct summand if and only if $V(I)$ is a clopen set.

REMARK 1. A Baer ring A is necessarily semiprime: assume x nilpotent, and let n be the smallest non negative integer such that $x^n = 0$. We want to show that $n = 1$, *i.e.* $x = 0$. For, otherwise, we have $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^{n-1})$; since A is Baer, $\mathrm{Ann}(x) = (e)$, $\mathrm{Ann}(x^{n-1}) = (f)$, with e, f idempotents; since $x \in \mathrm{Ann}(x^{n-1})$ then $x = xf$, which implies $x^{n-1} = x^{n-1}f = 0$, contraddicting the minimality of n.

REMARK 2. The ring $A = K[x, y]/(xy)$, where K is a field and x, y are indeterminates over K, is a ring whose minimal spectrum is compact, but it is not a Baer ring. A is a noetherian ring; then $\mathrm{Min}(A)$

is finite, hence compact (and discrete). It is easy to see that A is a semiprime ring with no non trivial idempotents. Using the fact that $K[x, y]$ is a unique factorization domain, it can be shown that $\mathrm{Ann}(x + (xy))$ is generated by $(y + (xy))$, so that A is not a Baer ring.

REMARK 3. If X is a topological space, $C(X)$ denotes the ring of all real valued continuos functions on X; X is said to be an F-space when every prime ideal of $C(X)$ contains a unique minimal prime ideal [GJ, 14.25]. X is said to be basically disconnected if the closure of every cozero-set is an open set [GJ, 1H]. One can easily prove that $C(X)$ is a Baer ring if and only if X is basically disconnected. There exist F-spaces X that are not basically disconnected, for instance $\beta R - R$ [GJ, 6M, 14.0]. Hence there are rings in which every prime ideal contains a unique minimal prime ideal, without being Baer rings.

REFERENCES

[DMO] G. De Marco - A. Orsatti, *Commutative rings in which every prime ideal is contained in a unique maximal ideal*, Proc. Amer. Math. Soc., **30** (1971), pp. 459-466.

[GJ] L. Gillman - M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J. (1960).

[HJ] M. Henriksen - M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc., **115** (1965), pp. 110-130.

[K] J. Kist, *Two characterizations of commutative Baer rings*, Pacific Journal of Math., **50** (1974), pp. 125-134.

[Q] Y. Quentel, *Sur la compacité du spectre minimal d'un anneau*, Bull. Soc. math. France, **99** (1971), pp. 265-272.

Manoscritto pervenuto in Redazione il 2 dicembre 1975.