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Partially Elliptic Pseudodifferential Operators and the WF_x of Distributions.

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Introduction.

The aims of this article are the following:

1) the generalization of the notion of differential operator $P(D_x, D_y)$ elliptic in the x-variable to the case of pseudodifferential operators;

2) the definition of $WF_x(u)(u \in \mathcal{D}'(A_x \times A_y))$.

Such problems are pointed out at the end of the introduction in [4].

Our results are the following:

a) a characterization of x-partially elliptic pseudodifferential operators by the construction of a partial (left or right) parametrix, modulo regularizing pseudodifferential operators

b) a characterization of $WF_x(u)$.

With notations and symbols of § 1:

a') if $A \in \Psi^{\alpha}(A_x \times A_y)$ is x-partially elliptic in A_x there exist B and $B_1 \in \Psi^{-\alpha}(A_x \times A_y)$ such that:

$$A \circ B \equiv B_1 \circ A \equiv I(x) \otimes G(y, D_y) \bmod \Psi^{-\infty}(A_x \times A_y)$$

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Lavoro eseguito mentre uno degli Autori godeva di una borsa di studio del C.N.R. per laureandi.

where $I(x)$ is the identity operator on $\mathcal{D}'(A_x)$ and $G(y, D_y)$ is in $\Psi^{-\infty}(A_y)$;

b') if $A = A(x, y, D_x, D_y) \in \Psi_0^\infty(A_x \times A_y)$ and it is x-partially elliptic in A_x then:

i) $\forall u = u(x, y) \in \mathcal{D}'(A_x \times A_y)$, $WF_x(u) = WF_x(Au)$;

ii) $WF_x(u) = \bigcap C_{A,x}$ $\forall A \in \Psi_0^\infty$ such that Au is regular in the x-variable in A_x.

§ **1.** – We shall write $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_n)$ for the coordinate in R^{m+n} and $(\xi, \eta) = (\xi_1, \dots, \xi_m, \eta_1, \dots, \eta_n)$ for the dual coordinate. Let A_x and A_y be opens in R^m and in R^n respectively and let $S^*(A_x \times A_y)$ be the cosphere bundle over $A_x \times A_y$ (this is the quotient of $T^*(A_x \times A_y)$ modulo the equivalence relation $(x, y, \xi, \eta) \sim (x', y', \xi', \eta') \Leftrightarrow (x, y) = (x', y')$ and there is $t > 0$ such that $(\xi', \eta') = t(\xi, \eta)$. $S^*(A_x \times A_y) \cong A_x \times A_y \times S_{m+n-1}$, where S_{m+n-1} denotes the unit sphere in R_{m+n}).

We denote by $\Psi^\alpha(A_x \times A_y)$ the space of the pseudodifferential operators of order $\leqslant \alpha$ and by $S^\alpha(A_x \times A_y)$ the space of their symbols. The union and the intersection of $\Psi^\alpha(A_x \times A_y)$ and $S^\alpha(A_x \times A_y)$ for all α, will be denoted by $\Psi^\infty(A_x \times A_y)$, $\Psi^{-\infty}(A_x \times A_y)$ and $S^\infty(A_x \times A_y)$, $S^{-\infty}(A_x \times A_y)$ respectively.

From [4] we have the following:

DEFINITION 1. *A differential operator* $P = P(D_x, D_y) = \sum_{\alpha,\beta} a^{\alpha,\beta} D_x^\alpha D_y^\beta$, $a^{\alpha,\beta} \in \boldsymbol{C}$ ($\boldsymbol{C}$ *denotes the complex field*), $|\alpha| \leqslant m_x$, $|\beta| \leqslant m_y$ $(\alpha, \beta) \in N^m \times N^n$, $(m_x, m_y) \in N \times N$ $|\alpha| = \sum_{i=1}^m \alpha_i$ $|\beta| = \sum_{i=1}^n \beta_i$ *is partially elliptic in the x-variable, if* $\forall f \in \mathcal{D}'(R^{m+n})$ *which is solution of* $Pf = 0$ *in* A, A *open in* R^{m+n}, *we have*: $f(x, y)$ *is analytic in* x *on* A (1).

From the proof of the second characterization (2) of the operators $P(D_x, D_y)$ which are partially elliptic in the x-variable we have that:

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(1) We say that $f(x, y)$ is analytic in $A \subseteq R^{m+n}$ in the x-variable if: $\forall A_1, A_2$ open in R^m and R^n respectively and such that $A_1 \times A_2 \subseteq A$ we have: $\forall \varphi \in \mathcal{D}(A_2)$ the distribution $\int f(x, y)\varphi(y)dy$ is analytic in A_1.

(2) The characterization is the following: $P(D_x, D_y)$ is partially elliptic in the x-variable if and only if $\sum_\alpha |P^\alpha(\xi, \eta)|^2 (1 + |\xi|^{2|\alpha|}) \sim \sum_\alpha |P^\alpha(\xi, \eta)|^2$ where $P^\alpha(\xi, \eta) = (\partial/\partial\xi, \partial/\partial\eta)_\alpha P$ and $A \sim B$ means: A/B and B/A are bounded quotients in R^{m+n} (theorem 2 of [4]).

if $P(\xi, \eta) = \sum_{\alpha,\beta} a^{\alpha,\beta} \xi^\alpha \eta^\beta$ and $m = \deg P(\xi, \eta)$ there exists $c \in R^+$ such that:

$$c \leqslant |\xi| \Rightarrow |\xi|^m \leqslant c|P(\xi, \eta)| . \tag{1}$$

This property suggests the idea for the extension of the notion of partial ellipticity to the pseudodifferential operators.

Let $a = a(x, y, \xi, \eta) \in S^\alpha(A_x \times A_y)$. Let $P_0 \equiv (x_0, y_0, \xi^0, \eta^0)$, $\xi^0 \neq 0$ be a point of $S^*(A_x \times A_y)$

DEFINITION 2. *We say that a is partially elliptic in the x-variable in P_0 or equivalently that a is x-partially elliptic in P_0 if:*

i) *there exists an open relatively compact neighbourhood Ω of (x_0, y_0) in $A_x \times A_y$*

ii) *there exists a neighbourhood Γ of $(\dot{\xi}_0, 0)$ $(\dot{\xi}_0 = \xi_0/|\xi_0|)$ in $R_m \times R_n$ such that $\Gamma \cap R_m$ is a conic neighbourhood of $\dot{\xi}_0$ and $\pi_\eta(\Gamma)$ is a compact in R_n;*

iii) *there exist $c_1, c_2 \in R^+$ such that:*

$$|\xi|^\alpha \leqslant c_1 |a(x, y, \xi, \eta)| \tag{2}$$

if $|\xi| \geqslant c_2$ $(x, y) \in \Omega$ $(\xi, \eta) \in \Gamma$.

REMARK 1. *If $a \in S^\alpha(A_x \times A_y)$ is x-partially elliptic in P_0 and if $r = r(x, y, \xi, \eta) \in S^{-\infty}(A_x \times A_y)$ then $a + r$ is x-partially elliptic in P_0.*
We can now give the following:

DEFINITION 3. *If $A \in \Psi^\infty(A_x \times A_y)$, then A is x-partially elliptic in P_0 if its symbol a is x-partially elliptic in P_0.*

REMARK 2. *If $a \in S^\alpha(A_x \times A_y)$ and $b \in S^\beta$ are the symbols of the pseudodifferential operators A and B respectively, then also $p \sim \sum_p (1/p!) \cdot \partial^p_{(\xi,\eta)} a D^p_{(x,y)} b$ which is the symbol of $A \circ B$, is x-partially elliptic in P_0.*

PROOF. Because of the x-partial ellipticity of a and b, we have that:

a) there exists an open relatively compact neighbourhood Ω Ω of (x_0, y_0);

b) there exists a neighbourhood Γ of $(\dot{\xi}^0, 0)$ in R_{m+n} such that $\Gamma \cap R_m$ is a conic neighbourhood of $\dot{\xi}^0$ and $\pi_\eta(\Gamma)$ is a compact in R_n;

c) there exist $c_1, c_2 \in R^+$ such that:

(3) $$|ab| \geqslant c_1 |\xi|^{\alpha+\beta}$$

if $|\xi| \geqslant c_2$ $(x, y) \in \Omega$ $(\xi, \eta) \in \Gamma$.

If $p(x, y, \xi, \eta)$ is the symbol of $A \circ B$, which is in $\Psi^{\alpha+\beta}(A_x \times A_y)$, then $ab - p$ is in $S^{\alpha+\beta-1}(A_x \times A_y)$, therefore:

(4) $$c(1 + |\xi|)^{\alpha+\beta-1} \geqslant |ab - p|$$

if $(x, y) \in \Omega$ $(\xi, \eta) \in \Gamma$.

If p is not x-partially elliptic $\forall 1/n$ $\exists (x_n, y_n, \xi_n, \eta_n)$, $|\xi_n| \to +\infty$, such that:

(5) $$|p| \leqslant 1/n |\xi_n|^{\alpha+\beta} \quad \text{if } (x, y) \in \Omega \ (\xi, \eta) \in \Gamma .$$

From (4) and (5) we have then:

(6) $$c(1 + |\xi_n|)^{\alpha+\beta} \geqslant |ab - p| \geqslant |ab| - |p| \geqslant c_1 |\xi_n|^{\alpha+\beta} - 1/n |\xi_n|^{\alpha+\beta}$$

and this is impossible.

With the same argument, *we can prove that if A is x-partially elliptic in $P_0(x_0, y_0, \xi^0, \eta^0)$ so is A^* (the adjoint of A) and that ${}^tA$ (the transpose of A) is x-partially elliptic in $P_0'(x_0, y_0, -\xi^0, -\eta^0)$.*

§ **2.** – Let a be x-partially elliptic in $P_0 \equiv (x_0, y_0, \xi^0, \eta^0)$ according to definition 2. Let $\Omega' = \pi_x(\Omega)$ and $\Gamma' = \Gamma \cap S_{m-1}$; let $g(x, \xi)$ be in $C_c^\infty(\Omega' \times \Gamma')$ and $\varphi(\xi)$ in $C^\infty(R_m)$ such that $0 \leqslant \varphi(\xi) \leqslant 1$, $\varphi(\xi) = 0$ if $|\xi| \leqslant c_2$ and $\varphi(\xi) = 1$ if $|\xi| \geqslant 2c_2$. Lastly let $\chi(y, \eta)$ be in $C_c^\infty(A_y \times R_n)$ such that $\pi_\eta(\operatorname{supp} \chi) \subseteq \pi_\eta(\Gamma)$, $\pi_y(\operatorname{supp} \chi) \subseteq \pi_y(\Omega)$ and the origin of R_n is in $\pi_\eta(\overset{\circ}{\operatorname{supp}} \chi)$ It is easy to show that $g(x, \xi)\varphi(\xi)\chi(y, \eta)/a$ is in $S^{-\alpha}(A_x \times A_y)$. Using now the method for construction of a parametrix of an elliptic differential operator, we can choose the terms of the formal series $\sum_{J=0}^{\infty} b_J(x, y, \xi, \eta)$ in such a way that:

i) $b_J \in S^{(-\alpha+J)}(A_x \times A_y)$

ii) we can choose the $\chi_J(\xi, \eta)$ functions in $C^\infty(R_{m+n})$ such that the series $\sum_J b_J \chi_J = b$ is convergent in $S^{-\alpha}(A_x \times A_y)$;

iii) $\sum_{p} (1/p!)\, \partial^{p}_{(\xi,\eta)} a D^{p}_{(x,y)} b \sim g(x,\xi)\varphi(\xi)\chi(y,\eta)$.

To get this result it is enough to put:

$$b_0 = g(x,\xi)\varphi(\xi)\chi(y,\eta)/a(x,y,\xi,\eta)$$

$$ab_1 + \sum_{|p|=1} \partial^{p}_{(\xi,\eta)} a D^{p}_{(x,y)} b_0 = 0$$

$$ab_2 + \sum_{|p|=2} \partial^{p}_{(\xi,\eta)} a D^{p}_{(x,y)} b_0 + \sum_{|p|=2} \partial^{p}_{(\xi,\eta)} a D^{p}_{(x,y)} b_1 + \sum_{|p|=1} \partial^{p}_{(\xi,\eta)} a D^{p}_{(x,y)} b_1 = 0 \qquad \text{etc.}$$

We have then that $b_J \in S^{-(\alpha+J)}(A_x \times A_y)$. We choose $\chi_J(\xi,\eta) \in \mathbb{C}^\infty(R_{m+n})$ as in [2], that is to say let $\chi(\xi,\eta)$ be in $\mathbb{C}^\infty(R_{m+n})$, $\chi(\xi,\eta) = 0$ if $|(\xi,\eta)| \leqslant c/2$ and $|\chi(\xi,\eta)| = 1$ if $|(\xi,\eta)| \geqslant c$; then we can select a sequence $t_J \to \infty$ increasing so rapidly that, if we put $\chi_J(\xi,\eta) = \chi(\xi,\eta/t_J)$, the series $\sum_J b_J \chi_J$ is convergent in $S^{-\alpha}(A_x \times A_y)$.

If we consider the properly supported [3] pseudodifferential operator B whose symbol is $b(x,y,\xi,\eta) (\operatorname{mod} S^{-\infty}(A_x \times A_y))$ and if we form the right compose $A \circ B$, it is easy to see, denoting by $\sigma(A \circ B)$ the symbol of $A \circ B$, that:

$$\sigma(A \circ B) = g(x,\xi)\varphi(\xi)\chi(y,\eta) + s \qquad s \in S^{-\infty}(A_x \times A_y)\,. \tag{6}$$

If we put then $\varphi(\xi) = 1 + \varphi_1(\xi)$ (so that $\operatorname{supp}\varphi_1$ is a compact in R_m) we obtain:

$$\sigma(A \circ B) = \big(g(x,\xi) \otimes 1(y,\eta)\big)\big(1(x,\xi) \otimes \chi(y,\eta)\big) + s_1 \tag{7}$$

where $1(x,\xi)\big(1(y,\eta)\big)$ is the identity function on $A_x \times R_m$ (on $A_y \times R_n$) and s_1 is in $S^{-\infty}(A_x \times A_y)$.

We can now state the following:

THEOREM 1. *Let $A = A(x,y,D_x,D_y) \in \Psi^\alpha(A_x \times A_y)$ be x-partially elliptic in $P_0 \equiv (x_0, y_0, \xi^0, \eta^0) \in S^*(A_x \times A_y)$, $\xi^0 \neq 0$. Let us denote by Ω the open set and by Γ the neighbourhood of definition 1. Let $O = \Omega \times (\Gamma \cap S_{m+n-1})$. Then given any relatively compact open subset O' of O,*

[3] The definition of properly supported pseudodifferential operator is in [1].

there exists $B = B(x, y, D_x, D_y)$ *in* $\Psi^{-\alpha}(A_x \times A_y)$, *such that:*

$$A \circ B \equiv I(x) \otimes G(y, Dy) \bmod \Psi^{-\infty}(O'^*) \tag{8}$$

where $I(x)$ *si the identity operator on* $\mathcal{D}'(A_x)$, $G(y, Dy)$ *is in* $\Psi^{-\infty}(Ay)$ *and the support of* $G(y, Dy)$'s *symbol is compact.*

PROOF. We must show that if $k(x, y, \xi, \eta) \in C_c^\infty(O')$ then:

$$k(x, y, \xi, \eta)[\sigma(A \circ B) - 1(x, \xi) \otimes \chi(y, \eta)] \quad \text{is in } S^{-\infty}(A_x \times A_y)\,.$$

Therefore substituting in the formula $\sigma(A \circ B)$ by its expression, we must have that:

$$k(x, \chi, \xi, \eta)\Big[\big(g(x, \xi) \otimes 1(y, \eta)\big)\big(1(x, \xi) \otimes \chi(y, \eta)\big) + s_1 - 1(x, \xi) \otimes \chi(y, \eta)\Big]$$

is in $S^{-\infty}(A_x \times A_y)$. We choose $g(x, \xi) = 1$ in the set of the (x, ξ) such that it exists (x, ξ) in $\pi_{x,\xi}(O')$. We then get:

$$k(x, y, \xi, \eta)\Big[\big(g(x, \xi) \otimes 1(y, \eta) - 1(x, y, \xi, \eta)\big)\big(1(x, \xi) \otimes \chi(y, \eta)\Big] + \\ + k(x, y, \xi, \eta) s_1(x, y, \xi, \eta)\,.$$

As $k(x,y, \xi, \eta)$ is in $C_c^\infty(O')$ outside O' the first addendum is null; furthermore, because of the choice of $g(x, \xi)$ it is null in O' too and so the first term is always null and the second term is obviously in $S^{-\infty}(A_x \times A_y)$. Q.E.D.

Observe that the construction of an operator B_1, with the same properties of B and such that $B_1 \circ A \equiv I(x) \otimes G(y, D_y) \bmod \Psi^{-\infty}(O'^*)$ can be done simply repeating, step by step, the above construction, obviously substituting $a(x, y, \xi, \eta)$ for $b_J(x, y, \xi, \eta)$ $(J = 0, 1, \ldots)$ and vice-versa. Observe moreover that the B's properties stated in the above theorem depend only formally on choice of the $\chi(y, \eta)$ and $\varphi(\xi)$ functions.

§ **3.** – THEOREM 2. *Let* $A = A(x, y, D_x, D_y) \in \Psi^\alpha(A_x \times A_y)$. *Let* O *an open set in* $S^*(A_x \times A_y)$; *let* O *be equal to* $\Omega \times \Gamma$ *where* Ω *is an open relatively compact set in* $A_x \times A_y$ *and* Γ *is the product of a cone in* R_m, *whose vertex is the origin, with a relatively compact neighbourhood of the origin in* R_n. *Let us assume that* (8) *is true for* A *on* O *for every* $G(y, D_y)$ *whose symbol* $\chi(y, \eta) \in C_c^\infty(A_y \times R_n)$ *and is such that* $\pi_y(\operatorname{supp} \chi) \subseteq \pi_y(\Omega)$,

$\pi_\eta(\operatorname{supp}\chi) \subseteq \pi_\eta(\Gamma)$ *and* $\pi_\eta(\overset{\circ}{\operatorname{supp}}\,\chi)$ *contains the origin in* R_n. *We have then that* A *is* x*-partially elliptic in every point* $P_0(x_0, y_0, \xi^0, \eta^0)$, $\xi^0 \neq 0$, *such that* $P_0'(x_0, y_0, \xi^0, 0)$ *is in* O.

PROOF. From (8) we have:

$$(9) \qquad k(x, y, \xi, \eta)[\sigma(A \circ B) - 1(x, \xi) \otimes \chi(y, \eta)] \quad \text{is in } S^{-\infty}(A_x \times A_y)$$

if $k(x, y, \xi, \eta)$ is in $C_c^\infty(O')$, where O' is an open relatively compact subset of O. As $A \circ B$ is in $\Psi^0(A_x \times A_y)$ and as $ab(x, y, \xi, \eta)$ is the first term of the series which formally determines the symbol of $A \circ B(\operatorname{mod} S^{-\infty}(A_x \times A_y))$ we get:

$$(10) \qquad k(x, y, \xi, \eta)[\sigma(A \circ B) - ab(x, y, \xi, \eta)] \quad \text{is in } S^{-1}(A_x \times A_y)$$

From (9) and (10) we obtain: $k(x, y, \xi, \eta)[ab(x, y, \xi, \eta) - 1(x, \xi) \otimes \chi(y, \eta)]$ is in $S^{-1}(A_x \times A_y)$ that is:

$$(11) \qquad |k(x, y, \xi, \eta)|\,|ab(x, y, \xi, \eta) - 1(x, \xi) \otimes \chi(y, \eta)| \leqslant c/|\xi|$$

if $(x, y) \in K \subset\subset A_x \times A_y$ and $\forall (\xi, \eta) \in R_m \times R_n$.

We now choose $O'' \subseteq O' \subseteq O$, O'' relatively compact in O', such that $|k(x, y, \xi, \eta)| \geqslant \gamma$ in O''. Let $\chi(y, \eta)$ be equal to 1 in $\pi_{y,\eta}(O'')$. Let $\Omega' = \pi_x(O'') \times \pi_y(O'')$. We get: $|ab(x, y, \xi, \eta) - 1| \leqslant c'/|\xi|$ if

$$|k(x, y, \xi, \eta)| \geqslant \gamma\,, \quad (x, y) \in \Omega' \quad (\xi, \eta) \in R_m \times \pi_\eta(O'') \quad \text{and} \quad (x, y, \xi, \eta) \in O''.$$

As $b(x, y, \xi, \eta)$ is in $S^{-\alpha}(A_x \times A_y)$ we have: $|b(x, y, \xi, \eta)| \leqslant c_{\Omega'}|\xi|^{-\alpha}$ if $(x, y) \in \Omega'$ and $\forall |\xi| \geqslant c$, $\eta \in \pi_\eta(O'')$. So for large $|\xi|$ we must have:

$$(12) \qquad |a(x, y, \xi, \eta)| \geqslant c_1 |\xi|^\alpha$$

if $(x, y) \in \Omega'$ $(\xi, \eta) \in \pi_\xi(O'') \times \pi_\eta(O'')$ and this is to say that A is x-partially elliptic in every point $P_0(x_0, y_0, \xi^0, \eta^0)$ such that $P_0' \equiv (x_0, y_0, \xi^0, 0)$ is in O. Q.E.D.

§ **4.** Let us study the connections between the x-partially elliptic pseudodifferential operators and the singularities of distributions.

DEFINITION 4. *Let* $u = u(x, y)$ *be in* $\mathcal{D}'(A_x \times A_y)$ *and* x_0 *a point in* A_x. *We say that* $u(x, y)$ *is regular in* x_0 *if* $\exists$ *a neighbourhood* V *of*

x_0 in A_x such that: $\forall\varphi\in C_c^\infty(A_y)$ the distribution $v(x)=\int u(x,y)\psi(y)\,dy$ is in $C^\infty(V)$.

It is easy to see that if $u(x,y)$ is regular in x_0 for every x_0 in A_x, then $u(x,y)$ is regular in the x-variable according to the characterization in [4].

DEFINITION 5. *Let $u(x,y)$ be in $\mathcal{D}'(A_x\times A_y)$. The set $WF_x(u)$ is the following: $P_0\equiv(x_0,y_0,\xi^0,\eta^0)$, $\xi^0\neq 0$, $P_0\in S^*(A_x\times A_y)$ is not in $WF_x(u)$ if:*

i) *there exists an open relatively compact neighbourhood Ω of (x_0,y_0) in $A_x\times A_y$;*

ii) *there exists a neighbourhood O of $\dot{\xi}_0$ in S_{m-1};*

iii) *there exists a relatively compact neighbourhood A of the origin in R_n such that: $\forall\varphi(x,y)\in C_c^\infty(\Omega)$, $\forall g(x,\xi)\in C_c^\infty(\pi_x(\Omega)\times O)$ and $\forall\chi(y,\eta)\in C_c^\infty(\pi_y(\Omega)\times A)$ we have: $\forall s\in R$ $\exists t=t(s)\in R$:*

$$(1+|\xi|^2)^s(1+|\eta|^2)^{t(s)}|G(x,D_x)\otimes X(y,D_y)(\varphi u)\hat{\ }(\xi,\eta)|^2 \quad \text{is in } L^1(R_{m+n}) \tag{13}$$

if $G(x,D_x)$ and $X(y,D_y)$ are the pseudodifferential operators whose symbols are $g(x,\xi)$ and $\chi(y,\eta)$ respectively.

DEFINITION 6. *If $u\in\mathcal{D}'(A_x,A_y)$, $WF_x(u)$ is called the x-wave-front set of u.*

From now on we shall suppose $u(x,y)$ in $\mathcal{E}'(A_x\times A_y)$, without loss of generality, as it can be easily proved.

THEOREM 3. *Let $A=A(x,y,D_x,D_y)\in\Psi_0^\infty(A_x\times A_y)$* [4]*. Let $u(x,y)$ be in $\mathcal{E}'(A_x\times A_y)$. Then:*

$$WF_x(Au)\subseteq WF_x(u)\,.$$

In the proof of the theorem we shall use the following:

LEMMA. *Let $A=A(x,y,D_x,D_y)\in\Psi_0^\alpha(A_x\times A_y)$. A maps continuously $H_c^{s,t}(A_x\times A_y)$ into $H^{s',t'}(R^{m+n})$, where* if *$\alpha\geq 0$, $s'=s-\alpha$, $t'=t-\alpha$ and if $\alpha<0$ we have at least $s'=s$, $t'=t$ (without proof).*

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[4] By $\Psi_0^\infty(A_x\times A_y)$ we denote the set of properly supported pseudodifferential operators.

PROOF OF THEOREM 3. Let $u(x, y)$ be in $\mathcal{E}'(A_x \times A_y)$. Let $P_0 \equiv (x_0, y_0, \xi^0, \eta^0) \notin WF_x(u)$. From the definition we have that there exists a neighbourhood Ω of (x_0, y_0), a neighbourhood O of $\dot{\xi}_0$ in S_{m-1} and a neighbourhood A of the origin in $\mathbf{R}_n$ such that: $\forall g(x, \xi)$ in $C_c^\infty(\pi_x(\Omega) \times O)$ and $\forall \chi(y, \eta) \in C_c^\infty(\pi_y(\Omega) \times A)$ we have: $\forall s\ \exists t(s) \in R$: $(G(x, D_x) \otimes X(y, D_y))u$ is in $H^{s,t(s)}(R^{m+n})$, if $G(x, D_x)$ and $X(y, D_y)$ are the pseudodifferential operators whose symbols are $g(x, \xi)$ and $\chi(y, \eta)$ respectively. Let us suppose $g(x, \xi) \otimes \chi(y, \eta)$ be equal to 1 in a neighbournood of $(x_0, y_0, \xi^0, \eta^0)$. By the above lemma we get:

$$A\big(G(x, D_x) \otimes X(y, D_y)u\big) \in H^{s',t'}(A_x \times A_y)\,.$$

Let us choose $g'(x, \xi) \otimes \chi'(y, \eta)$ such that its support is contained in the set of the points where $g(x, \xi) \otimes \chi(y, \eta)$ is equal to 1. Then $G'(x, D_x) \otimes X'(y, D_y) A\big(G(x, D_x) \otimes X(y, D_y)\big)u$ is in $H^{s',t'}(R^{m+n})$ as $G'(x, D_x) \otimes X'(y, D_y)$ is in $\Psi_0^0(A_x \times A_y)$; moreover because of the choice of $g'(x, \xi) \otimes \chi'(y, \eta)$ it is exactly $G'(x, D_x) \otimes X'(y, D_y)Au$. If we now choose the neighbourhoods of definition 5 in such a way that our request about the support of $g'(x, \xi) \otimes \chi'(y, \eta)$ is satisfied, we get at once that $P_0 \notin WF_x(Au)$. Q.E.D.

THEOREM 4. *Let $P_0 \equiv (x_0, y_0, \xi^0, \eta^0)$, $\xi^0 \neq 0$ be a point in $S^*(A_x \times A_y)$, let $A = I(x) \otimes G(y, D_y)$ where $I(x)$ is the identity operator on $\mathcal{D}'(A_x)$ and $G(y, D_y)$ has symbol $\chi(y, \eta)$ in $C_c^\infty(A_y \times R_n)$ and $\chi(y, n)$ is equal to 1 in a neighbourhood of $(y_0, 0) \in A_y \times R_n$.*

Let us suppose $P_0 \notin WF_x\big(I(x) \otimes G(y, D_y)u\big)$, then $P_0 \notin WF_x(u)$.

PROOF. By the hypothesis there exists a neighbourhood Ω of (x_0, y_0), a neighbourhood O of $\dot{\xi}_0$ in S_{m-1} and a neighbourhood A of the origin in R_n such that: $\forall g(x, \xi) \in C_c^\infty(\pi_x(\Omega) \times O)$ and $\forall \chi'(y, \eta) \in C_c^\infty(\pi_y(\Omega) \times A)$ we have that $\forall s \in R$, $\exists t(s) \in R$ such that:

$$\big(G(x, D_x) \otimes X'(y, D_y)\big)\big(I(x) \otimes G(y, D_y)\big)u \in H^{s,t(s)}(R^m \times R^n)\,.$$

If we choose now the neighbourhoods of the definition in such a way that the symbol $g(x, \xi) \otimes \chi'(y, \eta)$ have its support contained in the set of points where $1(x, \xi) \otimes \chi(y, \eta)$ is equal to 1, we get that $P_0 \notin WF_x(u)$. Q.E.D.

§ **5.** – From theorem 4 we have the following:

COROLLARY. *Let $A = A(x, y, D_x, D_y) \in \Psi^\alpha(A_x \times A_y)$ be x-partially elliptic in $P_0 \equiv (x_0, y_0, \xi^0, \eta^0)$. Then if $P_0 \notin WF_x(Au)$ it follows that $P_0 \notin WF_x(u)$.*

PROOF. As A is x-partially elliptic there exists a neighbourhood O of $P'_0 \equiv (x_0, y_0, \xi^0, 0)$ such that: $\exists B \in \Psi^{-\alpha}(A_x \times A_y)$ such that:

$$B \circ A \equiv I(x) \otimes G(y, D_y) \bmod \Psi^{-\infty}(O^*)$$

provided that the operator $G(y, D_y)$ has its symbol $\chi(y, \eta)$ in $C_c^\infty(A_y \times R_n)$. As $P_0 \notin WF_x(Au)$ from theorem 3 we get that $P_0 \notin WF_x(B(Au))$ and this is to say that $P_0 \notin WF_x(I(x) \otimes G(y, D_y)u)$. From theorem 4 we obtain that $P_0 \notin WF_x(u)$. Q.E.D.

Assuming now A x-partially elliptic on O, open set in $S^*(A_x \times A_y)$, we have: $WF_x(Au) \cap O = WF_x(u) \cap O$. If $O = S^*(A_x \times A_y)$ we get:

$$WF_x(u) = WF_x(Au)$$

DEFINITION 7. *Let* $A \in \Psi_0^\infty(A_x \times A_y)$. *We call characteristic set of* A *in the x-variable the following set*: $C_{A,x} = S^*(A_x \times A_y) \setminus \bigcup \{O$: O *is an open set in* $S^*(A_x \times A_y)$ *and* A *is x-partially elliptic on* $O\}$.

THEOREM 5. *Let* $u(x, y) \in \mathcal{E}'(A_x \times A_y)$. *Then*:

$$WF_x(u) = \bigcap_A C_{A,x}$$

for every $A \in \Psi_0^\infty(A_x \times A_y)$ *such that* Au *is regular in the x-variable on* A_x

PROOF. If Au is x-regular on A_x we get that $WF_x(Au) = \emptyset$.

Then from the corollary we have: $WF_x(u) \cap O = \emptyset$, for every O where A is x-partially elliptic; so $WF_x(u) \subseteq \cap C_{A,x}$. To complete the proof it is enough to show that given $P_0 \equiv (x_0, y_0, \xi^0, \eta^0)$, $P_0 \notin WF_x(u)$, it exists a pseudodifferential operator A such that:

a) Au is regular in the x-variable on A_x;

b) A is x-partially elliptic in P_0 (that is to say $P_0 \notin C_{A,x}$).

As $P_0 \notin WF_x(u)$ from the definition there exist a neighbourhood Ω of (x_0, y_0), a neighbourhood O of $\dot{\xi}_0$ in S_{m-1}, a neighbourhood A of the origin in R_n such that: $\forall g(x, \xi) \in C_c^\infty(\pi_x(\Omega) \times O)$, $\forall \chi(y, \eta)$ in $C_c^\infty(\pi_y(\Omega) \times A)$ the distribution $G(x, D_x) \otimes X(y, D_y)u$ (if $G(x, D_x)$ and $X(y, D_y)$ are the operators whose symbols are $g(x, \xi)$ and $\chi(y, \eta)$ respectively) is regular in the x-variable on A_x (according to the characterization in [4]). Let us prove that the operator $G(x, D_x) \otimes$

$\otimes X(y, D_y)$ is x-partially elliptic in P_0, and this will be the last step. As $G(x, D_x) \otimes X(y, D_y)$ is in $\Psi^0(A_x \times A_y)$ we must prove that there exist a neighbourhood Ω' of (x_0, y_0) and a neighbourhood Γ' of $(\xi_0, 0)$ such that

$$|g(x, \xi) \otimes \chi(y, \eta)| \geqslant c$$

if $(x, y) \in \Omega$ $(\xi, \eta) \in \Gamma$ and $|\xi| \geqslant c_1$.

To do that we can choose $\Omega' \subseteq \Omega$ and $\Gamma' \subseteq \Gamma$ such that $g(x, \xi) \otimes \otimes \chi(y, \eta) \neq 0$ in $\Omega' \times \Gamma'$. Q.E.D.

REMARK 3. *If $P(D_x, D_y)$ is a differential operator with constant coefficients and if it is partially elliptic in the x-variable according to definition* 1, *then $P(D_x, D_y)$ is x-partially elliptic according to definition* 2 *in every point (ξ_0, η_0), $\xi_0 \neq 0$. Definition* 2 *is however more extensive*:

The operator $P(D_x, D_y)$ such that $P(\xi, \eta) = \xi^2 + \eta$ which is not partially elliptic according to definition 1 (see [4]), is x-partially elliptic according to definition 2. As a matter of fact suppose $\xi^0 \neq 0$, say $\xi^0 > 0$, and consider the neighbourhood V of $(1, 0)$, defined as follows:

$$V = \{(\xi, \eta) \colon \xi > 0, \ |\eta| < \tfrac{1}{2}\}\,.$$

We now get at once

$$|\xi|^2 \leqslant 2|\xi^2 + \eta|$$

if $|\xi| \geqslant 1$ and $(\xi, \eta) \in V$.

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