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Connections on 1-jets principal fiber bundles

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0. Introduction.

Let $l = (E, X, \pi)$ be a differentiable fiber bundle. It is well-known that, both in the finite and infinite-dimensional context, the set $\bar{E} = J^1(E)$ *of* 1-*jets of sections of* l admits a differentiable fiber bundle structure $p: \bar{E} \to E$. If $l = (E, G, X, \pi)$ is a principal fiber bundle, then P. Garcia [3] proved that $\bar{l} = (\bar{E}, G, \bar{E}/G, \bar{\pi})$ is a principal fiber bundle with its connections being in 1-1 correspondence with the equivariant sections of p.

In addition, there exists on $\bar{E}$ a $\mathfrak{g}$-valued 1-form θ, the so-called *structure* 1-*form*, which is the connection form of a canonical connection on $\bar{l}$ and satisfying the following universal property: for every connection σ of l (regarded as a section of p) and its corresponding form θ^σ, equality $\theta^\sigma = \sigma^* \cdot \theta$ holds.

The aim of the present note is twofold:

I) to show that the main results of [3] are valid within the context of Banachable principal bundles. In doing so, we give in section 2 the infinite-dimensional version of the main points of [3], using methods of Banachable vector bundles.

II) to show that the above mentioned result on the universal property of θ can be reversed, so that an « iff » condition can be stated. Roughly speaking: each connection σ with corresponding form θ^σ in-

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duces a uniquely determined connection on $\bar{l}_{\sigma(E)}$, with connection form θ satisfying $\theta^\sigma = \sigma^* \cdot \theta$, for every σ and θ^σ. *θ turns out to be precisely the structure* 1-*form.* This is described in section 3.

The key to our approach is the notion of (f, φ, h)-related connections, briefly studied in section 1 (cf. also [6; p. 40]).

Manifolds and bundles are modelled on Banach spaces and, for the sake of simplicity, differentiability is of class C^∞ (smoothness). We mainly follow the terminology and notations of [1], [5], although we try to be as close as possible to [3], which is our main source of motivation.

1. Related connections.

Let $l = (E, G, X, \pi)$ be a principal fiber bundle (p.f.b. for short) and let $\mathfrak{g}$ be the Lie algebra of G.

DEFINITION 1.1. *A* (*smooth*) *connection on l is a smooth splitting of the* (*direct*) *exact sequence of vector bundles* (v.b. *for short*)

$$(*) \qquad 0 \to E \times \mathfrak{g} \xrightarrow{v} TE \xrightarrow{T\pi!} \pi^*(TX) \to 0$$

that is, there exists either a G-v.b.-morphism $c\colon \pi^*(TX) \to TE$ such that $T\pi! \circ c = \mathrm{id}_{\pi^*(TX)}$, or a G-v.b.-morphism $V\colon TE \to E \times \mathfrak{g}$ such that $V \circ v = \mathrm{id}_{E\times\mathfrak{g}}$. Here $T\pi!$ is the v.b.-morphism defined by the universal property of pull-backs and v is given by $v(p, A) = T^2_{(p,1)}\delta(A)$, if δ denotes the (right) action of G on P and $\mathfrak{g} \cong T_1 G$ (1 is the identity element of G). V and c are G-v.b.-morphisms in the sense they preserve the natural action of G on the corresponding bundles (cf. [7; *p.* 67]).

As well-known, $TE = VE \oplus HE$, where $VE = \mathrm{Im}(v)$ and $HE = \mathrm{Im}(c)$ are respectively the *vertical* and *horizontal* subbundles of TE. Hence, each vector $u \in T_e E$ has the unique expression $u = u^v + u^h$, with the exponents v and h denoting vertical and horizontal parts respectively.

DEFINITION 1.2. *Let $l = (E, G, X, \pi)$ and $\bar{l} = (\bar{E}, \bar{G}, \bar{X}, \bar{\pi})$ be two p.f.b's and let (f, φ, h) be a p.f.b-morphism of l into $\bar{l}$. Two connections c and $\bar{c}$, on l and $\bar{l}$ respectively, are said to be (f, φ, h)-related iff $Tf \circ c = \bar{c} \circ (f \times Th)$.*

If $\tilde{\varphi}$ is the Lie algebra homomorphism induced by φ, and ω, $\bar{\omega}$ the connection forms of c and $\bar{c}$ respectively, then we can prove (cf. [9], [10; *p.* 76]):

THEOREM 1.3. *The following conditions are equivalent*:

1) c and $\bar{c}$ are (f, φ, h)-related

2) $$\bar{V} \circ Tf = (f \times \tilde{\varphi}) \circ V$$

3) $$Tf(u^h) = (Tf(u))^{\bar{h}}$$

4) $$Tf(u^v) = (Tf(u))^{\bar{v}}$$

5) $$\tilde{\varphi} \cdot \omega \quad = f^* \cdot \bar{\omega}\,.$$

The main result of this section is to show that each connection c on l and each p.f.b.-morphism of l into $\bar{l}$ induce a properly related connection $\bar{c}$ on $\bar{l}$. We need the following lemma proved in [7; p. 66].

LEMMA 1.4. *Let $\boldsymbol{B}, \boldsymbol{E}, \boldsymbol{F}$ be Banach spaces and let U be an open subset of $\boldsymbol{B}$. If $f\colon U \to L(\boldsymbol{E}, \boldsymbol{F})$ is a map such that, for each $e \in \boldsymbol{E}$, each map $x \mapsto f(x)\, e$ is of class C^∞, then f is of class C^∞.*

PROPOSITION 1.5. *Let $\pi\colon E \to X$ and $\bar{\pi}\colon \bar{E} \to X$ be two v.b's over the same base X and (g, id_X) a v.b.-morphism of $\bar{\pi}$ into π. If in addition, for each $x \in X$, there exists $f_x \in L(E_x, \bar{E}_x)$ such that $g_x \circ f_x = \mathrm{id}_{E_x}$, then the couple (f, id_X), with $f\colon E \to \bar{E}$ given by $f|E_x := f_x$, is a v.b-morphism of π into $\bar{\pi}$.*

PROOF. In virtue of [5; p. 37] it is sufficient to prove *VBM*-2. Since (g, id_X) is a v.b.-morphism, for every x_0 there is a neighborhood U of x_0 such that

$$U \ni x \mapsto r_x \circ g_x \circ \bar{r}_x^{-1} \in L(\bar{\boldsymbol{E}}, \boldsymbol{E})$$

is a smooth map (the letter r is used to denote local trivializations). By the assumption, the restriction of g_x on $\mathrm{Im}(f_x)$ is a toplinear isomorphism $g_x\colon f_x(E_x) \to E_x$ with f_x as its inverse. Hence, for each $u \in \boldsymbol{E}$ there exists $\bar{u} \in \bar{\boldsymbol{E}}$ with $\bar{r}_x \circ g_x \circ r_x^{-1}(\bar{u}) = u$; thus the map

$$U \ni x \mapsto \bar{r}_x \circ f_x \circ r_x^{-1}(u) \in \bar{\boldsymbol{E}}$$

is smooth, as having constant value $\bar{u}$. The late together with the previous Lemma imply the smoothness of

$$U \ni x \mapsto \bar{r}_x \circ f_x \circ r_x^{-1} \in L(\boldsymbol{E}, \boldsymbol{E})$$

which completes the proof. ●

THEOREM 1.6. *Let $l = (E, G, X, \pi)$ and $\bar{l} = (\bar{E}, \bar{G}, \bar{X}, \bar{\pi})$ be two p.f.b.'s and let (f, φ, h) be a p.f.b.-morphism of l into $\bar{l}$ with h being local diffeomorphism. Then every connection c on l determines a unique (f, φ, h)-related connection $\bar{c}$ on $\bar{l}$.*

PROOF. Let $\bar{e}$ be an arbitrary element of $\bar{E}$ with $\bar{\pi}(\bar{e}) = \bar{x}$. We can always find $e \in E$ and $\bar{s} \in \bar{G}$ such that $f(e) = \bar{e}\cdot\bar{s}$. If $\pi(e) = x$, then $h(x) = \bar{x}$. Using the fact that $T_x h\colon T_x X \to T_{\bar{x}}\bar{X}$ is a toplinear isomorphism, we define the continuous linear map $\bar{c}(\bar{e}, \cdot)\colon T_{\bar{x}}\bar{X} \to T_{\bar{e}}\bar{E}$ given by

$$\bar{c}(\bar{e}, \cdot)(\bar{v}) := (T_{\bar{e}} R_{\bar{s}} \circ T_e f \circ c)\cdot(e, T_{\bar{x}}\bar{h}(\bar{v})) \tag{1.1}$$

where $R_{\bar{s}}$ denotes the right translation (by $\bar{s}$) on $\bar{E}$ and $\bar{h}$ is the inverse of h. The above map is independent of the choice of e and $\bar{s}$, for a given $\bar{e}$. We define the global map $\bar{c}\colon \pi^*(T\bar{X}) \to T\bar{E}$ by $\bar{c}(\bar{e}, \bar{v}) = \bar{c}(\bar{e}, \cdot)(\bar{v})$. Since $T\bar{\pi}! \circ \bar{c} = \mathrm{id}_{\bar{\pi}^*(T\bar{X})}$, in virtue of Prop. 1.5, we conclude that $\bar{c}$ is the right splitting morphism of a connection on $\bar{l}$.

Setting $\bar{e} = f(e)$ (which implies that $\bar{s}$ is the identity element of $\bar{G}$) we see that (1.1) yields

$$[\bar{c}\circ(f \times Th)]\cdot(e, v) = \bar{c}\big(f(e), T_x h(v)\big) = (Tf \circ c)\cdot(e, v)$$

which proves the relatedness of $\bar{c}$ and c.

Finally, $\bar{c}$ is unique, for if c' is another (f, φ, h)-related with c connection, then Def. 1.2 implies that

$$c' \circ (f \times Th) = \bar{c} \circ (f \times Th) \tag{1.2}$$

Now, for an arbitrary $(\bar{e}, \bar{v})$ in $\bar{\pi}^*(T\bar{X})$, we have $\bar{v} \in T_{\bar{x}}\bar{X}$, where $\bar{x} = \bar{\pi}(\bar{e})$. As before, there are $e \in E$ and $\bar{s} \in G$ with $e = f(e)\cdot\bar{s}$. If $x = \pi(e)$, then $h(x) = \bar{x}$ and the local diffeomorphism determines $v \in T_x X$ with $\bar{v} = T_x h(v)$. Hence, taking into account the invariance of c' and $\bar{c}$, (1.2) implies

$$c'(\bar{e},\bar{v}) = c'\big(f(e)\cdot\bar{s},\, T_xh(v)\big) = c'\big(f(e),\, T_xh(v)\big)\cdot\bar{s} = \\ = \bar{c}\big(f(e),\, T_xh(v)\big)\cdot\bar{s} = \bar{c}\big(f(e)\cdot\bar{s},\, T_xh(v)\big) = \bar{c}(\bar{e},\bar{v})$$

and the proof is complete. ●

As in the finite case, a connection on l can be determined by an appropriate connection form ω on P, satisfying the well-known properties. Here we set $\omega_e(u) = \mathrm{pr}_2\circ V\cdot(u)$, with $e\in E$ and $u\in T_eE$. In virtue of Def. 1.1, we easily obtain the following formula:

$$u - c\big(\mathcal{T}_E(u),\, T\pi(u)\big) = T^1_{(e,1)}\delta\cdot\big(\omega_e(u)\big) \tag{1.3}$$

for every $e\in E$ and $u\in T_eE$, and with $\mathcal{T}_E$ denoting the projection of the tangent bundle $TE\to E$. The previous formulas (1.1) and (1.3), applied for the connection form $\bar{\omega}$ of the connection $\bar{c}$ of Theorem 1.6, give the following useful formula:

$$\bar{\omega}_{\bar{e}}(\bar{u}) = (T^1_{(\bar{e},1)}\delta)^{-1}\cdot\Big(\bar{u} - (T_{\bar{e}}R_{\bar{s}}\circ T_ef\circ c)\cdot\big(e,\, T\bar{h}\circ T\bar{\pi}(\bar{u})\big)\Big) \tag{1.4}$$

where $\bar{e} = f(e)\cdot\bar{s}$.

2. The 1-jet principal fiber bundle.

In this section we briefly present P. Garcia's [3] results needed for the purpose of the note, and modified according to our infinite-dimensional point of view.

Let $1 = (E, G, X, \pi)$ be a Banachable p.f.b. If s is a section of 1, we denote by $\bar{s}_x$ the 1-*jet of* s at x. The fiber bundle structure of the 1-*jet bundle of sections of* l, denoted by $\bar{E}$, is standard (cf. [3], [2], [8]). The fact we are dealing with principal bundles implies the following:

THEOREM 2.1. *G acts on the right of $\bar{E}$ freely and differentiably by*

$$\bar{E}\times G \ni (\bar{s}_x, a) \mapsto \bar{s}_x\cdot a := (\overline{s\cdot a})_x \in \bar{E}$$

and the quadruple $\bar{l} = (\bar{E}, G, \bar{E}/G, \bar{\pi})$ is a p.f.b.

PROOF. The finite-dimensional proof of [3; p. 232-234] is easily extended to our context. ●

If $p: \bar{E} \to E$ is the canonical p.f.b-morphism given by $p(\bar{s}_x) = s(x)$, then we have:

THEOREM 2.2. *For each* p.f.b. $l = (E, G, X, \pi)$, *there exists a bijective correspondence of the set of sections of* $p: \bar{E} \to E$ *onto the set of connections of* l.

PROOF. The proof of [3; p. 236] has the following infinite-dimensional version, involving v. b-morphisms.

Let σ be an arbitrary section of p. If s is a section such that $\sigma(e) = \bar{s}_x^e$ $(\pi(e) = x)$, we set $c(e, \cdot) = T_x s$ and we define $c: \pi^*(TX) \to TE$ by $c(e, v) = c(e, \cdot)\cdot(v)$. Since $T\pi ! \circ c = \mathrm{id}_{\pi^*(TX)}$ we conclude that, in virtue of Def. 1.1 and Prop. 1.5, c is a connection.

Conversely, for a given connection c we define the section $\sigma: E \to \bar{E}$ of p as follows: for each $e \in E$ there exists a section with prescribed differential equal to $c(e, \cdot)$ *i.e.* we can find a section s^e passing through e such that $T_x s^e = c(e, \cdot)$. We set $\sigma(e) := \bar{s}_x^e$. The smoothness of σ derives from the local structure of E.

Finally, the desired bijectivity is checked as in [3]. ●

DEFINITION 2.3. *The* 1-*form* θ *defined by*

$$\theta: \bar{E} \ni \bar{e} \rightsquigarrow \theta_{\bar{e}}: T_{\bar{e}}\bar{E} \ni \bar{u} \mapsto \theta_{\bar{e}}(\bar{u}) := Tp(\bar{u}) - T(s \circ \pi)\cdot\big(Tp(\bar{u})\big) \in V_e E$$

is called the structure 1-*form of* $\bar{E}$.

NOTES. I) In the above definition we have set $\bar{e} = \bar{s}_x^e$ and, for the sake of simplicity, we have omitted the subscript of differentials.

II) Since each vertical space $V_e E$ is identified with the Lie algebra $\mathfrak{g}$ (cf. Def. 1.1), θ *can be considered as a* $\mathfrak{g}$-*valued form on* E.

The following result is also in [3; p. 238]. We sketch here a simple proof, involving v.b.-morphisms and the conventions of our framework.

THEOREM 2.4. θ *is a differentiable* 1-*form which defines a connection on* $\bar{l}$ *and satisfies the universal property for all the connections* σ*'s of* l *and their connection forms* θ^σ.

PROOF. As in the proof of Theorem 2.2, we construct the v.b.-morphism $c: \pi^*(TX) \to TE$ by $c(e, v) = T_{\pi(e)} s(v)\cdot c$ defines a connection on l corresponding to some σ. Hence, the map

$$\omega: E \ni e \rightsquigarrow \omega_e: T_e E \ni u \mapsto \omega_e(u) := u - c\big(e, T\pi(u)\big)$$

is a differential 1-form on E and with values in VE, since it is induced by the v.b.-morphism

$$\mathrm{id}_{TE} - c\circ(\mathrm{id}_E \times T\pi)\colon TE \to VE$$

(cf. [1; p. 81]). Moreover, identifying VE with $E\times \mathfrak{g}$, we see that ω is the connection form of c (equivalently of σ), thus we can write $\omega = \theta^\sigma$. Definition 2.3 implies that $\theta = p^*\cdot\theta^\sigma$ on $\sigma(E)$; hence, θ is a $\mathfrak{g}$-valued connection form on $\bar{l}$, by local arguments.

Also, for the given σ (or c) we have that

$$\sigma^*\cdot\theta = \sigma^*\cdot(p^*\cdot\theta^\sigma) = (p\circ\sigma)^*\cdot\theta^\sigma = \theta^\sigma\,.$$

The above procedure is valid for each connection σ, so the proof is complete. ●

Since E is a p.f.b., each section $\sigma\colon E\to\bar{E}$ is a p.f.b.-morphism and defines a differentiable map $\sigma_G\colon X\to\bar{E}/G$ so that the following diagram is commutative

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \bar{E} \arrow[d, "\bar{\pi}"] \\ X \arrow[r, "\sigma_G"] & \bar{E}/G \end{tikzcd}$$

σ_G turns out to be a section of $p_G\colon \bar{E}/G\to X$ (cf. [3; p. 234]).

Under the above notations and the terminology of section 1, we have:

Theorem 2.4 (Restated). *The structure* 1*-form* θ *induces on* $\bar{l}$ *a canonical connection which is* $(\sigma, \mathrm{id}_G, \sigma_G)$*-related with every connection* σ *of* l.

3. The main result.

Theorem 3.1. *There exists on each* $\bar{l}_{\sigma(E)}$ *a unique connection, which is* $(\sigma, \mathrm{id}_G, \sigma_G)$*-related with each connection* σ *on* l. *The corresponding connection form is precisely the structure* 1*-form* θ *and satisfies the universal equality* $\theta^\sigma = \sigma^*\cdot\theta$, *for every connection* σ *on* l.

Proof. Let c be an arbitrary connection on l, which corresponds to the section σ and has connection form θ^σ. As a consequence of

Theorem 1.6, there exists a unique connection, say $\bar{c}$, on $\bar{l}_{\sigma(E)}$ which is $(\sigma, \mathrm{id}_G, \sigma_G)$-related with c. Let us denote by θ the connection form of $\bar{c}$. Under the present data, (1.4) takes the form

$$(3.1)\quad \theta_{\bar{s}_x}(\bar{u}) = (T^1_{(\bar{s}_x,1)}\bar{\delta})^{-1}\cdot\Big(\bar{u} - \big(T_{s(x)}(\sigma)\circ c\big)\cdot\big(s(x),\, T\sigma_G^{-1}\circ T\bar{\pi}\cdot(\bar{u})\big)\Big).$$

Indeed, this is the case, for if $\bar{s}_x$ is an arbitrary element of $\sigma(E)$, there exists $e\in E$ with $\bar{s}_x = \sigma(e)$. Hence, $e = s(x)$ and the element $\bar{s}\in\bar{G}$ of Theorem 1.6 is now the identity 1 of G. Setting

$$T^1_{(\bar{s}_x,1)}\bar{\delta} = \bar{v}_{\bar{s}_x}$$

(cf. Def. 1.1) and taking into account the commutativity of the diagram

$$\begin{tikzcd} \bar{E} \arrow[r, "p"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ \bar{E}/G \arrow[r, "p_G"] & X \end{tikzcd}$$

we write (3.1) as

$$(3.2)\qquad \theta_{\bar{s}_x}(\bar{u}) = \bar{v}_{\bar{s}_x}^{-1}\cdot\Big(\bar{u} - \big(T_{(sx)}(\sigma)\circ c\big)\cdot\big(s(x),\, T(\pi\circ p)\cdot(\bar{u})\big)\Big).$$

On the other hand, we easily check that the p.f.b.-morphism p satisfies $Tp\circ\bar{v} = v\circ(p\times\mathrm{id}_g)$; thus, since $\bar{v}$ and v are toplinear isomorphisms on the fibers, we have that

$$\bar{v}_{\bar{s}_x}^{-1} = v_{s(x)}^{-1}\circ T_{\bar{s}_x}p$$

and (3.2) yields:

$$\theta_{\bar{s}_x}(\bar{u}) = v_{s(x)}^{-1}\Big(T_{\bar{s}_x}p(\bar{u}) - \big(T_{\bar{s}_x}p\circ T_{s(x)}(\sigma)\circ c\big)\cdot\big(s(x),\, T(\pi\circ p)\cdot(\bar{u})\big)\Big)$$

or

$$\theta_{\bar{s}_x}(\bar{u}) = v_{s(x)}^{-1}\cdot\Big(T_{\bar{s}_x}p(\bar{u}) - c\big(s(x),\, T_{s(x)}\pi\big(T_{\bar{s}_x}p(\bar{u})\big)\big)\Big)$$

or, in virtue of Theorem 1.6

$$\theta_{\bar{s}_x}(\bar{u}) = v_{s(x)}^{-1}\cdot\Big(T_{\bar{s}_x}p(\bar{u}) - T_x s\big(T_{\bar{s}_x}p(\bar{u})\big)\Big).$$

Since $v_{s(x)}$ is exactly the identification $g\cong V_{s(x)}E$, we see that θ is precisely the structural 1-form.

By the $(\sigma, \mathrm{id}_G, \sigma_G)$-relatedness we have that $\theta^\sigma = \sigma^* \cdot \theta$ if θ^σ is the connection form of σ (or c). The same argument assures the uniqueness of θ. Finally, since the above construction of θ is valid for all σ's, we see that $\sigma^\sigma = \sigma^* \cdot \theta$ is universally satisfied. ●

The above Theorem, compared with Theorem 2.4 (restated), can be considered as its converse; hence, for the sake of completeness we state the following:

THEOREM 3.2. *Let θ be a $\mathfrak{g}$-valued differentiable 1-form on $\bar{E}$. Then the following conditions are equivalent:*

I) *$\theta|_{\sigma(E)}$ is the structural 1-form on $\sigma(E)$.*

II) *For each connection σ of l (regarded as a section of $p\colon \bar{E} \to E$), the corresponding connection form θ^σ satisfies $\theta^\sigma = \sigma^* \cdot \theta$.*

III) *θ is the connection form of a uniquely determined connection on $\bar{l}$, which is $(\sigma, \mathrm{id}_G, \sigma_G)$-related with every connection σ of l.*

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