RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

S. ZAIDMAN

The weak Cauchy problem for abstract differential equations

Rendiconti del Seminario Matematico della Università di Padova, tome 56 (1976), p. 1-21

<http://www.numdam.org/item?id=RSMUP_1976__56__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 56 (1977)

The Weak Cauchy Problem for Abstract Differential Equations.

S. ZAIDMAN (*)

Introduction.

We consider the weak Cauchy problem in arbitrary Banach space for equations $(d/dt - A)u = 0$, as were defined by Kato-Tanabe. After proving some elementary relationships, we obtain a result which shows how uniqueness of Cauchy problem for strong solutions in the second dual space implies uniqueness of the weak Cauchy problem.

A simple result by Barbu-Zaidman (Notices A.M.S., April 1973, 73T-B120) gets then a new proof, and an uniqueness result for weakened solutions by Liubic-Krein gets a partial extension.

The last result is a certain extension of Barbu-Zaidman result to non-reflexive B-spaces, using as a main tool in the proof Phillips's theorem on dual semi-groups in B-spaces.

§ 1. – Let $\mathfrak{X}$ be a given Banach space, and $\mathfrak{X}^*$ be its dual space. If A is a linear closed operator with dense domain $\mathfrak{D}(A) \subset \mathfrak{X}$, mapping $\mathfrak{D}(A)$ into $\mathfrak{X}$, then, the dual operator A^* is defined on the set $\mathfrak{D}(A^*) = \{x^* \in \mathfrak{X}^*$, s.t. $\exists y^* \in \mathfrak{X}^*$, satisfying

$$(1.1) \qquad \langle x^*, Ax \rangle = \langle y^*, x \rangle \qquad \forall x \in \mathfrak{D}(A)\} .$$

(*) Indirizzo dell'A.: Dépt. de Math., Université de Montréal - Case Postale 6128 - Montréal, Canada.

This research is supported through a grant of the N.R.C., Canada.

By definition $A^*x^* = y^*$, and A^* is a well-defined linear operator from $\mathfrak{D}(A^*)$ into $\mathfrak{X}^*$.

Furthermore, the domain $\mathfrak{D}(A^*)$ is a total set in $\mathfrak{X}^*$; this means that, given any element $x \in \mathfrak{X}$, $x \neq \theta$, $\exists x^* \in \mathfrak{D}(A^*)$, such that $x^*(x) \neq 0$; hence, if $x^*(x) = 0$, $\forall x^* \in \mathfrak{D}(A^*)$, then $x = \theta$.

Let now be given a finite interval $-\infty < a < b < +\infty$ on the real axis; a class of « test-functions » associated to the operator $\mathfrak{D}(A^*)$ and to the given interval, denoted here by $K_{A^*}[a, b)$, consists of continuously differentiable functions $\phi^*(t)$, $a \leqslant t \leqslant b \to \mathfrak{X}^*$, which are $= \theta$ near b (that is $\phi^*(t) = \theta$ for $b - \delta \leqslant t \leqslant b$, where δ depends on ϕ^*); furthermore, $\phi^*(t)$ belong to $\mathfrak{D}(A^*)$, $\forall t \in [a, b]$, and $(A^*\phi^*)(t)$ is $\mathfrak{X}^*$-continuous on $[a, b]$.

Obviously, if $\varphi(t)$ is scalar-valued, $C^1[a, b]$-function, and $\varphi = 0$ near b, and if ϕ^* is any element in $\mathfrak{D}(A^*)$, then $\varphi(t)\phi^*$ belongs to $K_{A^*}[a, b)$.

Let us consider now the Bochner space $L^p_{\mathrm{loc}}([a, b); \mathfrak{X})$, where p is any real $\geqslant 1$, consisting of strongly measurable $\mathfrak{X}$-valued functions f defined on $[a, b)$, such that $\int_a^c \|f(t)\|^p_{\mathfrak{X}}\, dt < \infty$ for any $c < b$.

The weak forward Cauchy problem is here defined as follows: given any element $u_a \in \mathfrak{X}$ and any function $f(t) \in L^p_{\mathrm{loc}}([a, b); \mathfrak{X})$, find a function $u(t) \in L^p_{\mathrm{loc}}([a, b); \mathfrak{X})$ verifying

$$(1.2) \qquad -\langle \varphi^*(a), u_a \rangle - \int_a^b \left\langle \frac{d\varphi^*}{dt}, u(t) \right\rangle dt = \int_a^b \langle (A^*\varphi^*)(t), u(t) \rangle\, dt + \int_a^b \langle \varphi^*(t), f(t) \rangle\, dt\,, \qquad \forall \varphi^* \in K_{A^*}[a, b)\,.$$

In similar way define a weak backward Cauchy problem: The class $K_{A^*}(a, b]$ is defined like $K_{A^*}[a, b)$, with the only difference that the test-functions must be null near a, instead of being null near b.

There is also the space $L^p_{\mathrm{loc}}((a, b]; \mathfrak{X})$ of $\mathfrak{X}$-mesurable functions such that $\int_c^b \|f\|^p_{\mathfrak{X}}\, dt < \infty$, $\forall c > a$, $c < b$.

Then given any element $u_b \in \mathfrak{X}$, and again $f \in L^p_{\mathrm{loc}}((a, b]; \mathfrak{X})$, find

$u(t) \in L^p_{\text{loc}}((a, b]; \mathfrak{X})$, satisfying

$$(1.3)\qquad -\langle \varphi^*(b), u_b\rangle - \int_a^b \left\langle \frac{d\varphi^*}{dt}, u(t)\right\rangle dt = \int_a^b \langle (A^*\varphi^*)(t), u(t)\rangle \,dt + \int_a^b \langle \varphi^*(t), f(t)\rangle \,dt , \qquad \forall \varphi^* \in K_{A^*}(a, b] .$$

Remark. This definitions are slightly more general than the weakened Cauchy problem as defined for example in S. G. Krein [3]; extending his definition from the interval $[0, T]$ to an arbitrary interval $[a, b]$, we say that $u(t)$, $a \leqslant t \leqslant b \to \mathfrak{X}$ is a weakened solution of

$$(1.4)\qquad \dot{u}(t) = Au(t) + f(t) , \qquad u(a) = u_a \in \mathfrak{X} ,$$

where $f(t)$, $a \leqslant t \leqslant b \to \mathfrak{X}$ is $\mathfrak{X}$-continuous, if: $u(t)$ is $\mathfrak{X}$-continuous on the closed interval $[a, b]$; $u(t)$ is $\mathfrak{X}$-differentiable with continuous derivative on the half-open interval $(a, b]$; $u(t) \in \mathfrak{D}(A)$ on same $(a, b]$;

$$u'(t) = Au(t) + f(t) , \qquad a < t \leqslant b , \ u(a) = u_0 .$$

The following result holds:

Proposition 1.1. *If $u(t)$ is a weakened solution of* (1.4), *then* (1.2) *is also verified.*

Consider the equality $u'(t) - Au(t) = f(t)$, valid on the half-open interval $a < t \leqslant b$. Then take any test-function $\phi^*(t) \in K_{A^*}[a, b)$ (ϕ^* is null near b). We get obviously

$$(1.5)\qquad \langle \phi^*(t), u'(t)\rangle - \langle \phi^*(t), Au(t)\rangle = \langle \phi^*(t), f(t)\rangle , \qquad a < t \leqslant b .$$

Also we see that

$$(1.6)\qquad \frac{d}{dt}\langle \phi^*(t), u(t)\rangle = \left\langle \frac{d\phi^*}{dt}(t), u(t)\right\rangle + \left\langle \phi^*(t), \frac{du}{dt}(t)\right\rangle , \qquad a < t \leqslant b .$$

If we integrate (1.6) between $a+\varepsilon$ and b, $\forall\varepsilon>0$, we obtain

$$-\langle\phi^*(a+\varepsilon), u(a+\varepsilon)\rangle=\int_{a+\varepsilon}^{b}\{\langle\dot\phi^*(t), u(t)\rangle+\langle\phi^*(t), \dot u(t)\rangle\}\,dt\,.$$

Let us integrate now (1.5) between $a+\varepsilon$ and b, and remark also that $\langle\phi^*(t), Au(t)\rangle=\langle A^*\phi^*(t), u(t)\rangle$, $a<t\leqslant b$. We get

$$\int_{a+\varepsilon}^{b}\langle\phi^*(t), u'(t)\rangle\,dt=-\langle\phi^*(a+\varepsilon), u(a+\varepsilon)\rangle-\int_{a+\varepsilon}^{b}\langle\dot\phi^*(t), u(t)\rangle\,dt=$$

$$=\int_{a+\varepsilon}^{b}\langle A^*\phi^*(t), u(t)\rangle\,dt+\int_{a+\varepsilon}^{b}\langle\phi^*(t), f(t)\rangle\,dt\,.$$

By continuity of all functions here involved, one obtains, when $\varepsilon\to0$

$$-\langle\phi^*(a), u(a)\rangle-\int_{a}^{b}\langle\dot\phi^*(t), u(t)\rangle\,dt-\int_{a}^{b}\langle A^*\phi^*(t), u(t)\rangle\,dt=\int_{a}^{b}\langle\phi^*(t), f(t)\rangle\,dt\,.$$

A converse result is also given in the following.

PROPOSITION 1.2. *Let us assume*: $f(t)$, $a\leqslant t\leqslant b\to\mathfrak{X}$, *be strongly continuous*; $u_a\in\mathfrak{X}$ *be arbitrarily given. Then* $u(t)$, $a\leqslant t\leqslant b\to\mathfrak{X}$ *be a* $\mathfrak{X}$*-continuous function, which is continuously differentiable for* $a<t\leqslant b$, *and belongs to* $\mathfrak{D}(A)$ *for* $t\in(a, b]$. *Let also* (1.2) *be satisfied. Then it follows that* $u'-Au=f$ *on* $a<t\leqslant b$, *and also* $u(a)=u_a$.

In order to prove this simple fact, we shall first introduce in (1.2) test-functions of the special form $\phi^*(t)=\nu(t)x^*$ where $x^*\in\mathfrak{D}(A^*)$ and $\nu(t)$ is scalar-valued continuously differentiable function which $=0$ near a and near b. It results then, if $[a_1, b_1]\subset(a, b)$ contains $\operatorname{supp}\phi^*$

$$(1.7)\qquad -\int_{a_1}^{b_1}\langle\dot\phi^*(t), u(t)\rangle\,dt=\int_{a_1}^{b_1}\langle(A^*\phi^*)(t), u(t)\rangle\,dt+\int_{a_1}^{b_1}\langle\phi^*(t), f(t)\rangle\,dt\,.$$

As $u(t)$ is continuously differentiable on $[a_1, b_1]$, and $\phi^*(a_1)=\phi^*(b_1)=0$ (intervals (a, a_1), (b_1, b) are in the null set of ϕ^*: hence, by continuity,

$\phi^*(a_1) = \phi^*(b_1) = \theta$ also), it results

$$-\int_{a_1}^{b_1} \langle \dot{\phi}^*(t), u(t) \rangle \, dt = \int_{a_1}^{b_1} \langle \phi^*(t), \dot{u}(t) \rangle \, dt .$$

Also because $u(t) \in \mathfrak{D}(A)$ for $t \in [a_1, b_1]$, it is $\langle \phi^*(t), Au(t) \rangle = \langle A^* \phi^*(t), u(t) \rangle$. Hence, relation (1.7) becomes

$$\int_{a_1}^{b_1} \langle \phi^*(t), \dot{u}(t) \rangle \, dt = \int_{a_1}^{b_1} \langle \phi^*(t), (Au)(t) \rangle \, dt + \int_{a_1}^{b_1} \langle \phi^*(t), f(t) \rangle \, dt \tag{1.8}$$

or, as $\phi^*(t)$ is here $= \nu(t) x^*$,

$$\int_{a_1}^{b_1} \langle x^*, u'(t) \rangle \nu(t) \, dt = \int_{a_1}^{b_1} \langle x^*, (Au)(t) \rangle \nu(t) \, dt + \int_{a_1}^{b_1} \langle x^*, f(t) \rangle \nu(t) \, dt \tag{1.9}$$

or

$$\int_{a_1}^{b_1} \langle x^*, u'(t) - Au(t) - f(t) \rangle \nu(t) \, dt = 0 . \tag{1.10}$$

By continuity of the scalar function $\langle x^*, u'(t) - Au(t) - f(t) \rangle$ in $[a_1, b_1]$, letting $\nu(t)$ to vary, we get $\langle x^*, u'(t) - Au(t) - f(t) \rangle = 0$ in $[a_1, b_1]$, $\forall x^* \in \mathfrak{D}(A^*)$.

(If $\int_{\alpha}^{\beta} \phi(t) \nu(t) \, dt = 0$, $\forall \nu \in C_0^1(\alpha, \beta)$, $\phi \in C[\alpha, \beta] \Rightarrow \phi = 0$ on (α, β); if not, $\exists \xi \in (\alpha, \beta)$, $\phi(\xi) > 0$ say; in $(\xi - \delta, \xi + \delta)$, $\phi > 0$; take $0 \leqslant \nu$, $\nu = 1$ on $(\xi - \delta/2, \xi + \delta/2)$, $= 0$ outside $(\xi - \delta, \xi + \delta)$, $\in C^1$; then

$$\int_{\alpha}^{\beta} \phi \nu \, dt = \int_{\xi - \delta}^{\xi + \delta} \phi \nu \, dt > \int_{\xi - \delta/2}^{\xi + \delta/2} \phi \, dt > 0 , \qquad \text{absurde} .$$

If $\phi = 0$ on (α, β), $\Rightarrow \phi = 0$ on $[\alpha, \beta]$.)

Now, if we fix $t \in [a_1, b_1]$, and vary x^* over the total set $\mathfrak{D}(A^*)$, we get $u'(t) = Au(t) + f(t)$.

This is true for any $t \in [a_1, b_1]$, hence for any $t \in (a, b)$ too. But $u'(t)$, $f(t)$, hence $Au(t)$ are continuous on $t = b$; so we obtain $u'(b) = Au(b) + f(b)$ also to be valid.

We still must prove that $u(a) = u_a$.

Consider again the relation (1.2), for general test-functions $\phi^*(t) \in K_{A^*}[a, b)$. Take an arbitrary small $\varepsilon > 0$, and get

$$- \langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \dot{\phi}^*(t), u(t) \rangle \, dt - \int_{a+\varepsilon}^{b} \langle \dot{\phi}^*(t), u(t) \rangle \, dt =$$

$$= \int_a^{a+\varepsilon} \langle (A^*\phi^*)(t), u(t) \rangle \, dt + \int_{a+\varepsilon}^{b} \langle (A^*\phi^*)(t), u(t) \rangle \, dt + \int_a^{b} \langle \phi^*(t), f(t) \rangle \, dt$$

we have also,

$$\int_{a+\varepsilon}^{b} \langle \dot{\phi}^*(t), u(t) \rangle \, dt = \int_{a+\varepsilon}^{b} \frac{d}{dt} \langle \phi^*(t), u(t) \rangle \, dt - \int_{a+\varepsilon}^{b} \langle \phi^*(t), \dot{u}(t) \rangle \, dt =$$

$$= - \langle \phi^*(a+\varepsilon), u(a+\varepsilon) \rangle - \int_{a+\varepsilon}^{b} \langle \phi^*(t), \dot{u}(t) \rangle \, dt \, ,$$

and

$$\int_{a+\varepsilon}^{b} \langle (A^*\phi^*)(t), u(t) \rangle \, dt = \int_{a+\varepsilon}^{b} \langle \phi^*(t), Au(t) \rangle \, dt \, ;$$

so, we get

$$- \langle \phi^*(a), u_a \rangle - \int_a^{a+\varepsilon} \langle \dot{\phi}^*(t), u(t) \rangle \, dt + \langle \phi^*(a+\varepsilon), u(a+\varepsilon) \rangle +$$

$$+ \int_{a+\varepsilon}^{b} \langle \phi^*(t), \dot{u}(t) \rangle \, dt = \int_{a+\varepsilon}^{b} \langle \phi^*(t), Au(t) \rangle \, dt + \int_a^{b} \langle \phi^*(t), f(t) \rangle \, dt + \int_a^{a+\varepsilon} \langle A^*\phi^*, u \rangle \, dt \, .$$

But $\dot{u}(t) = Au(t) + f(t)$ on $a + \varepsilon \leqslant t \leqslant b$, as was proved above. Hence, it remains

$$-\langle \phi^*(a), u_a\rangle - \int_a^{a+\varepsilon} \langle \dot\phi^*(t), u(t)\rangle\, dt + \langle \phi^*(a+\varepsilon), u(a+\varepsilon)\rangle =$$

$$= \int_a^{a+\varepsilon} \langle A^*\phi^*(t), u(t)\rangle\, dt + \int_a^{a+\varepsilon} \langle \phi^*(t), f(t)\rangle\, dt\,.$$

If now let $\varepsilon \to 0$, it remains only, using continuity of u on $[a, b]$, that

$$\langle \phi^*(a), u(a) - u_a\rangle = 0\,, \qquad \forall \phi^* \in K_{A^*}[a, b)\,.$$

We can now take $\phi^*(t) = v_0(t)x^*$, where $v_0(t) \in C^1[a, b]$, equals 1 near a, and $= 0$ near b, and $x^* \in \mathfrak{D}(A^*)$. Hence

$$\langle x^*, u(a) - u_a\rangle = 0 \qquad \forall x^* \in \mathfrak{D}(A^*)$$

which is a total set in $\mathfrak{X}^*$, and again, it will be $u(a) = u_a$. Q.E.D.

§ 2. – In this section we shall prove that uniqueness of Cauchy problem for strong solutions on an interval $[a, b]$ in the second dual space $\mathfrak{X}^{**}$, implies uniqueness of the weak Cauchy problem in the same interval, in the original space $\mathfrak{X}$.

If A is linear, closed operator with dense domain in the B-space $\mathfrak{X}$, we saw that the dual operator A^* is linear, defined on a total set in $\mathfrak{X}^*$; also A^* is closed on this set; in fact, let $x_n^* \in \mathfrak{D}(A^*)$, $x_n^* \to x_0^* \in \mathfrak{X}^*$, $A^*x_n^* \to y_0^* \in \mathfrak{X}^*$. From relations $\langle x_n^*, Ax\rangle = \langle A^*x_n^*, x\rangle$, $\forall x \in \mathfrak{D}(A)$, we get, as $n \to \infty \langle x_0^*, Ax\rangle = \langle y_0^*, x\rangle$, $\forall x \in \mathfrak{D}(A)$. Hence, by definition of A^*, it is $x_0^* \in \mathfrak{D}(A^*)$, $A^*x_0^* = y_0^*$, so A^* is closed.

Let us assume from now on the supplementary.

HYPOTHESIS. *A^* is an operator with dense domain in $\mathfrak{X}^*$.*

(REMARK. This holds allways when $\mathfrak{X}$ is a reflexive B-space; the proof is similar to a classical one in Hilbert spaces).

Then, the second dual operator $A^{**} = (A^*)^*$ will be a well defined operator on a total set $\mathfrak{D}(A^{**}) \subset \mathfrak{X}^{**}$ the second dual space of $\mathfrak{X}$. More precisely $\mathfrak{D}(A^{**}) = \{\psi^{**} \in \mathfrak{X}^{**}$, such that $\exists x^{**} \in \mathfrak{X}^{**}$, satisfying relation $\langle \psi^{**}, A^*\phi^*\rangle = \langle x^{**}, \phi^*\rangle$, $\forall \phi^* \in \mathfrak{D}(A^*)\}$ and if $\psi^{**} \in \mathfrak{D}(A^{**})$,

$A^{**}\psi = x^{**}$. We also know the existence of a canonical map J: $\mathfrak{X} \to \mathfrak{X}^{**}$, which is linear and isometric; precisely, any element $x \in \mathfrak{X}$ defines a linear continuous functional f^{**} on $\mathfrak{X}^*$, by: $\langle f^{**}, x^* \rangle = \langle x^*, x \rangle$, $\forall x^* \in \mathfrak{X}^*$. Then put $Jx = f^{**}$, so that $\langle x^*, x \rangle = \langle Jx, x^* \rangle$, $\forall x^* \in \mathfrak{D}(A^*)$. Let now $u(t)$, $a \leqslant t \leqslant b$, be a $C^1[a, b; \mathfrak{X}]$ function such that $u(t) \in \mathfrak{D}(A)$, $\forall t \in [a, b]$ and $u'(t) = Au(t)$ on $[a, b]$. This is a strong solution on $[a, b]$, and $u(a)$ belongs necessarily to $\mathfrak{D}(A)$. Then $(Ju)(t)$ is a $C^1[a, b; \mathfrak{X}^{**}]$ function, as easily seen, and $(d/dt)(Ju) = J(du/dt)$. We prove now following

THEOREM 2.1. *Let us assume that for any function $u(t) \in C^1([a, b], \mathfrak{X})]$ such that*

i) $(Ju)(t) \in \mathfrak{D}(A^{**})$, $a \leqslant t < b$,

ii) $(d/dt)(Ju) - A^{**}(Ju) = 0$ on $[a, b)$,

iii) $(Ju)(a) = \theta$,

it is $(Ju)(t) = \theta$, $\forall t \in [a, b]$. Then, there is unicity of the forward weak Cauchy problem on $[a, b]$.

PROOF. What we must prove is the following: $v(t) \in L^p_{\text{loc}}([a, b); \mathfrak{X})$, and

$$(2.1) \qquad -\int_a^b \langle \dot\phi^*(t), v(t) \rangle \, dt = \int_a^b \langle (A^*\phi^*)(t), v(t) \rangle \, dt \,, \qquad \forall \phi^* \in K_{A^*}[a, b) \,,$$

implies $v(t) = \theta$ almost everywhere on $[a, b]$.

Now, using a suggestion by professor S. Agmon (in Pisa, Italy), we start by extending $v(t)$ to $(-\infty, b)$, as follows: $\tilde{v}(t) = v(t)$ for $a \leqslant t < b$, $\tilde{v}(t) = \theta$ for $-\infty < t < a$. It holds now the following

LEMMA 2.1. *The extended function $\tilde{v}(t)$ verifies the integral identity*

$$(2.2) \qquad \int_{-\infty}^b \langle \dot\psi^*(t) + (A^*\psi^*)(t), \tilde{v}(t) \rangle \, dt = 0$$

for any function $\psi^*(t)$, $-\infty < t \leqslant b \to \mathfrak{X}^*$, continuously differentiable there, such that $\psi^*(t) \in \mathfrak{D}(A^*)$, $\forall t \in (-\infty, b]$, $A^*\psi^*(t)$ is $\mathfrak{X}^*$-continuous; support ψ^* is compact in $(-\infty, b)$ (i.e. $\psi^* = \theta$ near b and near $-\infty$).

In fact, (2.2) is the same as

$$\int_a^b \langle \dot{\psi}^*(t) + (A^*\psi^*)(t), v(t)\rangle \, dt = 0 \,. \tag{2.3}$$

But the restriction to $[a, b]$ of the above considered test function $\psi^*(t)$ is obviously in the class $K_{A^*}[a, b)$, (because it was null near b, and had all regularity required properties).

Hence, by (2.1), the lemma is proved.

A second, needed result (already announced in our paper [6]) is as follows:

Take any scalar function $\alpha_\varepsilon(t) \in C^1(-\infty, \infty)$, which $= 0$ for $|t| \geqslant \varepsilon$; for any $w(t) \in L^p_{\text{loc}}(-\infty, b; \mathfrak{X})$ ($\mathfrak{X}$-mesurable on $(-\infty, b)$, such that $\int_\alpha^\beta \|w\|^p_{\mathfrak{X}} \cdot$ $\cdot dt < \infty$, $\forall \alpha > -\infty$, $\beta > \alpha$, $\beta < b$), we can consider the mollified function

$$(w * \alpha_\varepsilon)(t) = \int_{t-\varepsilon}^{t+\varepsilon} w(\tau)\, \alpha_\varepsilon(t-\tau)\, d\tau$$

which is well-defined for $-\infty < t < b - \varepsilon$, is strongly continuously differentiable, and

$$\frac{d}{dt}(w^* \alpha_\varepsilon) = \int_{t-\varepsilon}^{t+\varepsilon} w(\tau)\, \dot{\alpha}_\varepsilon(t-\tau)\, d\tau \,, \qquad -\infty < t < b - \varepsilon \,.$$

We have

LEMMA 2.2. *If* $w(t) \in L^p_{\text{loc}}(-\infty, b; \mathfrak{X})$ *verifies the integral identity*:

$$\int_{-\infty}^b \langle \dot{\psi}^*(t) + (A^*\psi^*)(t), w(t)\rangle \, dt = 0 \tag{2.4}$$

$\forall \psi^*$ *as in Lemma* 2.1, *then, it is* $J(w * \alpha_\varepsilon) \in \mathfrak{D}(A^{**})$, *and* $(d/dt) J(w * \alpha_\varepsilon) = A^{**}(J(w * \alpha_\varepsilon))$ *holds*, $\forall t \in (-\infty, b - \varepsilon)$ *where* J *is the canonical map of* $\mathfrak{X}$ *in* $\mathfrak{X}^{**}$.

Take in fact any fixed $t_0 \in (-\infty, b-\varepsilon)$, and consider then the functions $\psi^*_{t_0,\varepsilon}(t) = \alpha_\varepsilon(t_0 - t)x^*$, where $x^* \in \mathfrak{D}(A^*)$. These are good test functions because $\alpha_\varepsilon(t_0 - t) = 0$ for

$$|t - t_0| \geqslant \varepsilon\,, \qquad \text{hence in any case,} \qquad \alpha_\varepsilon(t_0 - t) = 0$$

near b and near $-\infty$.

There is also $(d/dt)\psi^*_{t_0,\varepsilon} = -\dot{\alpha}_\varepsilon(t_0 - t)x^*$. Writing now (2.4), we get

$$\int_{-\infty}^{b} \langle \dot{\alpha}_\varepsilon(t_0 - t)x^*, w(t)\rangle\, dt = \int_{-\infty}^{b} \alpha_\varepsilon(t_0 - t)\langle A^*x^*, w(t)\rangle\, dt$$

or also

$$\left\langle A^*x^*, \int_{-\infty}^{b} \alpha_\varepsilon(t_0 - t)\, w(t)\, dt \right\rangle = \left\langle x^*, \int_{-\infty}^{b} \dot{\alpha}_\varepsilon(t_0 - t)\, w(t)\, dt \right\rangle, \qquad \forall x^* \in \mathfrak{D}(A^*)\,,$$

or

$$\langle A^*x^*, (w^*\alpha_\varepsilon)(t_0)\rangle = \langle x^*, (w^*\alpha_\varepsilon)'(t_0)\rangle\,, \qquad \forall x^* \in \mathfrak{D}(A^*)\,.$$

Here, if we introduce the canonical imbedding operator J, we have:

$$\langle J(w^*\alpha_\varepsilon)(t_0), A^*x^*\rangle = \langle J(w^*\alpha_\varepsilon)'(t_0), x^*\rangle\,, \qquad \forall x^* \in \mathfrak{D}(A^*)\,.$$

Now if we use definition of $\mathfrak{D}(A^{**})$ and of A^{**}, we see that $J(w^*\alpha_\varepsilon)\cdot$ $\cdot(t_0) \in \mathfrak{D}(A^{**})$, and

$$A^{**}\big(J(w * \alpha_\varepsilon)(t_0)\big) = J(w * \alpha_\varepsilon)'(t_0) = \frac{d}{dt} J(w * \alpha_\varepsilon)(t_0)$$

which is the desired Lemma 2.2.

We pass now to the final steps of the proof.

Take $w(t) = \tilde{v}(t)$ the function used in Lemma 2.1; as $v(t) \in L^p_{\text{loc}}\cdot$ $\cdot([a, b); \mathfrak{X})$ and $\tilde{v} = \theta$ for $t < a$, it is obvious that

$$\tilde{v}(t) = w(t) \in L^p_{\text{loc}}[(-\infty, b); \mathfrak{X}]\cdot$$

$$\cdot\left(\int_{\alpha}^{\beta} \|w(t)\|^p\, dt = \int_{a}^{\beta} \|v(t)\|^p\, dt < \infty \text{ for } \beta < b,\ \alpha < a\right).$$

Let us apply Lemma 2.2 to $\tilde{v}(t)$. We obtain that $(\tilde{v}*\alpha_\varepsilon)(t)$ is well-defined on $-\infty<t<b-\varepsilon$ where is continuously differentiable; also $J(\tilde{v}*\alpha_\varepsilon)\in\mathfrak{D}(A^{**})$ and

$$\text{(2.5)}\qquad \frac{d}{dt}\big(J(\tilde{v}*\alpha_\varepsilon)\big)=A^{**}\big(J(\tilde{v}*\alpha_\varepsilon)\big)\qquad \text{holds on } -\infty<t<b-\varepsilon\,.$$

Remark also that $(\tilde{v}*\alpha_\varepsilon)(t)=\theta$ for $t\leqslant a-\varepsilon$, because it is

$$(\tilde{v}*\alpha_\varepsilon)(t)=\int_{t-\varepsilon}^{t+\varepsilon}\tilde{v}(\tau)\,\alpha_\varepsilon(t-\tau)\,d\tau$$

and $\tilde{v}(\tau)=\theta$ for $a-\varepsilon\leqslant\tau\leqslant a$.

Hence, (2.5) holds on $a-\varepsilon\leqslant t\leqslant b-\varepsilon$, and also $(\tilde{v}*\alpha_\varepsilon)(a-\varepsilon)=\theta$ so $J(\tilde{v}*\alpha_\varepsilon)(a-\varepsilon)=\theta$.

Now, if $(\tilde{v}*\alpha_\varepsilon)(t)=Z(t)$, we see that, in the space $\mathfrak{X}^{**}$, it is:

$$(JZ)'(t)=A^{**}JZ(t)\qquad \text{on } [a-\varepsilon,\,b-\varepsilon),\ \text{and } JZ(a-\varepsilon)=\theta\,.$$

Put then $t=\sigma-\varepsilon$ and $Z(t)=Z(\sigma-\varepsilon)=u(\sigma)$; when $a-\varepsilon\leqslant t\leqslant b-\varepsilon$, we get $a-\varepsilon\leqslant\sigma-\varepsilon\leqslant b-\varepsilon$, or $a\leqslant\sigma\leqslant b$; also $Ju'(\sigma)=JZ'(t)$, so that $(Ju)'(\sigma)=A^{**}(Ju)(\sigma)$ in $\mathfrak{X}^{**}$, $a\leqslant\sigma<b$, and $Ju(a)=(JZ)(a-\varepsilon)=\theta$. Applying the hypothesis of the theorem, it follows that $u(t)=\theta$ on $[a,b]$, hence, $Z(t)=\theta$ on $[a-\varepsilon,\,b-\varepsilon]$, that is

$$(\tilde{v}*\alpha_\varepsilon)(t)=\theta\qquad\text{on}\qquad[a-\varepsilon,\,b-\varepsilon]\,.$$

Now, take a sequence of functions $\alpha_n(t)$ which are non-negative, $=0$ for $|t|\geqslant 1/n$, continuously differentiable, such that

$$\int_{-1/n}^{1/n}\alpha_n(\sigma)\,d\sigma=1\,.$$

We obtain then, in the usual way, as for scalar-valued functions, the relation:

$$\lim_{n\to\infty}\int_{a_1}^{b_1}\|v(t)-(\tilde{v}*\alpha_n)(t)\|^p\,dt=0\,,\qquad \forall b_1<b,\ a_1>a\,.$$

But, for n big enough, $b-1/n>b_1$; so $\tilde{v}*\alpha_n=\theta$ on $[a_1, b_1]$, hence

$$\int_a^{b_1}\|v(t)\|^p\,dt=0 \qquad \forall b_1<b \Rightarrow v(t)=\theta \text{ a.e. on } (a, b)\,.$$

§ 3. – We shall give now some applications of Theorem 2.1. To start with, we give a proof of the following result (see [2]). « Let $\mathfrak{X}$ be a reflexive B-space; A be the infinitesimal generator of a strongly continuous semi-group of class C_0; A^* be the dual operator to A. Let $u(t)$, $0\leqslant t\leqslant T\to\mathfrak{X}$ be a strongly continuous function, verifying the integral identity

$$\int_0^T\langle\dot{\varphi}^*(t)+(A^*\varphi^*)(t), u(t)\rangle\,dt=0 \tag{3.1}$$

for any function $\varphi^*(t)$, $0\leqslant t\leqslant T\to\mathfrak{X}^*$, which is continuously differentiable in $\mathfrak{X}^*$, belongs to $\mathfrak{D}(A^*)$, $\forall t\in[0, T]$, $(A^*\varphi^*)$ is $\mathfrak{X}^*$-continuous, $0\leqslant t\leqslant T$, and $\varphi^*(t)$ is null near 0 and near T. Let also be $u(0)=\theta$; then $u(t)=\theta$, $0\leqslant t\leqslant T$. »

Let us remark first that A is linear closed with dense domain in $\mathfrak{X}$ as any generator of a C_0 semi-group. By reflexivity of $\mathfrak{X}$ (which means, as usual, that $J(\mathfrak{X})=\mathfrak{X}^{**}$), it follows that $\mathfrak{D}(A^*)$ is dense in $\mathfrak{X}^*$, and that $A^{**}(Jx)=J(Ax)$, $\forall x\in\mathfrak{D}(A)$, and $J(\mathfrak{D}(A))=\mathfrak{D}(A^{**})$, (see [9]).

We shall see now that hypothesis i)-ii)-iii) of Theorem 2.1 are verified.

Take hence $u(t)\in C^1\{[0, T];\mathfrak{X}\}$; assuming that $Ju\in\mathfrak{D}(A^{**})=J(\mathfrak{D}(A))$ means: $\forall t\in[0, T]$, $\exists v(t)\in\mathfrak{D}(A)$, such that $Jv(t)=Ju(t)$; as J^{-1} exists, $\Rightarrow v(t)=u(t)$; hence $u(t)\in\mathfrak{D}(A)$, $0\leqslant t\leqslant T$. Also, $A^{**}\cdot(Ju(t))=J(Au(t))$; We assumed in ii) that $(d/dt)Ju-A^{**}(Ju)=\theta$ on $[0, T]$. But $(d/dt)Ju=J(du/dt)$, as $u\in C^1\{[0, T];\mathfrak{X}\}$. Hence ii) becomes $J(du/dt)-J(Au)=\theta$ on $[0, T]$ which implies $u'-Au=\theta$ on $[0, T]$.

Furthermore iii) implies obviously that $u(0)=\theta$, again because J^{-1} exists $(\mathfrak{X}^{**}\to\mathfrak{X})$.

Now, the well-known unicity result for strong solutions of $(d/dt-A)w=0$ when A is generator of *a* C_0-semi-group (see for example [7], theorem 2.2.2) implies that $u(t)=\theta$ on $[0, T]$, so $Ju(t)=\theta$ on $[0, T]$ too. Hence, all conditions of theorem 2.1 are fulfilled, and by now we can conclude that:

If the relation

$$\int_0^T \langle \dot{\phi}^*(t) + (A^*\phi^*)(t), u(t)\rangle\, dt = 0$$

holds $\forall \phi \in K_{A^*}[0, T)$, *then* $u = \theta$ on $[0, T]$ (in fact, u-continuous is in L^p_{loc}, and $u = \theta$ a.e. on $[0, T] \Rightarrow u = \theta$ everywhere on $[0, T]$). Hence, it remains to check precisely that

$$\text{(3.2)} \qquad \int_0^T \langle \dot{\phi}^*(t) + (A^*\phi^*)(t), u(t)\rangle\, dt = 0 \qquad \forall \phi^* \in K_{A^*}[0, T)\,.$$

Remember that our hypothesis here is slightly different: we assume in fact that it is

$$\text{(3.3)} \qquad \int_0^T \langle \dot{\phi}^*(t) + (A^*\phi^*)(t), u(t)\rangle\, dt = 0$$

for test-functions regular as those in $K_{A^*}[0, T)$ but null near 0 as well as near T, which forms a subclass of $K_{A^*}[0, T)$ (denoted usually as $K_{A^*}(0, T)$). We added however the condition $u(0) = \theta$. So, it remains to prove that (3.2) holds.

Take henceforth an arbitrary $\phi^*(t) \in K_{A^*}[0, T)$. Then consider, for any $\varepsilon > 0$, a scalar-valued function $\nu_\varepsilon(t) \in C^1[0, T]$, which $= 0$ for $0 \leqslant t \leqslant \varepsilon$, and $= 1$ for $2\varepsilon \leqslant t \leqslant T$, satisfying also an estimate $|\dot{\nu}_\varepsilon(t)| \leqslant c/\varepsilon$, $0 \leqslant t \leqslant T$.

Then the product $\nu_\varepsilon(t)\phi^*(t)$ is also $= \theta$ near $t = 0$, so it is in the subclass of admissible here test-functions. We get from (3.3) the following equality

$$\text{(3.4)} \qquad \int_0^T \langle \dot{\nu}_\varepsilon \phi^* + \nu_\varepsilon \dot{\phi}^* + \nu_\varepsilon A^*\phi^*, u\rangle\, dt = 0\,, \quad \forall \varepsilon > 0\,,\ \phi^* \in K_{A^*}[0, T)\,.$$

Obviously (3.4) reduces to the following

$$\int_\varepsilon^{2\varepsilon} \dot{\nu}_\varepsilon \langle \phi^*, u\rangle\, dt + \int_\varepsilon^{2\varepsilon} \langle \nu_\varepsilon \dot{\phi}^*, u\rangle\, dt + \int_{2\varepsilon}^T \langle \dot{\phi}^*, u\rangle\, dt +$$
$$+ \int_\varepsilon^{2\varepsilon} \nu_\varepsilon \langle A^*\phi^*, u\rangle\, dt + \int_{2\varepsilon}^T \langle A^*\phi^*, u\rangle\, dt = 0\,, \quad \forall \varepsilon > 0\,,\ \forall \phi^* \in K_{A^*}[0, T)\,.$$

Now, for $\varepsilon \to 0$, the first integral is estimated as

$$\left| \int_{\varepsilon}^{2\varepsilon} \dot{\nu}_\varepsilon \langle \phi^*, u \rangle \, dt \right| \leqslant \frac{c}{\varepsilon} \sup_{\varepsilon \leqslant t \leqslant 2\varepsilon} |\langle \phi^*, u \rangle| \cdot \varepsilon ;$$

as $u(0) = \theta$, $u(t) \to \theta$ when $t \to 0$, hence $\sup_{\varepsilon \leqslant t \leqslant 2\varepsilon} |\langle \phi^*, u \rangle| \leqslant K \sup_{\varepsilon \leqslant t \leqslant 2\varepsilon} \|u(t)\| \to 0$ with ε. The other integrals containing ε are easily handled so that we obtain

$$\int_0^T \langle \dot{\phi}^*, u \rangle \, dt + \int_0^T \langle A^* \phi^*, u \rangle \, dt = 0 \, , \qquad \forall \phi^* \in K_{A^*}[0, T)$$

which finishes our proof.

REMARK. The original proof of [2] was given using the adjoint semi group theory in reflexive spaces in a very natural way. We shall see later on a similar proof for the non-reflexive case (§ 5).

§ 4. – We shall deal here with the following unicity result for weakened solutions (see [3], Theorem 3.1, p. 81):

« Let be A a linear operator in the B-space $\mathfrak{X}$, such that $R(\lambda; A) = (\lambda - A)^{-1} \in \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$ for λ real $\geqslant \lambda_0$, and

$$\overline{\lim_{\lambda \to +\infty}} \frac{\ln \|R(\lambda)\|}{\lambda} = h_A < \infty \, .$$

Let $u(t)$ be a weakened solution of $u' - Au = \theta$ on the interval $0 \leqslant t \leqslant T$, such that $u(0) = \theta$, and assume also that $h_A < T$. Then $u(t) = \theta$ in $0 \leqslant t \leqslant T - h_A$. »

A slight generalization is possible, replacing $[0, T]$ by an arbitrary real interval $[a, b]$.

THEOREM 4.1. *Under the same hypothesis on A, and if $h_A < b - a$, any weakened solution $u(t)$ of $u' - Au = 0$ on $a \leqslant t \leqslant b$, such that $u(a) = \theta$, is $= \theta$ on $a \leqslant t \leqslant b - h_A$.*

We can in fact take $T = b - a$ in the above theorem; so if $u(0) = \theta$, we get $u(t) = \theta$ on $[0, b - a - h_A]$.

To prove theorem 4.1, let us put $u(t+a)=u_a(t)$; it maps the interval $0\leqslant t\leqslant b-a$ into $\mathfrak{X}$. Also it is $\dot{u}_a(t)=u'(t+a)=Au(t+a)=$ $=Au_a(t)$ for $0<t\leqslant b-a$.

Hence $u_a(t)$ is weakened solution on $0\leqslant t\leqslant b-a$, and $u_a(0)=$ $=u(a)=\theta$; so, $u_a(t)=\theta$ on $0\leqslant t\leqslant b-a-h_A$ that is $u(t+a)=\theta$ for $0\leqslant t\leqslant b-a-h_A$, hence for $a\leqslant t+a\leqslant b-h_A$, which gives $u(t)=\theta$ for $a\leqslant t\leqslant b-h_A$.

Now we shall see a partial extension of Theorem 4.1 in general B-spaces, taking weak solutions instead of weakened. Precisely, we propose ourselves to prove the following

THEOREM 4.2. *Let A be a linear operator in the B-space $\mathfrak{X}$, such that $(\lambda-A)^{-1}\in\mathfrak{L}(\mathfrak{X},\mathfrak{X})$ for λ real $\geqslant\lambda_0$ and assume also that*

$$\overline{\lim_{\lambda\to\infty}}\frac{\ln\|R(\lambda;A)\|}{\lambda}=h_A<\infty\,.$$

Let also be $\mathfrak{D}(A^*)$ a dense subset of $\mathfrak{X}^*$, and $\mathfrak{D}(A)$ be dense in $\mathfrak{X}$ (*). *Assume finally that*

$$\int_a^b\langle\phi^*+A^*\phi^*,u\rangle\,dt=0$$

$\forall\phi^*\in K_{A^*}[a,b)$, *where* $u\in L^p_{\text{loc}}([a,b);\mathfrak{X})$. Then, $u=\theta$ *a.e. on* $a\leqslant t\leqslant$ $\leqslant b-h_A$, *provided* $h_A<b-a$.

Let us start the proof by remembering Phillips's fundamental results (see [4], [8]) concerning resolvents of dual operators.

« Let T be linear closed operator with dense domain $\mathfrak{D}(T)\subset\mathfrak{X}$, and T^* be its dual operator (acting on a total set in $\mathfrak{X}^*$, $\mathfrak{D}(T^*)$). Then the resolvent sets $\varrho(T)$ and $\varrho(T^*)$ coincide; also, for any $\lambda\in\varrho(T)$, it is $(R(\lambda;T))^*=R(\lambda;T^*)$. »

Apply this result to our operator A which is linear closed in $\mathfrak{X}$, because we assume that $R(\lambda;A)$ exists $\in\mathfrak{L}(\mathfrak{X},\mathfrak{X})$ for $\lambda\geqslant\lambda_0$, λ real, and $\mathfrak{D}(A)$ is dense by hypothesis. We obtain that for λ real $\geqslant\lambda_0$,

---

(*) The existence of $(\lambda-A)^{-1}\in\mathfrak{L}(\mathfrak{X},\mathfrak{X})$ does not implies in general, that $\mathfrak{D}(\lambda-A)=\mathfrak{D}(A)$ is dense in $\mathfrak{X}$.

It suffices to consider $\mathfrak{X}=C[0,1]$; $A=d^2/dx^2$ defined on functions in $C^2[0,1]$ which vanish for $x=0$ and $x=1$. Considering the equation $u''=f$, $\forall f\in C[0,1]$, we find a unique solution $u\in\mathfrak{D}(A)$, depending continuously on f. However, $\mathfrak{D}(A)$ is not dense in $\mathfrak{X}$.

$R(\lambda, A^*)$ also $\in \mathfrak{L}(\mathfrak{X}^*, \mathfrak{X}^*)$, and $R(\lambda; A^*) = [R(\lambda; A)]^*$. We know also that $\|[R(\lambda; A)]^*\| = \|[R(\lambda; A)]\|$ hence $\|R(\lambda; A^*)\| = \|R(\lambda; A)\|$ and consequently

$$\overline{\lim_{\lambda \to \infty}} \frac{\ln \|R(\lambda; A^*)\|}{\lambda} = h_A \text{ too} .$$

Now, $\mathfrak{D}(A^*)$ is also dense in $\mathfrak{X}^*$, and A^* is closed. It follows that $R(\lambda; A^{**}) \in \mathfrak{L}(\mathfrak{X}^{**}, \mathfrak{X}^{**})$, $\forall \lambda$ real $\geqslant \lambda_0$, and for these λ, $\|R(\lambda; A^{**})\| = \|R(\lambda; A\|$ so,

$$\overline{\lim_{\lambda \to \infty}} \frac{\ln \|R(\lambda; A^{**})\|}{\lambda} = h_A < \infty \text{ too}.$$

Now we shall apply theorem 2.1 on the interval $a \leqslant t \leqslant b - h_A$. Let us consider consequently a function $u(t) \in C^1[a, b - h_A; \mathfrak{X}]$, such that $Ju \in \mathfrak{D}(A^{**})$, $a \leqslant t \leqslant b - h_A$, $(d/dt)(Ju) - A^{**}(Ju) = 0$ on $a \leqslant t \leqslant b - h_A$, and $(Ju)(a) = \theta$.

Let us apply now theorem 4.1 taking A^{**} instead of A which is possible by the above (remarking also that here the solutions are strong which is better than weakened). It follows that $Ju(t) = \theta$ on $a \leqslant t \leqslant b - h_A$. Hence theorem 2.1 is applicable on $[a, b - h_A]$ and we get uniqueness of weak solutions, as desired.

§ 5. – In this section we present a variant of the unicity result considered in § 3, which is valid in more general, non-reflexive B-spaces.

Let us start by remembering Phillips's theorem on dual semi-groups (see [4], [5], [8]).

Consider in the B-space $\mathfrak{X}$, a linear closed operator A with domain $\mathfrak{D}(A)$ dense in $\mathfrak{X}$, and assume that A generates a semi-group of class (C_0) of linear continuous operators $T(t)$, $0 \leqslant t < \infty \to \mathfrak{L}(\mathfrak{X}, \mathfrak{X})$.

Now, as previously, the dual operator A^* of A is a closed linear transformation on $\mathfrak{D}(A^*) \subset \mathfrak{X}^*$ to $\mathfrak{X}^*$. We know that $\mathfrak{D}(A^*)$ is a total set in $\mathfrak{X}^*$, but in general $\mathfrak{D}(A^*)$ is not dense in $\mathfrak{X}^*$ so that A^* is not necessarily the infinitesimal generator of a strongly continuous semi-group in $\mathfrak{X}^*$.

Therefore it is convenient to consider the so called $\odot$-dual space $\mathfrak{X}^\odot$ of $\mathfrak{X}$, defined by $\mathfrak{X}^\odot = \overline{\mathfrak{D}(A^*)}$ (closure in $\mathfrak{X}^*$). In the case of reflexive $\mathfrak{X}$, we have $\mathfrak{X}^\odot = \mathfrak{X}^*$, else $\mathfrak{X}^\odot$ may be a proper subset of $\mathfrak{X}^*$.

Let us define now the operator $A^\odot$ to be the restriction of the dual operator A^* to the domain

(5.1) $\mathfrak{D}(A^\odot) = [x^* \in \mathfrak{X}^*, x^* \in \mathfrak{D}(A^*)$ such that $A^* x^* \in \mathfrak{X}^\odot]$.

Furthermore, let $T^*(t)$ be, for any $t \geqslant 0$, the dual operator of $T(t)$, and then $T^\odot(t)$ be the restriction of $T^*(t)$ to $\mathfrak{X}^\odot$; then $T^\odot(t) \in \mathfrak{L}(\mathfrak{X}^\odot, \mathfrak{X}^\odot)$, $t \geqslant 0$, and it is a semi-group of class (C_0) having $A^\odot$ as infinitesimal generator.

Our aim is to prove the following

THEOREM 5.1. *Let $u(t)$ be a continuous function. $0 \leqslant t \leqslant T$ to $\mathfrak{X}$, such that $u(0) = \theta$, and satisfying relation*

(5.1) $$\int_0^T \langle \dot{\phi}^\odot + A^\odot \phi^\odot, u(t) \rangle \, dt = 0$$

for any function $\phi^\odot(t)$, $0 \leqslant t \leqslant T \to \mathfrak{D}(A^\odot)$, $\phi^\odot \in C^1[0, T; \mathfrak{X}^\odot]$, $A^\odot \phi^\odot \in C[0, T; \mathfrak{X}^\odot]$, $\phi^\odot = \theta$ near 0 and near T. Then $u(t) = \theta$ on $[0, T]$.

REMARK. Before giving the proof, let us consider the particular case of reflexive space $\mathfrak{X}$. Then $\mathfrak{X}^\odot = \mathfrak{X}^*$, $A^\odot = A^*$, so we find again the previously proved theorem in § 3.

PROOF OF THE THEOREM. We have firstly

LEMMA 5.1. *The relation*

(5.2) $$\int_0^T \langle \dot{\phi}^\odot + A^\odot \phi^\odot, u \rangle \, dt = 0$$

is verified for the more general class of test-function: $\phi^\odot(t) \in C^1[0, T; \mathfrak{X}^\odot]$, $\phi^\odot(t) \in \mathfrak{D}(A^\odot)$, $(A^\odot \phi^\odot)(t) \in C[0, T; \mathfrak{X}^\odot]$, $\phi^\odot(T) = \theta$.

Let us consider, $\forall \varepsilon > 0$, a scalar-valued function $v_\varepsilon(t)$, continuously differentiable on $0 \leqslant t \leqslant T$, $= 0$ for $0 \leqslant t \leqslant \varepsilon$, $T - \varepsilon \leqslant t \leqslant T$, $= 1$ for $2\varepsilon \leqslant t \leqslant T - 2\varepsilon$, such that $|v'_\varepsilon(t)| \leqslant c/\varepsilon$, $0 \leqslant t \leqslant T$, $|v_\varepsilon(t)| \leqslant 1$, $0 \leqslant t \leqslant T$; then $v_\varepsilon(t) \phi^\odot(t)$ is a test-function as required in theorem 5.1, because it vanishes near $t = 0$ and near $t = T$. We can write henceforth the relation (5.2) for $v_\varepsilon \phi^\odot$, and obtain the following:

$$\int_0^T \langle \dot{v}_\varepsilon \phi^\odot + v_\varepsilon \dot{\phi}^\odot, u \rangle \, dt = -\int_0^T v_\varepsilon \langle A^\odot \phi^\odot, u \rangle \, dt \, .$$

The right-hand integral splits as

$$-\int_{2\varepsilon}^{T-2\varepsilon}\langle A^{\odot}\phi^{\odot}, u\rangle\, dt - \int_{\varepsilon}^{2\varepsilon} \nu_\varepsilon \langle A^{\odot}\phi^{\odot}, u\rangle\, dt - \int_{T-2\varepsilon}^{T-\varepsilon} \nu_\varepsilon \langle A^{\odot}\phi^{\odot}, u\rangle\, dt$$

and is readily seen that

$$\lim_{\varepsilon\to 0} -\int_0^T \nu_\varepsilon \langle A^{\odot}\phi^{\odot}, u\rangle\, dt = -\int_0^T \langle A^{\odot}\phi^{\odot}, u\rangle\, dt\,.$$

The left-hand side integral equals

$$\int_0^T \dot{\nu}_\varepsilon \langle \phi^{\odot}, u\rangle\, dt + \int_0^T \nu_\varepsilon \langle \dot{\phi}^{\odot}, u\rangle\, dt = I_1 + I_2\,.$$

Actually it results

$$I_1 = \int_{\varepsilon}^{2\varepsilon} \dot{\nu}_\varepsilon \langle \phi^{\odot}, u\rangle\, dt + \int_{T-2\varepsilon}^{T-\varepsilon} \dot{\nu}_\varepsilon \langle \phi^{\odot}, u\rangle\, dt = I_3 + I_4\,.$$

Now, $\lim_{\varepsilon\to 0} I_3 = 0$, essentially because $|\dot{\nu}_\varepsilon| \leqslant c/\varepsilon$, and $u(0) = \theta$. Also $\lim_{\varepsilon\to 0} I_4 = 0$, essentially because $|\dot{\nu}_\varepsilon| \leqslant c/\varepsilon$, and $\phi^{\odot}(T) = \theta$, As for I_2, it is obviously seen to converge to $\int_0^T \langle \dot{\phi}^{\odot}, u\rangle\, dt$, as $\varepsilon \to 0$. Hence, altogether, for $\varepsilon \to 0$ we get

$$\int_0^T \langle \dot{\phi}^{\odot}, u\rangle\, dt + \int_0^T \langle A^{\odot}\phi^{\odot}, u\rangle\, dt = 0\,,$$

and the Lemma is proved.

We can continue now the proof of our theorem.

Let us take an arbitrarily given function $k^{\odot}(t) \in C^1[0, T; \mathfrak{X}^*]$. Then consider in the $\odot$-dual space $\mathfrak{X}^{\odot}$, the strong inhomogeneous Cauchy problem

$$\frac{d\psi^{\odot}}{dt} - A^{\odot}\psi^{\odot} = -k^{\odot}, \qquad \psi^{\odot}(0) = \theta . \tag{5.3}$$

Due to the fact that $A^{\odot}$ is the generator of a (C_0)-semigroup $T^{\odot}(t)$ in $\mathfrak{X}^{\odot}$, by a well-known result of Phillips ([7], Theorem 2.2.3), the problem (5.3) has a unique solution (given by the formula $\psi^{\odot}(t) = -\int_0^t T^{\odot}(t-\sigma)\cdot k^{\odot}(\sigma)\,d\sigma$, but this is not important here).

Consider now the function $\phi^{\odot}(t)$, defined for $0 \leqslant t \leqslant T$ through the relation $\phi^{\odot}(t) = \psi^{\odot}(T-t)$.

It is continuously differentiable in $\mathfrak{X}^{\odot}$ on $0 \leqslant t \leqslant T$; it belongs to $\mathfrak{D}(A^{\odot})$, $\forall t \in [0, T]$, and $(A^{\odot}\phi^{\odot})(t) = (A^{\odot}\psi^{\odot})(T-t)$ is continuous, $0 \leqslant t \leqslant T \to \mathfrak{X}^{\odot}$. Finally, $\phi^{\odot}(T) = \psi^{\odot}(0) = \theta$. Hence, $\phi^{\odot}(T)$ is an admissible test-function, and the relation $\int_0^T \langle \dot{\phi}^{\odot} + A^{\odot}\phi^{\odot}, u\rangle\,dt = 0$ is verified.

Furthermore, $d\phi^{\odot}/dt = -\dot{\psi}^{\odot}(T-t)$ and consequently we get:

$$\dot{\phi}^{\odot}(t) + A^{\odot}\phi^{\odot}(t) = -\dot{\psi}^{\odot}(T-t) + A^{\odot}\psi^{\odot}(T-t) = k^{\odot}(T-t)\,,$$

in view of (5.3). Hence, we obtained the identity

$$\int_0^T \langle k^{\odot}(T-t), u(t)\rangle\,dt = 0\,,$$

for any $k^{\odot} \in C^1[0, T; \mathfrak{X}^{\odot}]$, or, obviously, as $t \to T-t$ maps $C^1[0, T; \mathfrak{X}^{\odot}]$ onto itself,

$$\int_0^T \langle h^{\odot}(t), u(t)\rangle\,dt = 0 \qquad \forall h^{\odot} \in C^1[0, T; \mathfrak{X}^{\odot}]\,.$$

Take in particular $h^{\odot}(t) = \nu(t)x^*$, where $x^* \in \mathfrak{X}^{\odot}$. Then

$$\int_0^T \nu(t)\langle x^*, u(t)\rangle\,dt = 0\,, \qquad \text{if } \nu(t) \in C^1[0, T]\,.$$

As $\langle x^*, u\rangle$ is scalar-continuous on $[0, T]$, we obtain $\langle x^*, u(t)\rangle = 0$,

$\forall t \in [0, T]$. But we can let x^* to vary in the total set $\mathfrak{D}(A^*) \subset \mathfrak{X}^{\odot}$. It follows that $u(t) = \theta$, $\forall t \in [0, T]$.

This ends the proof of our theorem.

A simple corollary is the following

THEOREM 5.2. – *Let $u(t) \in C\{[0, T]; \mathfrak{X}\}$, such that $u(0) = \theta$ and assume that*

$$(5.4) \qquad \int_0^T \langle \dot{\phi}^* + A^* \phi^*, u \rangle \, dt = 0 ,$$

for any function $\phi^*(t)$, $0 \leqslant t \leqslant T \to \mathfrak{D}(A^*)$, belonging to $C^1([0, T]; \mathfrak{X}^*)$, such that $A^* \phi^* \in C([0, T]; \mathfrak{X}^*)$ and $\phi^* = \theta$ near 0 and near T. Then $u(t) = \theta$ on $[0, T]$.

In fact it suffices to remark that the class of test-functions considered here contains as a subset the class considered in the theorem 5.1, because $A^{\odot}$ is a certain restriction of A^* to an (eventually) smaller domain. Hence, the relation (5.2) is verified and theorem 5.2 implies $u = \theta$ on $[0, T]$.

We have also the following

THEOREM 5.3. *Let A be the generator of a (C_0) semi-group $T(t)$ in the B-space $\mathfrak{X}$, and A^*; $\mathfrak{D}(A^*) \subset \mathfrak{X}^* \to \mathfrak{X}^*$ be its dual operator, defined on the total set $\mathfrak{D}(A^*)$.*

Let $u(t)$ a continuous function. $0 \leqslant t \leqslant T \to \mathfrak{X}$, such that $u(0) = u_0$ given arbitrarily in $\mathfrak{X}$, and satisfying the relation

$$(5.5) \qquad \int_0^T \langle \dot{\phi}^* + A^* \phi^*, u \rangle \, dt = 0 , \qquad \forall \phi^*(t) \in K_{A^*}(0, T) \text{ (1)} .$$

Then $u(t)$ has the representation $u(t) = T(t) u_0$, $0 \leqslant t \leqslant T$.

Let us consider in fact the strongly continuous function $v(t)$, $0 \leqslant \leqslant t \leqslant T \to \mathfrak{X}$, given by $v(t) = T(t) u_0$. Then (5.5) is valid also for this function v.

In fact, let $(u_n)_1^\infty \subset \mathfrak{D}(A)$ be a sequence convergent to u_0. Let also

(1) This is the class of test-functions considered in Theorem 5.2.

$v_n(t) = T(t)u_n$, so that, as well-known, it is $\dot{v}_n = Av_n$, $0 \leqslant t \leqslant T$. Now

$$\int_0^T \langle \dot{\phi}^*, v_n \rangle \, dt = -\int_0^T \langle \phi^*, \dot{v}_n \rangle \, dt ,$$

as obviously seen. Furthermore is $\langle A^*\phi^*, v_n \rangle = \langle \phi^*, Av_n \rangle$, $\forall t \in [0, T]$. It follows

$$\int_0^T \langle \dot{\phi}^* + A^*\phi^*, v_n \rangle \, dt = -\int_0^T \langle \phi^*, \dot{v}_n \rangle \, dt + \int_0^T \langle \phi^*, Av_n \rangle \, dt = 0 \, .$$

when $n \to \infty$, $v_n(t) \to v(t)$ uniformly on $[0, T]$, as $\sup_{0 \leqslant t \leqslant T} \|T(t)\| = C_T < \infty$ so, it results:

$$\int_0^T \langle \dot{\phi}^* + A^*\phi^*, v \rangle \, dt = 0$$

too. If we take now $w(t) = u(t) - v(t)$, then (5.5) is verified for $w(t)$, and $w(0) = \theta$. By previous theorem, it follows $u(t) = v(t) = T(t)u_0$ on $0 \leqslant t \leqslant T$.

REFERENCES

[1] T. Kato - H. Tanabe, *On the abstract evolution equation*, Osaka Math. J., **14** (1962), 107-133.

[2] S. Zaidman, Notices A.M.S., p. A-328, April 1973.

[3] S. G. Krein, *Linear differential equations in Banach spaces*, Nauka, Moscow, 1967.

[4] R. S. Phillips, *The adjoint semi-group*, Pacific J. Math., **5** (1955), 269-283.

[5] E. Hille - R. S. Phillips, *Functional analysis and semi-groups*, Colloq. Publ., Amer. Math. Soc., 1957.

[6] S. Zaidman, *Remarks on weak solutions of differential equations in Banach spaces*, Bollettino U.M.I., (4) **9** (1974), 638-643.

[7] G. Ladas - V. Lakshmikantham, *Abstract differential equations*, Academic Press, New York and London, 1972.

[8] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.

[9] S. Goldberg, *Unbounded linear operators*, Mc-Graw-Hill book Company, 1966.

Manoscritto pervenuto in redazione il 28 febbraio 1975.